Port-Hamiltonian Macroscopic Modelling based on the Homogenisation Method: case of an acoustic pipe with a porous wall

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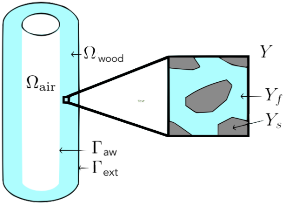
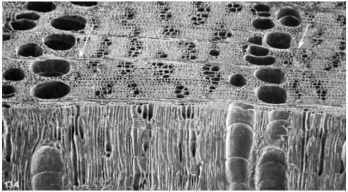
IFAC-LHMNC - Bensançon, France. June, 2024.

Motivation

In some woodwind musical instruments, **acoustic waves** are subjected to **volume dissipation inside the wall**





zoom $\times 80$ – Ulmus procera Salisb.
(extracted from Butterfiled et al., 1972)

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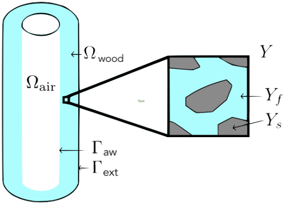

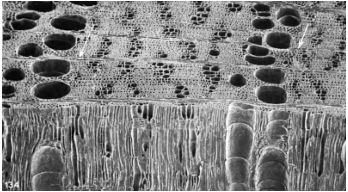

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Audible but slight effect	$\longrightarrow$	experiments require careful attention
Dependence on the pore geometry	$\longrightarrow$	complex model

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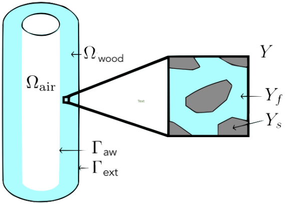

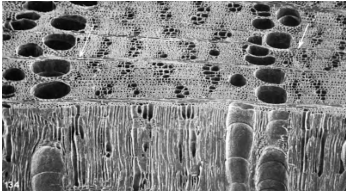

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Objective

Derive a model that is:

(i) Macroscopic $\longrightarrow$ Porosity effect experienced at the wall scale by acoustic waves inside the pipe

(ii) Guaranteed passive $\longrightarrow$ Port Hamiltonian Framework

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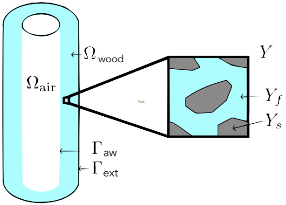

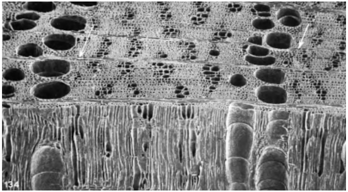

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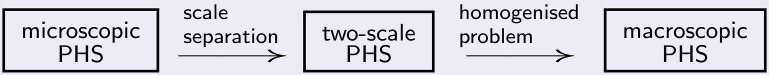
Work in the framework of PHS (≡ Port-Hamiltonian System)

1. METHOD : modelling acoustic waves in the porous wall *(+coupling with pipe)*



2. ILLUSTRATION on a straight acoustic pipe with a thick porous wall

Outline

A. Motivation

B. Fluid equations inside the porous wall

C. Scale separation method: periodic wall homogenisation

D. Macroscopic loss operators in the Laplace domain and application

E. Application to an acoustic pipe with a porous wall

F. Conclusion

Outline

Porous wall: Hypotheses

Domains

Porous wall = Solid part $\cup$ Fluid part

Fluid domain	$\Omega_\mathrm{f} \subset \mathbb{R}^d$:	bounded connected open set
Interface	$\Gamma_\mathrm{f} \subset \partial\Omega_\mathrm{f}$:	regular boundary between Ω_f and the solid part

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Fluid with small fluctuations

Field quantity	mass density $[\mathrm{kg.m}^{-3}]$	particle velocity $[\mathrm{m.s}^{-1}]$	pressure [Pa]	temperature [K]
total	ρ_tot	$\boldsymbol{v}_\mathrm{tot}$	P_tot	T_tot
at rest	$\rho_0 = 1.2$	$\boldsymbol{v}_0 = \mathbf{0}$	$P_0 = 101.3$	$T_0 = 293.15$ ($\approx 20^o$C)
fluctuation	$\rho = \rho_\mathrm{tot} - \rho_0$	$\boldsymbol{v} = \boldsymbol{v}_\mathrm{tot} - \mathbf{0}$	$P = P_\mathrm{tot} - P_0$	$T = T_\mathrm{tot} - T_0$

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Material characteristics

Solid part = isothermal ideally rigid wood $T_0 = 293.15\,\mathrm{K} \approx 20^o\mathrm{C}.$

Fluid part = air = perfect gas with law	$P_\mathrm{tot} = R_0\,\rho_\mathrm{tot}\,T_\mathrm{tot}$
gas constant	$R_0 = 288$ J.kg^{-1}.K^{-1}
specific heat capacity at constant pressure	$C_p = \frac{\gamma R_0}{\gamma-1} \approx 1004$ J.kg^{-1}.K^{-1}
heat capacity ratio	$\gamma = 1.402.$
thermal conductivity	$\kappa \approx 2.57 \times 10^{-2}$J.m^{-1}.s^{-1}.K^{-1},
shear viscosity	$\mu = 1.81 \times 10^{-5}$ kg.m^{-1}.s-1
bulk viscosity	$\zeta = 1.3 \times 10^{-5}$kg.m^{-1}.s^{-1}

Thermo-visco-acoustic equations

Under these assumptions, the governing equations can be approximated by the

Linearised Navier-Stokes equations *(see e.g. [RUY16, BP13])*

Mass conservation in Ω_f

$$\partial_t \rho + \rho_0 \operatorname{div} \boldsymbol{v} = 0, \tag{1a}$$

Momentum conservation in Ω_f

$$\rho_0 \partial_t \boldsymbol{v} = -\nabla P + \mu \Delta \boldsymbol{v} + \left(\zeta + \frac{\mu}{3}\right) \nabla(\operatorname{div} \boldsymbol{v}), \tag{1b}$$

Thermal conduction in Ω_f

$$\rho_0 C_p \partial_t T = \kappa \Delta T + \partial_t P, \tag{1c}$$

Gas state equation in Ω_f

$$P/P_0 = \rho/\rho_0 + T/T_0, \tag{1d}$$

No-slip condition on Γ_f :

$$\boldsymbol{v} = 0, \tag{1e}$$

Thermal contact on Γ_f:

$$T = 0. \tag{1f}$$

Port-Hamiltonian formulation (1/2)

Using the gas state equation (1d) to substitute ρ, equations (1a-1c) rewrite:

Conservation and conduction equations on effort variables P, $\boldsymbol{v}$, T

$$\underbrace{\begin{pmatrix} \frac{1}{P_0} & 0 & -\frac{1}{T_0} \\ 0 & \rho_0 & 0 \\ -\frac{1}{T_0} & 0 & \frac{\rho_0 C_p}{T_0} \end{pmatrix}}_{\mathcal{M}} \underbrace{\begin{pmatrix} \partial_t P \\ \partial_t \boldsymbol{v} \\ \partial_t T \end{pmatrix}}_{\partial_t \boldsymbol{e}} = \underbrace{\begin{pmatrix} 0 & -\operatorname{div} & 0 \\ -\nabla & \mathcal{A} & 0 \\ 0 & 0 & \frac{\kappa}{T_0}\Delta \end{pmatrix}}_{\mathcal{S}} \underbrace{\begin{pmatrix} P \\ \boldsymbol{v} \\ T \end{pmatrix}}_{\boldsymbol{e}}, \qquad (2)$$

with $\begin{cases} \mathcal{M} = \mathcal{M}^\intercal \succ 0 \quad \text{(positive as } \rho_0 C_p/(P_0 T_0) = \gamma/(\gamma-1) > 1) \\ \mathcal{A} = \mu\Delta + \left(\zeta + \frac{\mu}{3}\right)\nabla(\operatorname{div}(\cdot)) \ : \ \text{negative self-adjoint operator} \end{cases}$

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$$\underbrace{\mathcal{J} = -\mathcal{J}^*}_{\text{skew-symmetric}} = \begin{pmatrix} 0 & -\operatorname{div} & 0 \\ -\nabla & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \underbrace{\mathcal{R} = \mathcal{R}^*}_{\text{symmetric}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\mathcal{A} & 0 \\ 0 & 0 & \frac{\kappa}{T_0}(-\Delta) \end{pmatrix} \underbrace{\succeq 0}_{\text{pos.}}$$

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This defines a **linear dissipative port-Hamiltonian system**

(formulated with co-energy ($\equiv$effort) variables $\boldsymbol{e}$, see [VdSJ+14])

Energy: $E = \frac{1}{2}\langle \boldsymbol{e}, \mathcal{M}\boldsymbol{e}\rangle_{\Omega_\mathrm{f}} = \frac{1}{2}\int_{\Omega_f} \boldsymbol{e}^\intercal \mathcal{M}\boldsymbol{e}\,\mathrm{d}\Omega \geq 0$

Power balance: $\dot{E} = \langle \boldsymbol{e}, \mathcal{M}\partial_t\boldsymbol{e}\rangle_{\Omega_\mathrm{f}} \overset{(2)}{=} \langle \boldsymbol{e}, \mathcal{S}\boldsymbol{e}\rangle_{\Omega_\mathrm{f}} = -\underbrace{\langle \boldsymbol{e}, \mathcal{R}\boldsymbol{e}\rangle_{\Omega_\mathrm{f}}}_{\text{dissipated power}} \leq 0$ (if no flow or effort on $\partial\Omega_\mathrm{f}$)

Port-Hamiltonian formulation (2/2)

Consider the energy variable $\boldsymbol{\alpha}$ and Hamiltonian $H(\boldsymbol{\alpha})$ s.t. $\delta_{\boldsymbol{\alpha}} H = \boldsymbol{e}$

$$\boldsymbol{\alpha} := \mathcal{M}\,\boldsymbol{e} = \begin{pmatrix} \frac{P}{P_0} - \frac{T}{T_0} = \frac{\rho}{\rho_0} \\ \rho_0 \boldsymbol{v} \\ \rho_0 C_P \frac{T}{T_0} - \frac{P}{T_0} \end{pmatrix} \begin{array}{l} \text{relative mass density,} \\ \text{momentum density,} \\ \equiv\text{entropy density,} \end{array} \tag{3a}$$

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This yields the standard state-space Port-Hamiltonian System

microscopic PHS in Ω_f

$$\partial_t \boldsymbol{\alpha} = (\mathcal{J} - \mathcal{R})\,\delta_{\boldsymbol{\alpha}} H(\boldsymbol{\alpha}), \tag{4}$$

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$$= -\underbrace{\langle \delta_\alpha H, \mathcal{R}\,\delta_\alpha H\rangle_{\Omega_f}}_{\text{dissipated power} \geq 0} \leq 0 \text{ (if no flow or effort on } \partial\Omega_f)$$

Outline

Homogeneisation problem

1. Introduce a **scaling coefficient** $\varepsilon = L_p/L_m \ll 1$ to separate
   the microscopic pore length L_p at which dissipation takes place
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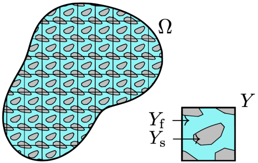

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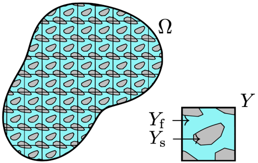

3. Distinguish scales using

   Macroscopic coordinate $\boldsymbol{x} \in \Omega$,

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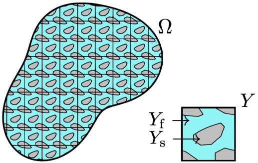

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4. In the governing equations, replace fields and coefficients by

$$(\rho, \boldsymbol{v}, P, T)\,\&\,(\boldsymbol{\alpha}, \boldsymbol{e}) \longrightarrow (\rho_\varepsilon, \boldsymbol{v}_\varepsilon, P_\varepsilon, T_\varepsilon)\,\&\,(\boldsymbol{\alpha}_\varepsilon, \boldsymbol{e}_\varepsilon)$$

$$\mu, \eta, \kappa \longrightarrow \varepsilon^2\mu,\ \varepsilon^2\eta,\ \varepsilon^2\kappa, \qquad \text{(Ansatz from Allard and Atalla, 2009)}$$

$\Rightarrow$ non-trivial limit velocity $v_{\varepsilon\to 0}$, compensating for shrinking pores by reducing friction

Aim of the procedure ("homogeneised effect")

Determine the macroscopic behaviour as $\varepsilon \to 0^+$

Two-scale asymptotic expansion

STEP 1: Assume that each field writes as a power series for the scale ε

$$\rho_\varepsilon(\boldsymbol{x},t) = \rho^{(0)}\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon},t\right) + \varepsilon\,\rho^{(1)}\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon},t\right) + \varepsilon^2\,\rho^{(2)}\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon},t\right) + \dots$$

$$\boldsymbol{v}_\varepsilon(\boldsymbol{x},t) = \boldsymbol{v}^{(0)}\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon},t\right) + \varepsilon\,\boldsymbol{v}^{(1)}\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon},t\right) + \varepsilon^2\,\boldsymbol{v}^{(2)}\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon},t\right) + \dots$$

Idem for P_ε, T_ε and $\boldsymbol{\alpha}_\varepsilon, \boldsymbol{e}_\varepsilon$

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STEP 2: Differential operators and chain rule applied to STEP1

$$\partial_t \equiv \partial_t, \qquad \operatorname{div} \equiv \operatorname{div}_x + \varepsilon^{-1}\operatorname{div}_y,$$

$$\nabla \equiv \nabla_x + \varepsilon^{-1}\nabla_y, \qquad \Delta \equiv \Delta_x + \varepsilon^{-1}(\operatorname{div}_x \nabla_y + \operatorname{div}_y \nabla_x) + \varepsilon^{-2}\Delta_y.$$

STEP 3: Inject power series into the governing equations

Mass conservation: $\partial_t \sum_{k=0}^{\infty} \varepsilon^k \rho^{(k)} + \rho_0 \sum_{k=0}^{\infty} \varepsilon^k \left(\operatorname{div}_x \boldsymbol{v}^{(k)} + \varepsilon^{-1}\operatorname{div}_y \boldsymbol{v}^{(k)}\right) = 0.$

& similar results for other equations

$\rightarrow$ The formal identification of the terms with homogeneous degree ε^k leads to a cascade of equations.

Detailed results for Linearised Navier-Stokes equations

The degrees ε^{-1} and ε^{0} are sufficient to describe the homogenised problem.

Mass conservation

$$\varepsilon^{-1}: \qquad \mathrm{div}_y\, \boldsymbol{v}^{(0)} = 0,$$

$$\varepsilon^{0}: \qquad \partial_t \rho^{(0)} + \rho_0\, \mathrm{div}_x\, \boldsymbol{v}^{(0)} + \rho_0 \mathrm{div}_y\, \boldsymbol{v}^{(1)} = 0.$$

Momentum conservation

$$\varepsilon^{-1}: \qquad \nabla_y P^{(0)} = 0,$$

$$\varepsilon^{0}: \qquad \rho_0 \partial_t \boldsymbol{v}^{(0)} = \mu \Delta_y \boldsymbol{v}^{(0)} + (\zeta + \mu/3)\, \nabla_y (\mathrm{div}_y\, \boldsymbol{v}^{(0)}) - \nabla_x P^{(0)} - \nabla_y P^{(1)}.$$

Thermal conduction

$$\varepsilon^{0}: \qquad \kappa \Delta_y T^{(0)} - \rho_0 C_p \partial_t T^{(0)} = -\partial_t P^{(0)}.$$

Gas state equation

$$\varepsilon^{0}: \qquad P^{(0)}/P_0 = \rho^{(0)}/\rho_0 + T^{(0)}/T_0.$$

Boundary conditions :

$$\forall (\boldsymbol{x}, \boldsymbol{y}) \in \Omega \times \partial Y_{\mathrm{f}}, \qquad \boldsymbol{v}^{(0)}(\boldsymbol{x}, \boldsymbol{y}, t) = \boldsymbol{v}^{(1)}(\boldsymbol{x}, \boldsymbol{y}) = 0,$$

$$\forall (\boldsymbol{x}, \boldsymbol{y}) \in \Omega \times \partial Y_{\mathrm{f}}, \qquad T^{(0)}(\boldsymbol{x}, \boldsymbol{y}, t) = 0.$$

Two-scale Port-Hamiltonian Systems

Two-scale PHS for co-energy variables $\boldsymbol{e}_\varepsilon$: $\qquad \mathcal{M}\,\partial_t \boldsymbol{e}_\varepsilon = (\mathcal{J}_\varepsilon - \mathcal{R}_\varepsilon)\boldsymbol{e}_\varepsilon,$

- constant matrix: $\mathcal{M}$
- ε-structured energy: $\frac{1}{2}\int_{\Omega_\varepsilon}\sum_{k_1,k_2\geq 0}\varepsilon^{k_1+k_2}\boldsymbol{e}^{(k_1)}\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon},t\right)^\mathsf{T}\mathcal{M}\boldsymbol{e}^{(k_2)}\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon},t\right)\mathrm{d}\Omega$
- ε-structured differential operators $\begin{cases} \mathcal{J}_\varepsilon &= -\mathcal{J}_\varepsilon^\star &= \sum_{n=-1}^{0}\varepsilon^n\mathcal{J}_n \\ \mathcal{R}_\varepsilon &= \mathcal{R}_\varepsilon^\star &= \sum_{n=0}^{2}\varepsilon^n\mathcal{R}_n \succeq 0 \end{cases}$

  with

n	$\mathcal{J}_n$	$\mathcal{R}_n$
-1	$\begin{pmatrix} 0 & -\operatorname{div}_y & 0 \\ -\nabla_y & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$0_{3\times 3}$
0	$\begin{pmatrix} 0 & -\operatorname{div}_x & 0 \\ -\nabla_x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\mathcal{A}_y & 0 \\ 0 & 0 & -\frac{\kappa}{T_0}\Delta_y \end{pmatrix}$
1	$0_{3\times 3}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\mathcal{A}_{xy} & 0 \\ 0 & 0 & -\frac{\kappa}{T_0}\Delta_{xy} \end{pmatrix}$
2	$0_{3\times 3}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\mathcal{A}_x & 0 \\ 0 & 0 & -\frac{\kappa}{T_0}\Delta_x \end{pmatrix}$

Two-scale PHS and Macroscopic PHS

Two-scale PHS for co-energy variables $\boldsymbol{e}_\varepsilon$: $\mathcal{M}\,\partial_t \boldsymbol{e}_\varepsilon = (\mathcal{J}_\varepsilon - \mathcal{R}_\varepsilon)\boldsymbol{e}_\varepsilon$,

Details ($\mathcal{S}_n = (\mathcal{J}_n - \mathcal{R}_n)$):

$$
\begin{aligned}
\varepsilon^{-1}: &\quad 0 = \mathcal{S}_{-1}\boldsymbol{e}^{(0)}, \\
\varepsilon^{0}: &\quad \mathcal{M}\partial_t \boldsymbol{e}^{(0)} = \mathcal{S}_{-1}\boldsymbol{e}^{(1)} + \mathcal{S}_0\boldsymbol{e}^{(0)}, \\
\varepsilon^{1}: &\quad \mathcal{M}\partial_t \boldsymbol{e}^{(1)} t = \mathcal{S}_{-1}\boldsymbol{e}^{(2)} + \mathcal{S}_0\boldsymbol{e}^{(1)} + \mathcal{S}_1\boldsymbol{e}^{(0)}, \\
\varepsilon^{k}, k \geq 2: &\quad \mathcal{M}\partial_t \boldsymbol{e}^{(k)} = \mathcal{S}_{-1}\boldsymbol{e}^{(k+1)} + \mathcal{S}_0\boldsymbol{e}^{(k)} + \mathcal{S}_1\boldsymbol{e}^{(k-1)} + \mathcal{S}_2\boldsymbol{e}^{(k-2)}.
\end{aligned}
$$

The homogenised problem (ε^{-1} & ε^{0}) defines a **Macroscopic PHS**

$$
\begin{pmatrix} \mathcal{M}\partial_t \boldsymbol{e}^{(0)} \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} \mathcal{S}_0 & \mathcal{S}_{-1} \\ \mathcal{S}_{-1} & 0 \end{pmatrix}}_{\text{negativeop.}} \begin{pmatrix} \boldsymbol{e}^{(0)} \\ \boldsymbol{e}^{(1)} \end{pmatrix},
$$

Remark: $\boldsymbol{e}^{(1)}$ serves as the Lagrange multiplier.

Two-scale PHS and Macroscopic PHS

Two-scale PHS for energy variables $\boldsymbol{\alpha}_\varepsilon$

$$\underbrace{\partial_t \boldsymbol{\alpha}_\varepsilon}_{\mathcal{M}\partial_t \boldsymbol{e}_\varepsilon} = \underbrace{(\mathcal{J}_\varepsilon - \mathcal{R}_\varepsilon)}_{\mathcal{S}_\varepsilon} \underbrace{\delta_{\boldsymbol{\alpha}_\varepsilon} H_\varepsilon(\boldsymbol{\alpha}_\varepsilon)}_{\mathcal{M}^{-1}\boldsymbol{\alpha}_\varepsilon = \boldsymbol{e}_\varepsilon},$$

with $\boldsymbol{\alpha}_\varepsilon := \mathcal{M}\, \boldsymbol{e}_\varepsilon$

and $H_\varepsilon(\boldsymbol{\alpha}_\varepsilon) := \frac{1}{2}\langle \mathcal{M}^{-1}\boldsymbol{\alpha}_\varepsilon, \boldsymbol{\alpha}_\varepsilon \rangle_{\Omega_\varepsilon}$,

The homogenised problem (ε^{-1} & ε^0) defines a Macroscopic PHS

$$\begin{pmatrix} \partial_t \boldsymbol{\alpha}^{(0)} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathcal{S}_0 & \mathcal{S}_{-1} \\ \mathcal{S}_{-1} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{M}^{-1}\boldsymbol{\alpha}^{(0)} \\ \mathcal{M}^{-1}\boldsymbol{\alpha}^{(1)} \end{pmatrix}.$$

Outline

Homogenised fields

Denote the fields with a hat symbol in the Laplace domain
$s \in \overline{\mathbb{C}_0^+} = \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}$ (or $s = \mathrm{i}\omega \in \mathrm{i}\mathbb{R}$ for the Fourier domain)

Fields $\hat{P}^{(0)}$, $\hat{T}^{(0)}$ and $\hat{\boldsymbol{v}}^{(0)}$

$$\hat{T}^{(0)}(\boldsymbol{x}, \boldsymbol{y}) = \hat{P}^{(0)}(\boldsymbol{x})\, \varphi(\boldsymbol{y}),$$

$$\hat{\boldsymbol{v}}^{(0)}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{d} \boldsymbol{w}_i(\boldsymbol{y}) \frac{\partial \hat{P}^{(0)}}{\partial x_i}(\boldsymbol{x}),$$

where φ and $(\boldsymbol{w}_i, q_i)$ are solutions to boundary problems in the unit cell :

$$\begin{cases} \kappa\, \Delta\varphi(\boldsymbol{y}) - s\rho_0 C_p\, \varphi(\boldsymbol{y}) = -s \ \text{ on } Y_\mathrm{f}, \\ \varphi|_{\partial Y_\mathrm{f}} = 0, \end{cases} \tag{5}$$

$$\begin{cases} \mu\Delta\boldsymbol{w}_i(\boldsymbol{y}) - \nabla q_i(\boldsymbol{y}) - s\rho_0\, \boldsymbol{w}_i(\boldsymbol{y}) = \boldsymbol{E}_i \ \text{ on } Y_\mathrm{f} \\ \mathrm{div}\, \boldsymbol{w}_i(\boldsymbol{y}) = 0 \ \text{ on } Y_\mathrm{f} \\ \boldsymbol{w}_i|_{\partial Y_\mathrm{f}} = 0. \end{cases} \tag{6}$$

$\boldsymbol{E}_i$: unit vector oriented in direction i

Effective fluid and loss coefficients in acoustic pipes

Define the permeability operators

$$\hat{\boldsymbol{A}}\boldsymbol{E}_i = \langle \boldsymbol{w}_i \rangle \text{ and } \hat{B} = \langle \varphi \rangle$$

with the average operator $\langle \cdot \rangle = \int_{Y_f} \cdot \, \mathrm{d}\boldsymbol{y} / \int_{Y_f} 1 \, \mathrm{d}\boldsymbol{y}$.

Result

(see [Thi23, chap. 6])

Fields $\hat{P}^{(0)}$, $\hat{T}^{(0)}$ and $\hat{\boldsymbol{v}}^{(0)}$ correspond to an effective fluid characterised by

$$\rho_e(s) = -\frac{1}{s} \hat{\boldsymbol{A}}^{-1}(s), \qquad \text{(density)}$$

$$K_e(s) = \frac{1}{1/P_0 - \hat{B}(s)/T_0}, \qquad \text{(compressibility)}$$

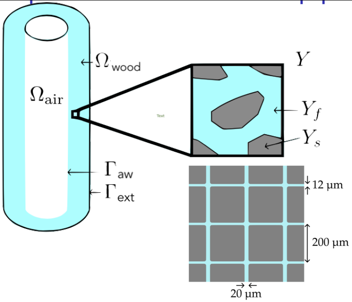
Loss coefficients (analog to those involved in acoustic pipes with boundary layers) that represent the macroscopic dissipation inside the porous wall are

$$K_t(s) = 1 - \rho_0 C_p \hat{B}(s), \qquad \text{(thermal)}$$

$$\boldsymbol{K}_v(s) = \boldsymbol{I}_d + \rho_0 s \hat{\boldsymbol{A}}(s), \qquad \text{(viscous)}$$

Outline

Wave equation in an acoustic pipe with a porous wall



Cylindrical pipe with a porous wall

$$\begin{aligned}
\Omega_{\text{air}} &= \{x \,|\, r < R\}, \\
\Omega_{\text{wood}} &= \{x \,|\, < r < R_{\text{ext}}\}, \\
\Gamma_{\text{ab}} &= \{x \,|\, r = R\}, \\
\Gamma_{\text{ext}} &= \{x \,|\, r = R_{\text{ext}}\}.
\end{aligned}$$

Wave propagation in $\Omega_{\text{air}} \cup \Omega_{\text{wood}}$ is governed by

$$\widetilde{\rho}_e(s,x)\, s\, \widetilde{v}(s,x) = -\nabla \hat{P}^{(0)}(s,x) \quad \& \quad s\, \hat{P}^{(0)}(s,x) = -\widetilde{K}_e(s,x)\, \text{div}\, \widetilde{v}(s,x)$$

$$\text{with} \left\{ \begin{array}{llll} \widetilde{v} = \hat{v}, & \widetilde{K}_e = \gamma P_0, & \widetilde{\rho}_e = \rho_0 & \text{in } \Omega_{\text{air}}, \\ \widetilde{v}(s,x) = \phi \langle \hat{v} \rangle (s,x), & \widetilde{K}_e(s,x) = \frac{K_e(s)}{\phi}, & \widetilde{\rho}_e(s,x) = \frac{\rho_e(s)}{\phi} & \text{in } \Omega_{\text{bois}} \end{array} \right.$$

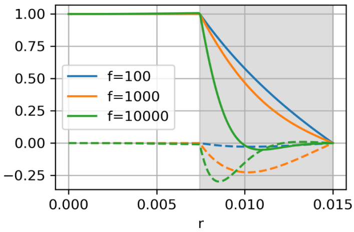
and porosity coefficient $\phi := |Y_f|/|Y|$.

Helmholtz equation (eliminating $\tilde{v}$)

$$s^2\, \widetilde{K}_e^{-1}(s,x)\, \hat{P}^{(0)}(s,x) - \text{div}\big(\widetilde{\rho}_e^{-1}(s,x)\, \nabla \hat{P}^{(0)}(s,x)\big) = 0 \text{ in } \Omega \;\; \& \;\; \hat{P}^{(0)} = 0 \text{ on } \Gamma_{\text{ext}}$$

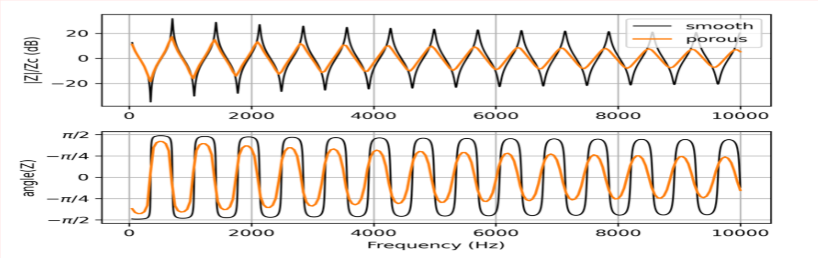
Numerical results

1. Pipe parameters: $R = 7.45\,\text{mm}$, $R_{\text{ext}} = 15.0\,\text{mm}$, porosity $\phi = 0.156$
2. Solve the cell problem (high-order finite element code *Montjoie*) (see [Dur21])
3. Derive the dispersion relationship of the first pressure mode (closest to the planar mode).



Radial profile of the first pressure mode: real (-) and imaginary(- -) parts

Normalised input impedance for the length $L = 240.5\,\text{mm}$



(black) perfect wall (Zwikker-Kosten model);
(orange) porous wall (academic geometry with exaggerated porosity)

Outline

Conclusion

Interest of combining the homogenisation method and the port-Hamiltonian formulation

$\rightarrow$ enables the passivity of microscopic phenomena to be characterised at several scales

$\rightarrow$ produces passive macroscopic descriptions

$\rightarrow$ interpretation of microscopic / two-scale / macroscopic energies and operators

Perspectives

Convergence analysis of the solutions

Examine this approach on nonlinear systems.

Thank you for your attention

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