GENERAL COVARIANT FORMULATION OF RELATIVISTIC HYPERELASTICITY

Boris KOLEV and Rodrigue DESMORAT [1]

LMPS
Université Paris-Saclay (France)

Authors have extended the Classical Continuum Mechanics modeling to Relativistic continuous media,

- for the modeling of the solid crust of neutron stars, at the astrophysics scale
  (Souriau, 1958, 1964, Synge, 1959, Rayner, 1963, Bennoun, 1965, Carter and Quintana, 1972, Lamoureux-Brousse, 1989, Kijowski and Magli, 1992, 1997, Beig and Schmidt, 2003, Epstein et al., 2006, Wernig-Pichler, 2006, Gundlach et al., 2011, Brown, 2021)
- but also at a local scale for mechanical engineering applications
  (Grot and Eringen, 1966, Maugin, 1978, Panicaud and Rouhaud, 2013, 2015)

- Let us formulate Relativistic Hyperelasticity and associated mechanical strains and stresses within the Variational Relativity framework.
- Let us obtain Netwon-Cartan classical limit $\underbrace{c \to \infty}_{\text{infinite light speed}}$ with gravity.

1. B. Kolev and R. Desmorat, *J. Mech. Phys. Solids,* 181, 105463, 2023.

Variational Relativity (Souriau, 1958, 1960, 1964)

The Variational Relativity geometric framework, introduced by Souriau, extremalizing the Lagrangian (i.e. Action)

$$\mathscr{L}\underbrace{[g,\Psi]}_{\text{fields}} = \underbrace{\mathscr{H}[g]}_{\text{Relativity}} + \underbrace{\mathscr{L}^{\text{matter}}[g,\Psi]}_{\text{Matter}} = \iiiint_{\mathscr{U}_{4D}=\mathscr{M}} L(g,\Psi,\underbrace{\ldots,\ldots,\ldots}_{\text{jets of fields}})\,\mathrm{vol}_g$$

is radically different from the ones

- by Lamoureux-Brousse (1989) (see also Rouhaud et al., 2013) which considers a perturbation of 4D universes,
- by Grot and Eringen (1966), Maugin (1972) and Epstein et al. (2006), which introduce an imbedding of a body-time (the Cartesian product $I \times \mathscr{B}$ of a time interval $I = [t_i, t_f]$ and the 3D body $\mathscr{B}$) into the Universe.

Variational Relativity does not assume *a priori* the definition of a (space)time.

Outline

1. (Gauge Theory) 4D General Covariant formulation of Hyperelasticity
2. Spacetimes : introduction of an observer/orthogonal decompositions
3. Relativistic Hyperelasticity in (curved) Schwarzschild spacetime
4. Galilean limit of Relativistic Hyperelasticity

OUTLINE

1. (Gauge Theory) 4D General Covariant formulation of Hyperelasticity
2. Spacetimes : introduction of an observer/orthogonal decompositions
3. Relativistic Hyperelasticity in (curved) Schwarzschild spacetime
4. Galilean limit of Relativistic Hyperelasticity

4D Formalism for Continuum Mechanics

(Souriau, 1958, 1960, 1964)

Souriau's modeling of perfect matter is inspired by Gauge Theory, where matter fields are described by sections of vector bundles (here trivial).

- A perfect matter field is a vector-valued function

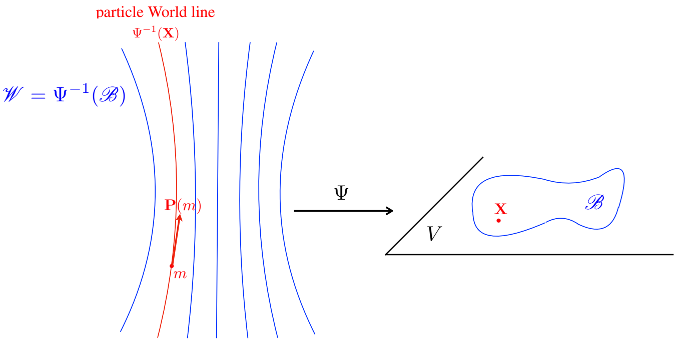
$$\Psi : \mathscr{M} \to V = \mathbb{R}^3,$$

  where $\mathscr{M}$, the Universe, is a manifold of dimension 4, equipped with a Lorentzian metric g of signature $(-,+,+,+)$.

- The notation Ψ for the matter field is on purpose,
  as usually used for the wave function in Quantuum Mechanics.

The body $\mathscr{B}$, the mass measure μ and the World tube $\mathscr{W}$

- Matter is described by a compact orientable submanifold with border, of dimension 3, the body $\mathscr{B} \subset \mathbb{R}^3$, which labels the particles.
- The body $\mathscr{B}$ is endowed with a volume form, the mass measure μ (or particle density).
- Ψ is assumed to be a submersion, so that $\mathscr{W} := \Psi^{-1}(\mathscr{B})$ is fibered by the World lines of the particles $\Psi^{-1}(\mathbf{X})$, $\mathbf{X} \in \mathscr{B}$.
- $\mathscr{W}$ is called the matter tube or World tube.



A REVERSE VIEW OF THE 3D FORMALISM

- In the classic 3D formalism, a configuration is an embedding

$$p\colon \mathscr{B} \to \mathscr{E}, \qquad \mathbf{X} \mapsto \mathbf{x} = p(\mathbf{X}),$$

of the body $\mathscr{B}$ in (Euclidean) space $\mathscr{E}$.

- In the present 4D formalism, a configuration is an application

$$\Psi\colon \mathscr{M} \to \mathscr{B}, \qquad m \mapsto \mathbf{X} = \Psi(m),$$

from the 4D Universe $\mathscr{M}$ to the 3D body $\mathscr{B}$.

An essential difference is that, in classical theory, the embedding p and its differential (the so-called deformation gradient)

$$\mathbf{F} = Tp\colon T\mathscr{B} \to T\mathscr{E}, \qquad \delta\mathbf{X} \mapsto \delta\mathbf{x} = \mathbf{F}\delta\mathbf{X},$$

are invertible, whereas in Variational Relativity Ψ and $T\Psi = d\Psi$ are not.

Current of matter P

- The pullback $\Psi^*\mu$ of mass measure μ is a 3-form on the Universe $\mathscr{M}$. As Ψ is a submersion, there exists a (quadri-)vector $\mathbf{P}$ which does not vanish on $\mathscr{W}$, such as

$$\Psi^*\mu = i_{\mathbf{P}}\mathrm{vol}_g = \mathbf{P}\cdot\mathrm{vol}_g.$$

  This vector field $\mathbf{P}$ is the current of matter.
- To describe perfect matter, Souriau also assumes that $\mathbf{P}$ is of type time,

$$\|\mathbf{P}\|_g^2 = g(\mathbf{P},\mathbf{P}) < 0 \quad \text{on the World tube } \mathscr{W}$$

- We then define the rest mass density and the (unit norm) quadrivelocity $\mathbf{U}$

$$\rho_r := \sqrt{-\|\mathbf{P}\|_g^2}, \quad \text{so that} \quad \mathbf{P} = \rho_r\mathbf{U}, \quad \text{where} \quad \|\mathbf{U}\|_g^2 = -1.$$

Since $\Psi^*\mu = i_{\mathbf{P}}\mathrm{vol}_g$ we get

Lemma (Relativistic version of mass conservation)

$$\mathrm{div}^g\,\mathbf{P} = \mathrm{div}^g(\rho_r\mathbf{U}) = 0.$$

Mass conservation

Lemma (Souriau, 1958)

Let $\mathbf{q}$ *be a fixed Riemannian metric on the body* $\mathscr{B}$*. Then, the rest mass density* ρ_r *can be expressed as*

$$\rho_r = \rho_{\mathbf{q}}(\Psi)\sqrt{\det\left[\mathbf{K}(\mathbf{q}\circ\Psi)\right]},$$

where Ψ *is the matter field,* $\mathbf{K}$ *is the conformation and* $\rho_{\mathbf{q}} = \dfrac{\mu}{\mathrm{vol}_{\mathbf{q}}}$.

Mass conservation in (Classical) Continuum Mechanics (on Ω_0)

$$\rho_0 = (\rho\circ\phi)J, \qquad J := \sqrt{\det(\mathfrak{g}^{-1}\mathbf{C})},$$

$\phi\colon \Omega_0 \to \Omega$ is the deformation and $\mathbf{C} := \phi^*\mathfrak{g}$, is right Cauchy–Green tensor.

The conformation $\mathbf{K}$

The conformation, denoted $\mathbf{K}$, is the cornerstone of the formulation of large-scale Relativistic Hyperelasticity

▷ as in the large (astrophysics) scale modeling of neutron stars (with a solid crust).

Definition (Souriau, 1958)

The conformation is the vector-valued function

$$\mathbf{K} := (T\Psi)\, g^{-1} (T\Psi)^{\star}.$$

- At any point $m \in \mathscr{W}$, $\mathbf{K}(m)$ is a positive definite quadratic form on $V^{\star}$ (since $\mathbf{U}$ is of type time).
- Since $\Psi : \mathscr{W} \to \mathscr{B}$ is not invertible, $\mathbf{K}$ is not the pushforward by Ψ of g^{-1}.

To be compared to the (3D) definition of right Cauchy-Green tensor

$$\mathbf{C} := \phi^{*}\mathfrak{g} = \mathbf{F}^{\star}\mathfrak{g}\,\mathbf{F} = (\mathbf{F}^{-1}\mathfrak{g}^{-1}\mathbf{F}^{-\star})^{-1}, \qquad \mathbf{F} := T\phi.$$

General covariance

The fundamental postulate of general relativity is that
the laws of physics are independent of the choice of coordinates.

- More precisely, this means that the Lagrangian $\mathscr{L}$ is invariant by any (local) diffeomorphism of the Universe $\varphi \in \mathrm{Diff}(\mathscr{M})$, *i.e.*,

$$\mathscr{L}[\varphi^* g, \varphi^* \Psi] = \mathscr{L}[g, \Psi],$$

  where φ^* means the pull-back by φ,

$$\varphi^* g = (T\varphi)^\star (g \circ \varphi)(T\varphi), \quad \text{and} \quad \varphi^* \Psi = \Psi \circ \varphi.$$

- General covariance is meaningful before the introduction of a spacetime.
- The infinitesimal version of General Covariance (using Lie derivative) was formulated by Noether in 1918.

The (General Relativity) Hilbert-Einstein functional $\mathscr{H}$ is general covariant, i.e.,

$$\mathscr{H}[\varphi^* g] = \mathscr{H}[g].$$

General covariance

The fundamental postulate of general relativity is that

the laws of physics are independent of the choice of coordinates.

- More precisely, this means that the Lagrangian $\mathscr{L}$ is invariant by any (local) diffeomorphism of the Universe $\varphi \in \mathrm{Diff}(\mathscr{M})$, *i.e.*,

$$\mathscr{L}[\varphi^* g, \varphi^* \Psi] = \mathscr{L}[g, \Psi],$$

where φ^* means the pull-back by φ,

$$\varphi^* g = (T\varphi)^\star \, (g \circ \varphi)(T\varphi), \quad \text{and} \quad \varphi^* \Psi = \Psi \circ \varphi.$$

- General covariance is meaningful before the introduction of a spacetime.
- The infinitesimal version of General Covariance (using Lie derivative) was formulated by Noether in 1918.

The (General Relativity) Hilbert-Einstein functional $\mathscr{H}$ is general covariant, i.e.,

$$\mathscr{H}[\varphi^* g] = \mathscr{H}[g].$$

As observed by Einstein : "$\operatorname{div}^g \mathbf{T} = 0$, this is mechanics !"

- The extremalization (w.r.t. g) of the Lagrangian $\mathscr{L} = \mathscr{H} + \mathscr{L}^{\text{matter}}$

$$2\frac{\delta \mathscr{L}}{\delta g} = 0 = \underbrace{2\frac{\delta \mathscr{H}}{\delta g}}_{\propto \text{ Einstein tensor } \mathbf{G}_g^\sharp} + \underbrace{2\frac{\delta \mathscr{L}^{\text{matter}}}{\delta g}}_{\text{opposite of stress–energy tensor } \mathbf{T}} = \frac{1}{\kappa}\mathbf{G}_g^\sharp - \mathbf{T}$$

gives back Einstein equation $\mathbf{G}_g^\sharp = \kappa\,\mathbf{T}$ (with matter), $\kappa = \dfrac{8\pi G}{c^4}$.

- A direct consequence of general covariance is the fundamental property

$$\operatorname{div}^g \mathbf{G}_g = 0, \qquad \mathbf{G}_g^\sharp := 2\kappa\frac{\delta \mathscr{H}}{\delta g}$$

and thus the stress–energy(–momentum) tensor $\mathbf{T}$ satisfies

Theorem

$$\operatorname{div}^g \mathbf{T} = 0, \qquad \mathbf{T} := -2\frac{\delta \mathscr{L}^{matter}}{\delta g}$$

General Covariance of Matter Lagrangian

Theorem (Souriau, 1958)

Suppose that the Lagrangian

$$\mathscr{L}^{matter}[g, \Psi] = \int L_0(g_m, \Psi(m), T_m\Psi)\, \mathrm{vol}_g$$

is general covariant. Then, its Lagrangian density can be written as

$$L_0(g, \Psi, T\Psi) = L(\Psi, \mathbf{K}),$$

for some function where $\mathbf{K} = (T\Psi)\, g^{-1} (T\Psi)^{\star}$ *is the conformation.*

In Classical Continuum Mechanics, for a Hyperelasticity law that satisfies the material indifference (objectivity) principle, the energy density depends on the deformation ϕ only through the right Cauchy–Green tensor $\mathbf{C} = \phi^{*} q$.

General Covariance of Matter Lagrangian

Theorem (Souriau, 1958)

Suppose that the Lagrangian

$$\mathscr{L}^{matter}[g, \Psi] = \int L_0(g_m, \Psi(m), T_m\Psi)\, \mathrm{vol}_g$$

is general covariant. Then, its Lagrangian density can be written as

$$L_0(g, \Psi, T\Psi) = L(\underbrace{\Psi}_{\textit{non homogeneity}}, \underbrace{\mathbf{K}}_{\textit{same role as } \mathbf{C}^{-1}}),$$

for some function where $\mathbf{K} = (T\Psi)\, g^{-1} (T\Psi)^{\star}$ *is the conformation*.

In Classical Continuum Mechanics, for a Hyperelasticity law that satisfies the material indifference (objectivity) principle, the energy density depends on the deformation ϕ only through the right Cauchy–Green tensor $\mathbf{C} = \phi^{*}\mathfrak{g}$.

Stress-energy tensor for perfect matter

Let us assume the standard decomposition (Souriau, 1958, De Witt, 1962)

$$L(\Psi, \mathbf{K}) = \rho_r c^2 + E(\Psi, \mathbf{K}) = \rho_r c^2 + \rho_r e(\Psi, \mathbf{K}),$$

where ρ_r is the rest mass density and E is the internal energy density. Then,

$$\mathbf{T} = -2\frac{\delta \mathscr{L}^{\text{matter}}}{\delta g} = L\, \mathbf{U} \otimes \mathbf{U} - \mathbf{\Sigma}, \quad \text{where}$$

$$\mathbf{\Sigma} := -2\rho_r\, g^{-1}(T\Psi)^\star \frac{\partial e}{\partial \mathbf{K}}(T\Psi) g^{-1}, \qquad \mathbf{\Sigma}.\mathbf{U}^\flat = 0,$$

which define $\mathbf{\Sigma}$ as a relativistic (mechanical) stress tensor.

Remark

The stress tensor $\mathbf{\Sigma}$ can be interpreted as a 3D object since $\mathbf{\Sigma}.\mathbf{U}^\flat = 0$.

OUTLINE

1 (Gauge Theory) 4D General Covariant formulation of Hyperelasticity

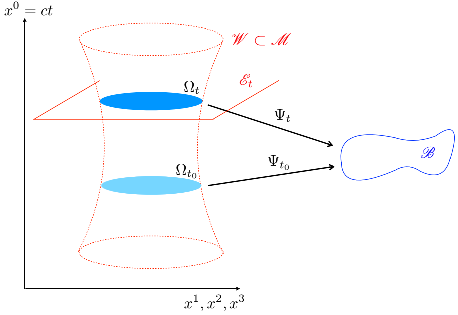
2 Spacetimes : introduction of an observer/orthogonal decompositions

3 Relativistic Hyperelasticity in (curved) Schwarzschild spacetime

4 Galilean limit of Relativistic Hyperelasticity

Observer as a time function $\hat{t}(x^\mu)$ on $\mathscr{W}$

(Arnowitt et al, 1962, York, 1979, Gourgoulhon, 2012)



- The World tube $\mathscr{W}$ is foliated by the (3D) hypersurfaces Ω_t which play the role of configurations $\Omega = \Omega_{p(t)}$ of Classical Continuum Mechanics.

Definition (Unit normal to Ω_t, oriented towards future, as $\mathbf{U} = \mathbf{P}/\rho_r$)

$$\mathbf{N} := -\frac{\operatorname{grad}^g \hat{t}}{\sqrt{-\left\|\operatorname{grad} \hat{t}\right\|_g^2}}, \qquad \langle \mathbf{U}, \mathbf{N} \rangle_g < 0.$$

Orthogonal decomposition of quadrivelocity $\mathbf{U}$

A spacetime structure is given on $\mathscr{W}$, with unit normal $\mathbf{N}$, which is assumed oriented in the same way as $\mathbf{U}$, i.e. (with g not necessarily the Minskowski metric)

$$\gamma := -\langle \mathbf{U}, \mathbf{N} \rangle_g > 0 \quad \text{(Generalized Lorentz factor).}$$

which allows us to define the mass density $\rho := \gamma \rho_r$.

Orthogonal decomposition of $\mathbf{U}$ with respect to $\mathbf{N}$

$$\mathbf{U} = \mathbf{U}^N + \mathbf{U}^\top, \qquad \begin{cases} \text{Normal component :} & \mathbf{U}^N = \gamma\, \mathbf{N}, \\ \text{Spatial (3D) component :} & \mathbf{U}^\top = \gamma \dfrac{\boldsymbol{u}}{c}. \end{cases}$$

Then, by $\|\mathbf{U}\|_g^2 = -1$:

$\boldsymbol{u}$ interpreted as the spatial (3D, Eulerian, relativistic) velocity

$$\mathbf{U} = \gamma \left(\mathbf{N} + \frac{\boldsymbol{u}}{c} \right), \qquad \gamma = 1 \Big/ \sqrt{1 - \frac{\|\boldsymbol{u}\|_g^2}{c^2}}.$$

Orthogonal decomposition of T

Orthogonal decomposition of the stress energy tensor $\mathbf{T}$ w.r.t. $\mathbf{N}$

$$\mathbf{T} = E_{\mathrm{tot}}\,\mathbf{N}\otimes\mathbf{N} + \frac{1}{c}\left(\mathbf{N}\otimes\boldsymbol{p} + \boldsymbol{p}\otimes\mathbf{N}\right) + \mathbf{s}.$$

- E_{tot} is the total energy density,
- $\boldsymbol{p}$ is the relativistic linear momentum,
- $\mathbf{s}$ is the spatial part of $\mathbf{T}$ (related to the mechanical stress field).

Definition (Relativistic Cauchy stress tensor $\boldsymbol{\sigma}$)

It is defined as the spatial part $\boldsymbol{\sigma} := \boldsymbol{\Sigma}^{\top}$ of

$$\boldsymbol{\Sigma} := -\mathbf{T} + L\,\mathbf{U}\otimes\mathbf{U}, \qquad \boldsymbol{\Sigma}.\mathbf{U}^{\flat} = 0.$$

$$\begin{cases} E_{\mathrm{tot}} = \gamma^2 L - \frac{1}{c^2}\boldsymbol{u}^{\flat}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{u}^{\flat}, \\ \boldsymbol{p} = \gamma^2 L\boldsymbol{u} - \boldsymbol{\sigma}\cdot\boldsymbol{u}^{\flat}, \\ \mathbf{s} = \frac{\gamma^2}{c^2}L\boldsymbol{u}\otimes\boldsymbol{u} - \boldsymbol{\sigma}, \end{cases} \qquad \text{where } L = \rho_r c^2 + E.$$

OUTLINE

1 (Gauge Theory) 4D General Covariant formulation of Hyperelasticity

2 Spacetimes : introduction of an observer/orthogonal decompositions

3 Relativistic Hyperelasticity in (curved) Schwarzschild spacetime

4 Galilean limit of Relativistic Hyperelasticity

(CURVED) SCHWARZSCHILD SPACETIME (1916)

- The Schwarzschild metric g is a static solution to Einstein vacuum equation $\mathbf{G}_g = 0$, around a spherical body of mass M and radius r_0.
- In cartesian isotropic coordinates $(x^0 = ct, x^i)$, for $\bar{r} \geq r_0 >> \bar{r}_s$ (planet),

$$g = -\mathcal{N}^2 c^2 \mathrm{d}t^2 + g^{3D},$$

$$\mathcal{N} = \sqrt{-g_{00}} = \frac{1 - \bar{r}_s/\bar{r}}{1 + \bar{r}_s/\bar{r}}, \quad \text{et} \quad g^{3D} = k\,\mathfrak{g} = k\,\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

where k and g^{3D} are function of the radius $\bar{r}$,

$$k = \left(1 + \frac{\bar{r}_s}{\bar{r}}\right)^4, \qquad \bar{r} := \sqrt{\delta_{ij}\, x^i x^j}, \qquad \bar{r}_s := \frac{GM}{2c^2}.$$

- $\bar{r} = 0$ at Earth center, $\bar{r} \approx r_0$ at its surface,
- The Minkowski metric $\boldsymbol{\eta} = -c^2\mathrm{d}t^2 + \mathfrak{g}$ corresponds to vanishing mass $M = 0$ (and vanishing Schwarzschild radius $\bar{r}_s = 0$).

Evaluation in Schwarzschild spacetime

- Spatial velocity and generalized Lorentz factor :

$$\boldsymbol{u} = -\frac{1}{\mathscr{N}}\,\mathbf{F}\,\frac{\partial \Psi}{\partial t}, \qquad \gamma = \frac{1}{\sqrt{1 - k\frac{\boldsymbol{u}^2}{c^2}}},$$

where $\boldsymbol{u}^2 = \|\boldsymbol{u}\|_{\mathfrak{g}}^2$ is square Euclidean norm and $k = \left(1 + \frac{\bar{r}_s}{\bar{r}}\right)^4$.

- Conformation :

$$\mathbf{K} = \mathbf{F}^{-1}\left(\frac{1}{k}\mathfrak{g}^{-1} - \frac{1}{c^2}\boldsymbol{u}\otimes\boldsymbol{u}\right)\mathbf{F}^{-\star}, \qquad \mathbf{F} = (T\Psi_t)^{-1}$$

- Stress–energy tensor (coordinates (t, x^i))

$$\mathbf{T} = \begin{pmatrix} \frac{1}{c^2\mathscr{N}^2}E_{\text{tot}} & \frac{1}{c^2\mathscr{N}}\boldsymbol{p}^\star \\ \frac{1}{c^2\mathscr{N}}\boldsymbol{p} & \mathbf{s} \end{pmatrix}.$$

COMPONENTS OF THE STRESS-ENERGY-MOMENTUM TENSOR $\mathbf{T}$

DEFINITION OF THE MECHANICAL STRESS $\boldsymbol{\sigma}$ AS THE SPATIAL PART OF $\boldsymbol{\Sigma}$

- Energy density :

$$E_{\text{tot}} = \gamma\rho c^2 + E\big(1 + k\gamma^2\frac{\boldsymbol{u}^2}{c^2}\big) - \frac{1}{c^2}\boldsymbol{u}^\flat \cdot \boldsymbol{\sigma} \cdot \boldsymbol{u}^\flat,$$

- Linear momentum :

$$\boldsymbol{p} = \Big(\gamma\rho c^2 + E\big(1 + k\gamma^2\frac{\boldsymbol{u}^2}{c^2}\big)\Big)\boldsymbol{u} - \boldsymbol{\sigma} \cdot \boldsymbol{u}^\flat,$$

- Stress (Relativistic Hyperelasticity constitutive equation) :

$$\mathbf{s} = \Big(\gamma\rho + \frac{1}{c^2}E\big(1 + k\gamma^2\frac{\boldsymbol{u}^2}{c^2}\big)\Big)\boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{\sigma},$$

$$\boldsymbol{\sigma} = -\frac{2}{\gamma k^2}\rho\, \mathfrak{g}^{-1}\mathbf{F}^{-\star}\frac{\partial e}{\partial \mathbf{K}}\mathbf{F}^{-1}\mathfrak{g}^{-1}.$$

where $\rho := \gamma\rho_r$ is the mass density.

- Conservation of current of matter **P**

$$\operatorname{div}^{g}\mathbf{P} = \frac{1}{\mathscr{N}}\frac{\partial \rho}{\partial t} + \operatorname{div}^{g^{3D}}(\rho\boldsymbol{u}) + \rho\boldsymbol{u}\cdot \mathrm{d}\ln\mathscr{N} = 0.$$

where $\rho := \gamma\rho_r$ is the mass density.

- Conservation of Hilbert stress-energy(-momentum) tensor **T**,

$$(\operatorname{div}^{g}\mathbf{T})^{\top} = \frac{1}{c^2\mathscr{N}}\frac{\partial \boldsymbol{p}}{\partial t} + \operatorname{div}^{g^{3D}}\mathbf{s} + \mathbf{s}\cdot \mathrm{d}\ln\mathscr{N} + \frac{E_{\text{tot}}}{k}\mathfrak{g}^{-1}\mathrm{d}\ln\mathscr{N} = 0.$$

$$(\operatorname{div}^{g}\mathbf{T})^{t} = \frac{1}{c^2\mathscr{N}^2}\frac{\partial E_{\text{tot}}}{\partial t} + \frac{1}{c^2\mathscr{N}}\operatorname{div}^{g^{3D}}\boldsymbol{p} + 2\frac{\boldsymbol{p}}{c^2\mathscr{N}}\cdot \mathrm{d}\ln\mathscr{N} = 0.$$

Remark

The lapse function is $\mathscr{N} = 1$, so that $\mathrm{d}\ln\mathscr{N} = 0$, for Minkowski spacetime $g = \boldsymbol{\eta}$, $g^{3D} = \mathfrak{g}$.

Outline

1 (Gauge Theory) 4D General Covariant formulation of Hyperelasticity

2 Spacetimes : introduction of an observer/orthogonal decompositions

3 Relativistic Hyperelasticity in (curved) Schwarzschild spacetime

4 Galilean limit of Relativistic Hyperelasticity

Newton–Cartan limit $\lambda = 1/c^2 \to 0$

We refer to Dixon (1975), Havas (1964), Künzle (1976), Duval–Künzle (1977), Duval (1985) for Galilean Structures and Newton-Cartan limit.

1. At order 0 in λ : the time component $(\mathrm{div}^{\lambda}\, \mathbf{T}^{\lambda})^{t} = 0$ corresponds to the expression of mass conservation of Classical Continuum Mechanics

$$\frac{\partial \overset{0}{\rho}}{\partial t} + \mathrm{div}(\overset{0}{\rho}\,\overset{0}{\boldsymbol{u}}) = 0.$$

2. At order 0 in λ : the spatial component $(\overset{\lambda}{\mathrm{div}}\, \mathbf{T}^{\lambda})^{\top} = 0$ corresponds to the linear momentum balance of Classical Continuum Mechanics

$$\frac{\partial}{\partial t}(\overset{0}{\rho}\,\overset{0}{\boldsymbol{u}}) + \mathrm{div}\left(\overset{0}{\rho}\,\overset{0}{\boldsymbol{u}} \otimes \overset{0}{\boldsymbol{u}} - \overset{0}{\boldsymbol{\sigma}}\right) - \overset{0}{\rho}\,\mathbf{g} = 0.$$

3. At order 1 in λ : the equation $\mathscr{N}(\overset{\lambda}{\mathrm{div}}\, \overset{\lambda}{\mathbf{T}})^{t} - \overset{\lambda}{\mathrm{div}}(c\overset{\lambda}{\mathbf{P}}) = 0$ corresponds to the internal energy balance in Classical Continuum Mechanics

$$\frac{\partial}{\partial t}\left(\overset{0}{E} + \frac{1}{2}\rho\overset{0}{\boldsymbol{u}}{}^{2}\right) + \mathrm{div}\left(\left(\overset{0}{E} + \frac{1}{2}\overset{0}{\rho}\,\overset{0}{\boldsymbol{u}}{}^{2}\right)\overset{0}{\boldsymbol{u}} - \overset{0}{\boldsymbol{\sigma}} \cdot \overset{0}{\boldsymbol{u}}{}^{\flat}\right) - \overset{0}{\rho}\,\mathbf{g} \cdot \overset{0}{\boldsymbol{u}} = 0.$$

Conclusion

Classical conservation laws recovered

1. is mass conservation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{u}) = 0.$$

2. is the equilibrium equation (with gravity)

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \nabla_{\boldsymbol{u}}\boldsymbol{u}\right) = \operatorname{div}\boldsymbol{\sigma} + \rho\mathbf{g},$$

3. is the first principle of thermodynamics with no heat transfert,

$$\rho\left(\frac{\partial e}{\partial t} + \nabla_{\boldsymbol{u}} e\right) = \boldsymbol{\sigma} : \mathbf{d}, \qquad \mathbf{d} := \frac{1}{2}\left(\nabla \boldsymbol{u}^{\flat} + (\nabla \boldsymbol{u}^{\flat})^{\star}\right),$$

$e := E/\rho$: specific internal energy and $\mathbf{d}$: classical strain rate tensor.

Conclusion

- We have presented a relativistic formulation of Hyperelasticity, within the (Souriau) framework of Variational Relativity.
- The conformation $\mathbf{K}$ plays a fundamental role (as a strain variable).
- We have defined the stress of a continuous medium in this context.
- Although the complete theory is 4D, the constitutive laws of Relativistic Hyperelasticity are 3D.
- Gravity has been taken into account (here in Schwarzschild spacetime).
- The Newton-Cartan formulation of Galilean relativity is obtained as the infinite light speed limit $c \to \infty$.

QU'EST CE QU'UNE FORME DIFFÉRENTIELLE ?

Une forme différentielle ω de degré k sur $\mathbb{R}^d$ (ou plus généralement une variété différentielle) est un champ de tenseurs d'ordre k qui est alterné

$$\omega_{r_1 \cdots r_j \cdots r_i \cdots r_k} = -\omega_{r_1 \cdots r_i \cdots r_j \cdots r_k}.$$

- une 1-forme sur $\mathbb{R}^3$ s'écrit : $\alpha = (\alpha_i) = P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$,
- une 2-forme sur $\mathbb{R}^3$ s'écrit :
  $\omega = (\omega_{ij}) = \omega_{12}\, \mathrm{d}x \wedge \mathrm{d}y + \omega_{13}\, \mathrm{d}x \wedge \mathrm{d}z + \omega_{23}\, \mathrm{d}y \wedge \mathrm{d}z$,
- une 3-forme sur $\mathbb{R}^3$ s'écrit : $\omega = (\omega_{ijk}) = \omega_{123}\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.

Ici $(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ est la base duale de la base canonique sur $\mathbb{R}^3$ et

$$\mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}x \otimes \mathrm{d}y - \mathrm{d}y \otimes \mathrm{d}x,$$
$$\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = (\mathrm{d}x \otimes \mathrm{d}y \otimes \mathrm{d}z)^a,$$

est le produit tensoriel alterné.

Retour

Forme volume riemannien

- Une forme volume sur $\mathbb{R}^d$ (ou une variété orientable de dimension d) est une d-forme (degré maximal) qui ne s'annule nulle part,

$$\mu = f\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^d, \quad \text{where } f(x^1, \ldots, x^d) \neq 0.$$

- Sur toute variété riemannienne (orientable) (M, g) il existe une unique forme volume, notée vol_g qui est caractérisée par le fait qu'elle vaut 1 sur toute base orthonormée directe.
- Exemple : Sur $\mathbb{R}^3$ muni de la métrique euclidienne

$$g = \mathfrak{g} = dx^2 + dy^2 + dz^2$$

dans les coordonnées canoniques (x, y, z), on a

$$\mathrm{vol}_{\mathfrak{g}} = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = (\mathrm{d}x \otimes \mathrm{d}y \otimes \mathrm{d}z)^a\,,$$

Retour

Pull-back et push-forward

Des variables lagrangiennes au variables euleriennes et inversement

$$p\colon \mathscr{B} \to \mathscr{E} = (\mathbb{R}^3, \mathfrak{g}) \qquad Tp\colon T\mathscr{B} \to T\mathscr{E}$$
$$\mathbf{X} \mapsto \mathbf{x} \qquad \delta\mathbf{X} \mapsto \delta\mathbf{x} = \mathbf{F}.\delta\mathbf{X}$$

Ces notions étendent les opérations suivantes sur les fonctions

$$(p^* f)(\mathbf{X}) = f(p(\mathbf{X})), \qquad f \in \mathrm{C}^\infty(\Omega_p, \mathbb{R}) \qquad \text{(pull-back)},$$
$$(p_* \mathscr{F})(\mathbf{x}) = \mathscr{F}(p^{-1}(\mathbf{x})), \qquad \mathscr{F} \in \mathrm{C}^\infty(\mathscr{B}, \mathbb{R}) \qquad \text{(push-forward)}$$

à tout type de champ de tenseurs.

- Pour des champs de tenseurs d'ordre 2 covariants $\mathbf{k} = (k_{ij})$ et $\mathbf{K} = (k_{IJ})$:

$$p^*\mathbf{k} = \mathbf{F}^\star(\mathbf{k} \circ p)\mathbf{F}, \qquad \mathbf{k} \text{ défini sur } \Omega_p \qquad \text{(pull-back)},$$
$$p_*\mathbf{K} = \mathbf{F}^{-\star}(\mathbf{K} \circ p^{-1})\mathbf{F}^{-1}, \qquad \mathbf{K} \text{ défini sur } \mathscr{B} \qquad \text{(push-forward)}$$

où $\mathbf{F} := Tp = \left(\frac{\partial x^i}{\partial X^J}\right)$.

Retour