# Micro-structures et défauts

## une approche géométrique

Mewen Crespo

Encadré par Guy Casale et Loïc Le Marrec

Université de Rennes – IRMAR

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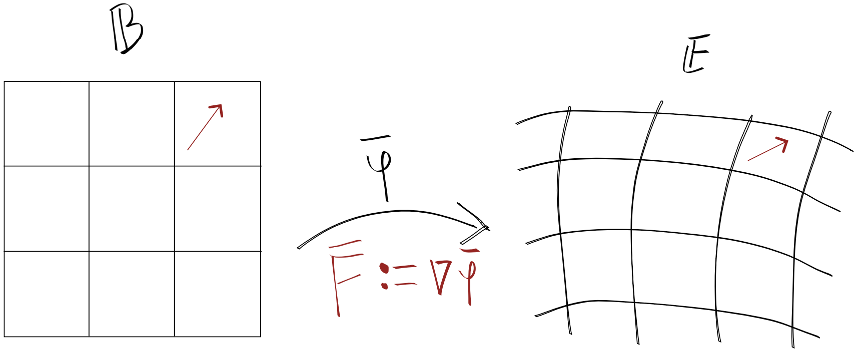
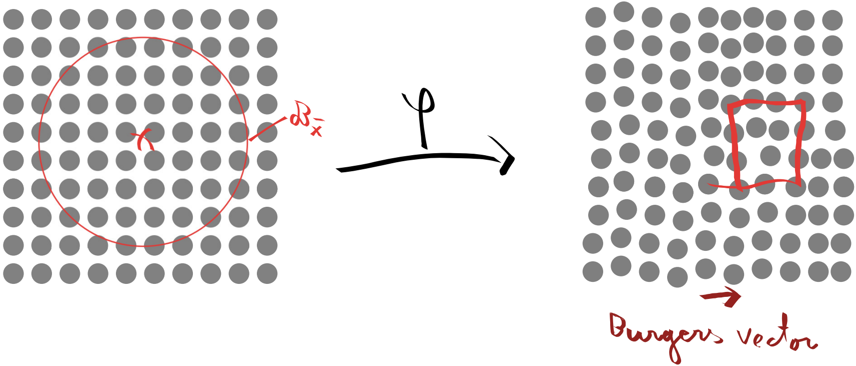
# Introduction — Mécanique classique




$$\operatorname{curl}\left(\overline{\mathbf{F}}\right) = \mathbf{0}$$

# Introduction – Échelle atomique

$$\bar{\varphi}(\bar{x}) := \frac{1}{|\bullet \in B_{\bar{x}}|} \sum_{\bullet \in B_{\bar{x}}} \varphi(\bullet)$$



# Introduction − Micromorphe

$\mathbf{P}_{\overline{X}} :=$ meilleure approximation linéaire de $\varphi_{|_{\mathcal{B}_{\overline{X}}}}$ en $\overline{X}$ ($\mathbf{P}$ est $\mathbf{F}^{\mathrm{v}}_{\mathrm{v}}$)
Idéalement, sans dislocation ou autre défaut, on a:

$$\mathbf{P}_{\overline{X}} \simeq \nabla_{\overline{X}} \overline{\varphi}$$

Si on linéarise, on a:

$$\mathrm{curl}\,(\mathbf{P}) \propto \overrightarrow{b}$$

En général, $\mathbf{P}$ est donc potentiellement différent de $\nabla_{\overline{X}} \overline{\varphi}$.

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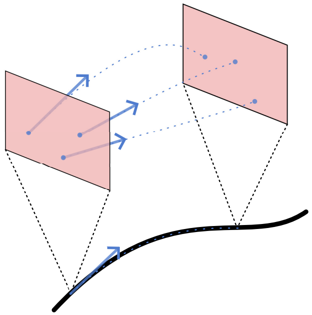
$\mathbf{N}(y) + \mathbf{P} \cdot y :=$ meilleur approximation quadratique ($\mathbf{N}$ est $\mathbf{F}^{\mathrm{v}}_{\mathrm{h}}$)
Idéalement, sans défaut, on a:

$$\mathbf{N}_{\overline{X}} \simeq \nabla_{\overline{X}} \mathbf{P} \approx \nabla_{\overline{X}} \nabla_{\overline{X}} \varphi$$

# Introduction – Transport parallèle



$$\left[\boldsymbol{\Gamma}_{\overline{X}}\right]^{j}_{ia} y^a \delta \frac{\partial}{\partial \overline{X}^i} = \left[\mathbf{P}^{-1}\right]^{j}_{k} \nabla_i \mathbf{P}^{k}_{a}$$ ( cas holonome $\mathbf{N} = \nabla \mathbf{P}$ )

$$= \left[\mathbf{P}^{-1}\right]^{j}_{k} \mathbf{N}^{k}_{ia}$$ (cas général)

# *La micro-métrique*

## *vue par Riemann-Cartan*

# Compatibilité métrique

$\mathbf{G} := \overline{\mathbf{F}}^{\mathrm{T}} \cdot \overline{\mathbf{F}}$

Compatibilité métrique (holonome):

$$\begin{aligned} \mathbf{0} &= \nabla^{\mathbf{\Gamma}} \mathbf{G} \\ &:= \left( \frac{\partial \mathbf{G}_{ab}}{\partial \overline{X}^i} - \mathbf{G}_{ka} \mathbf{\Gamma}^k_{ib} - \mathbf{G}_{kb} \mathbf{\Gamma}^k_{ia} \right) \mathrm{d}\overline{X}^i \otimes \mathrm{d}Y^b \otimes \mathrm{d}Y^a \end{aligned}$$

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$\mathfrak{C} := \mathbf{P}^{\mathrm{T}} \cdot \mathbf{P}$ ( $\mathfrak{C}$ est $\mathfrak{G}^{\mathrm{v}}_{\mathrm{v}}$ )

Compatibilité métrique (cas général):

$$\begin{aligned} \mathbf{0} &= \nabla^{\mathbf{\Gamma}} \mathfrak{C} \\ &:= \left( \frac{\partial \mathfrak{C}_{ab}}{\partial \overline{X}^i} - \mathfrak{C}_{ka} \mathbf{\Gamma}^k_{ib} - \mathfrak{C}_{kb} \mathbf{\Gamma}^k_{ia} \right) \mathrm{d}\overline{X}^i \otimes \mathrm{d}Y^b \otimes \mathrm{d}Y^a \end{aligned}$$

# *La pseudo-métrique*

## *telle que présentée dans notre preprint*

# Réinterprétation de la connexion

La connexion pour Cartan:

$$\begin{aligned} \boldsymbol{\Gamma}_{\overline{X}} : \quad \mathrm{T}_{\overline{X}}\mathbb{B} \times \mathcal{B}_{\overline{X}} &\longrightarrow \mathrm{T}\mathcal{B}_{\overline{X}} \\ (\overline{\mathbf{U}}, Y) &\longmapsto \boldsymbol{\Gamma}^{j}_{ia} \cdot \mathbf{U}^{i} \cdot Y^{a} \ \frac{\partial}{\partial Y^{j}} \end{aligned}$$

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$\mathcal{B} := \bigcup_{\overline{X} \in \mathbb{B}} \mathcal{B}_{\overline{X}} \simeq \mathbb{B} \times \mathbb{R}^3$

La connexion pour nous:

$$\begin{aligned} \boldsymbol{\Gamma}_{\overline{X}} : \quad \mathrm{T}_{\overline{X}}\mathbb{B} \times \mathcal{B}_{\overline{X}} &\longrightarrow \mathrm{T}\mathcal{B} \\ (\overline{\mathbf{U}}, Y) &\longmapsto \overline{\mathbf{U}}^{l} \frac{\partial}{\partial \overline{X}^{l}} + \boldsymbol{\Gamma}^{j}_{ia} \cdot \mathbf{U}^{i} \cdot Y^{a} \ \frac{\partial}{\partial Y^{j}} \end{aligned}$$

*i.e.* $\boldsymbol{\Gamma} \equiv \begin{bmatrix} \delta^{l}_{i} \\ \boldsymbol{\Gamma}^{j}_{ia} \end{bmatrix}$

# Longeurs macro et micro

Longeur macro: poussé en avant de $\mathbf{G} = \overline{\mathbf{F}}^{\mathrm{T}} \cdot \overline{\mathbf{F}}$ par $\boldsymbol{\Gamma}$

Changement de coordonnée macro/micro (i.e. soudure) :

$$\boldsymbol{\Theta}^{-1} := \overline{\mathbf{F}}^{-1} \cdot \mathbf{P}$$

Longeur micro: poussé en avant de $\mathbf{G} = \overline{\mathbf{F}}^{\mathrm{T}} \cdot \overline{\mathbf{F}}$ par $\boldsymbol{\Theta}^{-1}$

Longeur totale (dans les coordonnées induites par $\boldsymbol{\Gamma}$):

$$\mathfrak{G} := \begin{bmatrix} \overline{\mathbf{F}}^{\mathrm{T}} \cdot \overline{\mathbf{F}} & \overline{\mathbf{F}}^{\mathrm{T}} \cdot \mathbf{P} \\ \mathbf{P}^{\mathrm{T}} \cdot \overline{\mathbf{F}} & \mathbf{P}^{\mathrm{T}} \cdot \mathbf{P} \end{bmatrix}$$

Uniquement déterminée par:

$$\mathfrak{G} \cdot (\boldsymbol{\Gamma} - \boldsymbol{\Theta}) = \mathbf{0}$$

# Correspondance des compatibilités

## vers une généralisation de la géométrie de Riemann-Cartan

# Factorisation

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$$
\begin{aligned}
\mathbf{0} &= \mathfrak{G} \cdot (\boldsymbol{\Gamma} - \boldsymbol{\Theta}) \\
&= \begin{bmatrix} \overline{\mathbf{F}}^{\mathrm{T}} \cdot \overline{\mathbf{F}} & \overline{\mathbf{F}}^{\mathrm{T}} \cdot \mathbf{P} \\ \mathbf{P}^{\mathrm{T}} \cdot \overline{\mathbf{F}} & \mathbf{P}^{\mathrm{T}} \cdot \mathbf{P} \end{bmatrix} \cdot \begin{bmatrix} \delta_i^s \\ \boldsymbol{\Gamma}_{ia}^r - \boldsymbol{\Theta}_i^r \end{bmatrix}
\end{aligned}
$$

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$$
\begin{aligned}
\mathbf{0} &= \nabla^{\boldsymbol{\Gamma}} \mathfrak{C} \\
&= \left( \frac{\partial \mathfrak{C}_{ja}}{\partial \overline{X}^i} - \mathfrak{C}_{jr} \boldsymbol{\Gamma}_{ia}^r - \mathfrak{C}_{al} \boldsymbol{\Gamma}_{ij}^l \right) \mathrm{d}Y^j \otimes \mathrm{d}\overline{X}^i \otimes Y^a \\
0 &= \begin{bmatrix} -\frac{\partial \mathfrak{C}_{ja}}{\partial \overline{X}^s} + \mathfrak{C}_{al} \boldsymbol{\Gamma}_{sj}^l & \mathfrak{C}_{jr} \end{bmatrix} \cdot \begin{bmatrix} \delta_i^s \\ \boldsymbol{\Gamma}_{ia}^r \end{bmatrix}
\end{aligned}
$$

# Grossissement

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$$
\begin{aligned}
\mathbf{0} &= \nabla^{\boldsymbol{\Gamma}} \mathfrak{C} \\
&= \left( \frac{\partial \mathfrak{C}_{ja}}{\partial \overline{X}^i} - \mathfrak{C}_{jr} \boldsymbol{\Gamma}^r_{ia} - \mathfrak{C}_{al} \boldsymbol{\Gamma}^l_{ij} \right) \mathrm{d}Y^j \otimes \mathrm{d}\overline{X}^i \otimes Y^a \\
0 &= \begin{bmatrix} -\frac{\partial \mathfrak{C}_{ja}}{\partial \overline{X}^s} + \mathfrak{C}_{al} \boldsymbol{\Gamma}^l_{sj} & \mathfrak{C}_{jr} \end{bmatrix} \cdot \begin{bmatrix} \delta^s_i \\ \boldsymbol{\Gamma}^r_{ia} \end{bmatrix} \\
0 &= \begin{bmatrix} \mathfrak{C}_{tl} \boldsymbol{\Gamma}^l_{ja} \boldsymbol{\Gamma}^t_{sb} & -\frac{\partial \mathfrak{C}_{ra}}{\partial \overline{X}^j} + \mathfrak{C}_{al} \boldsymbol{\Gamma}^l_{jr} ? \\ -\frac{\partial \mathfrak{C}_{ja}}{\partial \overline{X}^s} + \mathfrak{C}_{al} \boldsymbol{\Gamma}^l_{sj} & \mathfrak{C}_{jr} \end{bmatrix} \cdot \begin{bmatrix} \delta^s_i \\ \boldsymbol{\Gamma}^r_{ia} \end{bmatrix} \\
&= \begin{bmatrix} \mathfrak{C}_{tl} \boldsymbol{\Gamma}^l_{ja} \boldsymbol{\Gamma}^t_{sb} - \frac{\partial \mathfrak{C}_{ra}}{\partial \overline{X}^j} \boldsymbol{\Theta}^r_s + \mathfrak{C}_{al} \boldsymbol{\Gamma}^l_{jr} \boldsymbol{\Theta}^r_s + \mathfrak{C}_{rl} \boldsymbol{\Theta}^l_j \boldsymbol{\Theta}^r_s - \mathfrak{C}_{rl} \boldsymbol{\Theta}^l_j \boldsymbol{\Gamma}^r_{sb} & -\frac{\partial \mathfrak{C}_{ra}}{\partial \overline{X}^j} + \mathfrak{C}_{al} \boldsymbol{\Gamma}^l_{jr} + \mathfrak{C}_{rl} \boldsymbol{\Theta}^l_j \\ -\frac{\partial \mathfrak{C}_{ja}}{\partial \overline{X}^s} + \mathfrak{C}_{al} \boldsymbol{\Gamma}^l_{sj} + \mathfrak{C}_{jl} \boldsymbol{\Theta}^l_s & \mathfrak{C}_{jr} \end{bmatrix} \cdot \begin{bmatrix} \delta^s_i \\ \boldsymbol{\Gamma}^r_{ib} - \boldsymbol{\Theta}^r_i \end{bmatrix}
\end{aligned}
$$

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$$
\mathbf{0} = \begin{bmatrix} \overline{\mathbf{F}}^{\mathrm{T}} \cdot \overline{\mathbf{F}} & \overline{\mathbf{F}}^{\mathrm{T}} \cdot \mathbf{P} \\ \mathbf{P}^{\mathrm{T}} \cdot \overline{\mathbf{F}} & \mathbf{P}^{\mathrm{T}} \cdot \mathbf{P} \end{bmatrix} \cdot \begin{bmatrix} \delta^s_i \\ \boldsymbol{\Gamma}^r_{ia} - \boldsymbol{\Theta}^r_i \end{bmatrix}
$$

# Forme simplifiée

$$\mathbf{0} = \begin{bmatrix} \left(\nabla\mathbf{P}_a + \overline{\mathbf{F}}\right)^{\mathrm{T}} \cdot \left(\nabla\mathbf{P}_b + \overline{\mathbf{F}}\right) & \left(\nabla\mathbf{P}_a^{\mathrm{T}} + \overline{\mathbf{F}}^{\mathrm{T}}\right) \cdot \mathbf{P} \\ \mathbf{P}^{\mathrm{T}} \cdot \left(\nabla\mathbf{P}_a + \overline{\mathbf{F}}\right) & \mathbf{P}^{\mathrm{T}} \cdot \mathbf{P} \end{bmatrix} \cdot (\boldsymbol{\Gamma} - \boldsymbol{\Theta})$$

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$$\mathbf{0} = \begin{bmatrix} \overline{\mathbf{F}}^{\mathrm{T}} \cdot \overline{\mathbf{F}} & \overline{\mathbf{F}}^{\mathrm{T}} \cdot \mathbf{P} \\ \mathbf{P}^{\mathrm{T}} \cdot \overline{\mathbf{F}} & \mathbf{P}^{\mathrm{T}} \cdot \mathbf{P} \end{bmatrix} \cdot (\boldsymbol{\Gamma} - \boldsymbol{\Theta})$$

# Corrolaires, observations et conclusion

# Comparaison

Soit:

$$\mathfrak{G}_{\text{holo}}^{\boldsymbol{\Gamma},\boldsymbol{\Theta}}(\mathfrak{C}) := \begin{bmatrix} \mathfrak{C}_{tl}\boldsymbol{\Gamma}^l_{ja}\boldsymbol{\Gamma}^t_{sb} - \dfrac{\partial \mathfrak{C}_{ra}}{\partial \overline{X}^j}\boldsymbol{\Theta}^r_s + \mathfrak{C}_{al}\boldsymbol{\Gamma}^l_{jr}\boldsymbol{\Theta}^r_s + \mathfrak{C}_{rl}\boldsymbol{\Theta}^l_j\boldsymbol{\Theta}^r_s - \mathfrak{C}_{rl}\boldsymbol{\Theta}^l_j\boldsymbol{\Gamma}^r_{sb} & -\dfrac{\partial \mathfrak{C}_{ra}}{\partial \overline{X}^j} + \mathfrak{C}_{al}\boldsymbol{\Gamma}^l_{jr} + \mathfrak{C}_{rl}\boldsymbol{\Theta}^l_j \\ -\dfrac{\partial \mathfrak{C}_{ja}}{\partial \overline{X}^s} + \mathfrak{C}_{al}\boldsymbol{\Gamma}^l_{sj} + \mathfrak{C}_{jl}\boldsymbol{\Theta}^l_s & \mathfrak{C}_{jr} \end{bmatrix}$$

Alors:

$$\nabla^{\boldsymbol{\Gamma}}\mathfrak{C} = \mathbf{0} \iff \mathfrak{G}_{\text{holo}}^{\boldsymbol{\Gamma},\boldsymbol{\Theta}}(\mathfrak{C}) \cdot (\boldsymbol{\Gamma} - \boldsymbol{\Theta}) = \mathbf{0}$$

$$\mathfrak{C} \text{ metric-compatible} \iff \mathfrak{G}_{\text{holo}}^{\boldsymbol{\Gamma},\boldsymbol{\Theta}}(\mathfrak{C}) \text{ pseudo-metric-compatible}$$

MAIS !

# Cas général – Pseudo-métrique

On a supposé $\mathbf{N} = \nabla \mathbf{P}$ dans cette formule

$$\mathfrak{G} \equiv \begin{bmatrix} \left(\mathbf{N}_a + \overline{\mathbf{F}}\right)^{\mathrm{T}} \cdot \left(\mathbf{N}_b + \overline{\mathbf{F}}\right) & \left(\mathbf{N}_a^{\mathrm{T}} + \overline{\mathbf{F}}^{\mathrm{T}}\right) \cdot \mathbf{P} \\ \mathbf{P}^{\mathrm{T}} \cdot \left(\mathbf{N}_a + \overline{\mathbf{F}}\right) & \mathbf{P}^{\mathrm{T}} \cdot \mathbf{P} \end{bmatrix}$$

Libérer $\mathbf{N} \neq \nabla \mathbf{P}$ a le même effet sur $\mathfrak{C}$ que de libérer $\mathfrak{N} \neq \dfrac{\partial \mathfrak{C}}{\partial \overline{X}}$:

$$\mathfrak{G} = \begin{bmatrix} \mathfrak{C}_{tl} \boldsymbol{\Gamma}^l_{ja} \boldsymbol{\Gamma}^t_{sb} - \mathfrak{N}_{raj} \boldsymbol{\Theta}^r_s + \mathfrak{C}_{al} \boldsymbol{\Gamma}^l_{jr} \boldsymbol{\Theta}^r_s + \mathfrak{C}_{rl} \boldsymbol{\Theta}^l_j \boldsymbol{\Theta}^r_s - \mathfrak{C}_{rl} \boldsymbol{\Theta}^l_j \boldsymbol{\Gamma}^r_{sb} & -\mathfrak{N}_{raj} + \mathfrak{C}_{al} \boldsymbol{\Gamma}^l_{jr} + \mathfrak{C}_{rl} \boldsymbol{\Theta}^l_j \\ -\mathfrak{N}_{jas} + \mathfrak{C}_{al} \boldsymbol{\Gamma}^l_{sj} + \mathfrak{C}_{jl} \boldsymbol{\Theta}^l_s & \mathfrak{C}_{jr} \end{bmatrix}$$

# Cas général – Métrique

$$\mathbf{0} = \mathfrak{N}_{jai} - \mathfrak{C}_{al}\mathbf{\Gamma}^{l}_{ij} - \mathfrak{C}_{jr}\mathbf{\Gamma}^{r}_{ib}$$

alors:

$$\begin{aligned}\nabla^{\mathbf{\Gamma}}\mathfrak{C} &:= \frac{\partial \mathfrak{C}_{ja}}{\partial \overline{X}^{i}} - \mathfrak{C}_{al}\mathbf{\Gamma}^{l}_{ij} - \mathfrak{C}_{jr}\mathbf{\Gamma}^{r}_{ib} \\ &= \frac{\partial \mathfrak{C}_{ja}}{\partial \overline{X}^{i}} - \mathfrak{N}_{jai} + \mathfrak{N}_{jai} - \mathfrak{C}_{al}\mathbf{\Gamma}^{l}_{ij} - \mathfrak{C}_{jr}\mathbf{\Gamma}^{r}_{ib} \\ &= \frac{\partial \mathfrak{C}_{ja}}{\partial \overline{X}^{i}} - \mathfrak{N}_{jai}\end{aligned}$$

# Cas général – Conclusion

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$$\mathfrak{G} \equiv \begin{bmatrix} \mathfrak{C}_{tl}\mathbf{\Gamma}^l_{ja}\mathbf{\Gamma}^t_{sb} - \mathfrak{N}_{raj}\mathbf{\Theta}^r_s + \mathfrak{C}_{al}\mathbf{\Gamma}^l_{jr}\mathbf{\Theta}^r_s + \mathfrak{C}_{rl}\mathbf{\Theta}^l_j\mathbf{\Theta}^r_s - \mathfrak{C}_{rl}\mathbf{\Theta}^l_j\mathbf{\Gamma}^r_{sb} & -\mathfrak{N}_{raj} + \mathfrak{C}_{al}\mathbf{\Gamma}^l_{jr} + \mathfrak{C}_{rl}\mathbf{\Theta}^l_j \\ -\mathfrak{N}_{jas} + \mathfrak{C}_{al}\mathbf{\Gamma}^l_{sj} + \mathfrak{C}_{jl}\mathbf{\Theta}^l_s & \mathfrak{C}_{jr} \end{bmatrix}$$

$$\equiv \begin{bmatrix} \mathfrak{C}_{tl}\mathbf{\Gamma}^l_{ja}\mathbf{\Gamma}^t_{sb} - \mathfrak{C}_{rl}\mathbf{\Gamma}^l_{ja}\mathbf{\Theta}^r_s + \mathfrak{C}_{rl}\mathbf{\Theta}^l_j\mathbf{\Theta}^r_s - \mathfrak{C}_{rl}\mathbf{\Theta}^l_j\mathbf{\Gamma}^r_{sb} & -\mathfrak{C}_{rl}\mathbf{\Gamma}^l_{ja} + \mathfrak{C}_{rl}\mathbf{\Theta}^l_j \\ -\mathfrak{C}_{jl}\mathbf{\Gamma}^l_{sa} + \mathfrak{C}_{jl}\mathbf{\Theta}^l_s & \mathfrak{C}_{jr} \end{bmatrix}$$

$$\mathfrak{G}^{\mathbf{\Gamma},\mathbf{\Theta}}(\mathfrak{C}) := \begin{bmatrix} \mathbf{\Theta}^l_j - \mathbf{\Gamma}^l_{ja} \\ \delta^l_j \end{bmatrix} \cdot \mathfrak{C}_{lr} \cdot \begin{bmatrix} \mathbf{\Theta}^r_s - \mathbf{\Gamma}^r_{sb} & \delta^r_s \end{bmatrix}$$

$$\begin{aligned} \mathfrak{C} \text{ metric-compatible } &\iff \mathfrak{G}^{\mathbf{\Gamma},\mathbf{\Theta}}_{\text{holo}}(\mathfrak{C}) \text{ pseudo-metric-compatible} \\ &\iff \mathfrak{G}^{\mathbf{\Gamma},\mathbf{\Theta}}(\mathfrak{C}) = \mathfrak{G}^{\mathbf{\Gamma},\mathbf{\Theta}}_{\text{holo}}(\mathfrak{C}) \\ &\Longleftarrow \mathbf{N}^j_{ia} = \nabla_i \mathbf{P}^j_a \end{aligned}$$

rem: $\mathfrak{G}^{\mathbf{\Gamma},\mathbf{\Theta}}(\mathfrak{C})$ est toujours compatible avec $\mathbf{\Gamma}$ et $\mathbf{\Theta}$

rem: compatible = physiquement acceptable. Donc si $\mathfrak{G}^{\mathbf{\Gamma},\mathbf{\Theta}}_{\text{holo}}(\mathfrak{C}) \neq \mathfrak{G}^{\mathbf{\Gamma},\mathbf{\Theta}}(\mathfrak{C})$, la "bonne" est $\mathfrak{G}^{\mathbf{\Gamma},\mathbf{\Theta}}(\mathfrak{C})$

rem: les indices $a$ et $b$ sont interchangeables

*Merci*

*pour votre attention*