Passive Systems, Port-Hamiltonian Systems

and Applications in Audio/Acoustics

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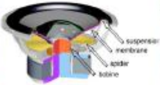

Context and motivation

1. **Model input-output multi-physics systems** for sound and musical applications:
    - Phenomena: mechanical, acoustic, electronic, magnetic, etc
    - Realism: nonlinearities, non ideal dissipations, etc
2. **Satisfy fundamental physical properties**:
    - causality, stability, passivity and more precisely ...
    - the **power balance** structured into conservative/dissipative/source parts
    - other natural invariants and symmetries (if any)
3. **Simulate such systems and preserve these properties** in the discrete time domain *(+accuracy+sound quality/Shannon-Nyquist principle)*
4. **Design code generators** from netlists for real-time applications
5. **Design correctors and controllers** to reach target behaviours

Outline

Stability and passivity in **nonlinear dynamical systems**

- Stability of an equilibrium point (autonomous system)
- Passivity of an input/output system (input/output system)

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→ **Lyapounov analysis**

Preamble (1/4): autonomous systems

$$\begin{aligned}\dot{x}(t) &= f\big(x(t)\big), \quad \text{for } t \geq 0, \quad \text{with } f : \mathbb{R}^n \to \mathbb{R}^n \quad (n \in \mathbb{N}^*)\\ x(0) &= x_0 \in \mathbb{R}^n\end{aligned}$$

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x can be defined on $J_{x_0} \subseteq \mathbb{R}$, an open maximal interval that contains 0, or on interval $J^+_{x_0} := J_{x_0} \cap \mathbb{R}_+$, for its restriction to positive times.

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Rk: $J_{x^*} = \mathbb{R}, \quad J_{x^*}^+ = \mathbb{R}^+$

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Stabilities of x^* (L: local, A: asymptotic, G: global)

(LS) if: $\forall R > 0, \exists r(R) > 0$ such that $\forall x_0 \in \mathbb{R}^n$,
$\|x_0 - x^*\| < r(R) \Rightarrow \|x(t) - x^*\| < R, \ \forall t \in J_{x_0}^+$

Lemma: if $\|x_0 - x^*\| < r(R)$, then $J_{x_0}^+ = \mathbb{R}^+$

(LAS) if: (LS) and $\exists r > 0$ s.t. $\|x_0 - x^*\| < r \Rightarrow \lim_{t \to +\infty} x(t) = x^*$

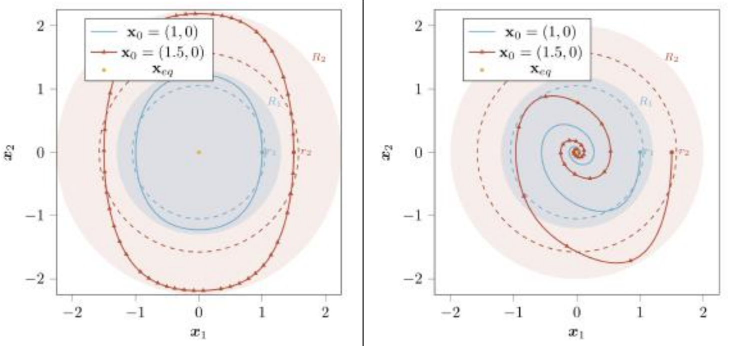
(GAS) if: (LAS) for all $r > 0$

Preamble (2/4): the Duffing oscillator

$$\ddot{y} + \alpha\dot{y} + (1+\beta y^2)y = 0$$

$$\dot{x}(t) = f\big(x(t)\big), \quad \text{with } x = [y, \dot{y}]^T,$$
$$\text{and } f(x) = [x_2, \; -\alpha x_2 - (1+\beta x_1^2)x_1]^T$$

LS (no damping: $\alpha = 0$) | LAS (damping: $\alpha > 0$)



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Preamble (3/4): Lyapounov analysis (of a system $\mathcal{S}: \dot{x} = f(x)$)

Definition (Hyp.: $x^* = 0$ and $\Omega \subseteq \mathbb{R}^n$ open set)

$V : \Omega \longrightarrow \mathbb{R}$ is a Lyapounov function of $\mathcal{S}$ if:

(i) V is $\mathcal{C}^1$-regular on Ω

(ii) $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$

(iii) $\frac{\mathrm{d}}{\mathrm{d}t} V \circ x(t) \leq 0$ for all trajectories of $\mathcal{S}$ in Ω

($\Leftrightarrow \nabla V(x)^T f(x) \leq 0$, for all x in Ω)

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Let $\mathcal{I}$ be the **largest subset of** $\{x \in \Omega \text{ s.t. } \nabla V(x)^T f(x) = 0\}$ (points leaving V invariant) that is **invariant under the flow** in positive time.
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Usual difficulty: find a Lyapounov function for a given nonlinear f

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Input/output system (u: input, y: output, $\dim u = \dim y \geq 1$)

$$\mathcal{S}: \quad \dot{x} = f\big(x, u\big), \quad y = h(x, u) \quad \text{and } x(0) = x_0$$

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Strict passivity: $\frac{\mathrm{d}}{\mathrm{d}t} V \circ x(t) \leq y(t)^T u(t) - \psi\big(x(t)\big)$

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→ Stability for $u = 0$

→ Stabilization for dissipative feedback-loop laws: $(u = -Ry \Rightarrow y^T u = -R\|y\|^2 \leq 0)$

→ In physics, a natural Lyapounov function is the energy