Port-Hamiltonian Systems in musical acoustics

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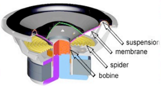
Frauenchiemsee, Bavaria, Germany



Port-Hamiltonian Systems

with

a component-based approach

(finite-dimensional case $\equiv$ ODEs)

A physical system is made of. . .

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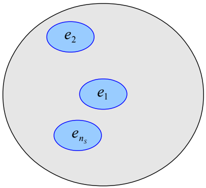

(i) Energy-storing components

$E = \sum_{n=1}^{N} e_n \geq 0$

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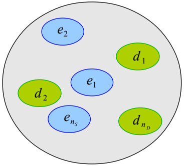

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(ii) Memoryless passive components

$P_{\mathbf{diss}} = \sum_{m=1}^{M} d_m > 0$ (dissipative) or $= 0$ (conservative)

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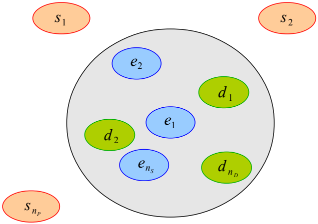

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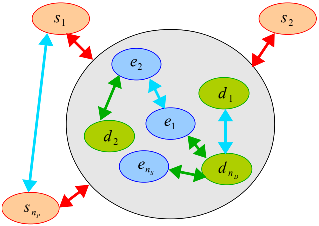

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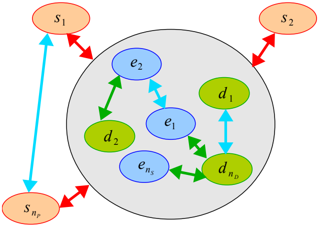

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$+$ Conservative connections $\rightarrow$ *sum of received powers is zero*

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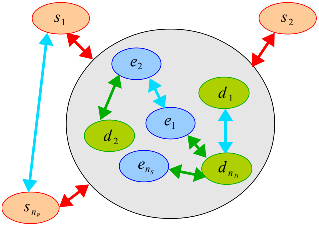
⚠ receiver convention



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$P_{\textbf{stored}} + \underbrace{P_{\textbf{diss}}}_{\geq 0} + P_{\textbf{ext}} = 0$ with $P_{\textbf{stored}} = \dot{E}$ **(power balance)**

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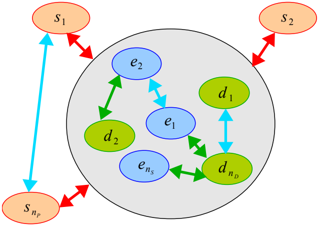

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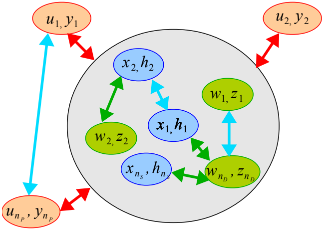

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(effort × flow : force × velocity, voltage × current, etc)

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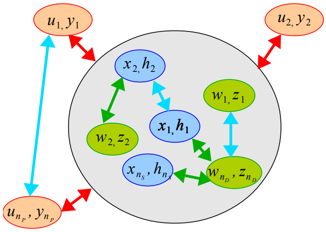

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PHS: Input-State-Output representation ($\boldsymbol{S}$: interconnection matrix)

$$\underbrace{\begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{w} \\ \mathbf{y} \end{bmatrix}}_{\mathbf{f}} = \underbrace{\begin{bmatrix} \boldsymbol{S}_{\mathbf{xx}} & \boldsymbol{S}_{\mathbf{xw}} & \boldsymbol{S}_{\mathbf{xu}} \\ * & \boldsymbol{S}_{\mathbf{ww}} & \boldsymbol{S}_{\mathbf{wu}} \\ * & * & \boldsymbol{S}_{\mathbf{yu}} \end{bmatrix}}_{\text{with } \boldsymbol{S} = -\boldsymbol{S}^{\mathsf{T}}} \underbrace{\begin{bmatrix} \nabla H(\mathbf{x}) \\ \mathbf{z}(\mathbf{w}) \\ \mathbf{u} \end{bmatrix}}_{\mathbf{e}} \qquad (1)$$

(*i*) storage → differential eq.
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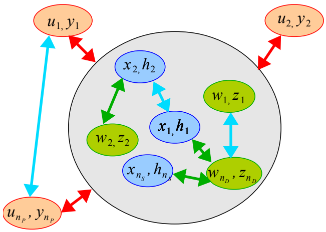

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Power balance: $\mathbf{e}^{\mathsf{T}}\mathbf{f} \overset{(1)}{=} \mathbf{e}^{\mathsf{T}}\boldsymbol{S}\mathbf{e} = 0$ as $\boldsymbol{S} = -\boldsymbol{S}^{\mathsf{T}}$ $\Rightarrow \mathbf{e}^{\mathsf{T}}\boldsymbol{S}\mathbf{e} = (\mathbf{e}^{\mathsf{T}}\boldsymbol{S}\mathbf{e})^{\mathsf{T}} = -(\mathbf{e}^{\mathsf{T}}\boldsymbol{S}\mathbf{e})$

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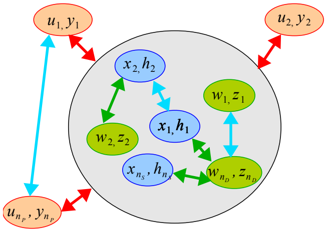

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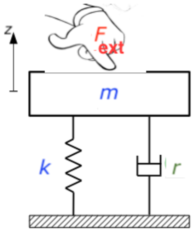
→ **Differential-Algebraic Formulation** *(with no constraint: PH-DAE [Maschke,Schaft])*

Example: damped mechanical oscillator excited by F_{ext} $(m\ddot{z} + r\dot{z} + kz = F_{\text{ext}})$

(no gravity)



- **4 separate components**

(i$_1$) mass $\boxed{\text{m}}$ of momentum $\pi = mv$ (energy: $\frac{1}{2}mv^2 = \frac{\pi^2}{2m}$),

(i$_2$) spring $\boxed{\text{sp}}$ of elongation ξ

(ii) damper $\boxed{\text{dp}}$ of velocity V_{dp}

(iii) actuator $\boxed{\text{ext}}$ applying a force F_{ext} (→your finger experiences $-F_{ext}$)

	state	energy H_n	flow **f**	effort **e**
m	$x_1 := \pi$	$\pi^2/(2m)$	$\dot{x}_1 = \dot{\pi}$	$H_1'(x_1) = x_1/m$
sp	$x_2 := \xi$	$k\,\xi^2/2$	$\dot{x}_2 = \dot{\xi}$	$H_2'(x_2) = k\,x_2$
dp	blue : force		$w := V_{\text{dp}}$	$z(w) := r\,w$
ext	red : velocity		$y := V_{\text{ext}}$	$u := -F_{\text{ext}}$

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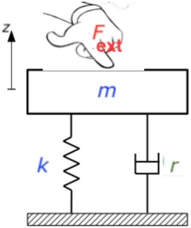

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- **assembled with rigid connections**
  - internal forces are balanced $F_{\mathrm{m}} + F_{\mathrm{sp}} + F_{\mathrm{dp}} + (-F_{\mathrm{ext}}) = 0$
  - all velocities are equal $V_{\mathrm{m}} = V_{\mathrm{sp}} = V_{\mathrm{dp}} = V_{\mathrm{ext}}$

$$\underbrace{\begin{array}{r} \dot{\pi} = F_{\mathrm{m}} \\ \dot{\xi} = V_{\mathrm{sp}} \\ \hline V_{\mathrm{dp}} \\ \hline V_{\mathrm{ext}} \end{array} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ w \\ y \end{bmatrix}}_{\mathbf{f}} = \underbrace{\begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{S} = -\boldsymbol{S}^\top} \underbrace{\begin{bmatrix} H_1'(x_1) \\ H_2'(x_2) \\ z(w) \\ u \end{bmatrix} \begin{array}{l} V_{\mathrm{m}} = \pi/m \\ F_{\mathrm{sp}} = k\xi \\ \hline F_{\mathrm{dp}} = r\,V_{\mathrm{dp}} \\ \hline -F_{\mathrm{ext}} \end{array}}_{\mathbf{e}}$$

→ Formulation (1) with $H(\mathbf{x}) = H_1(x_1) + H_2(x_2)$ *(ODE: with $z = \xi$)*
→ $\boldsymbol{S} = -\boldsymbol{S}^\top$ is canonical *(no mechanical coefficients)*