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ÉCOLE D'INGÉNIEURS

Analysis of one-dimensional structure using Lie groups approach

Marwan Hariz

Joint work with Loïc Le Marrec and Jean Lerbet

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- $\mathfrak{D}(\mathcal{E}) = \{X : \mathcal{E} \longrightarrow E, \exists\, \omega_X \in E / X(b) = X(a) + \omega_X \times \overrightarrow{ab} \quad \forall\, a, b \in \mathcal{E}\}$ where $\times$ is the standard cross product in E.

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- The Lie bracket is defined in $\mathfrak{D}(\mathcal{E})$ by

$$\forall\, a \in \mathcal{E} \qquad [X, Y](a) = \omega_X \times Y(a) - \omega_Y \times X(a)$$

Klein form

The Klein form $[[\cdot|\cdot]]$ on $\mathfrak{D}(\mathcal{E})$ is the map defined by:

$$\begin{array}{rrcl} [[\cdot|\cdot]]: & \mathfrak{D}(\mathcal{E}) \times \mathfrak{D}(\mathcal{E}) & \longrightarrow & \mathbb{R} \\ & (X, Y) & \longrightarrow & [[X|Y]] = \omega_X \cdot Y + \omega_Y \cdot X \end{array}$$

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- $[[\cdot|\cdot]]$ is invariant by applying the adjoint map.
- $[[\cdot|\cdot]]$ plays a crucial role in energy-related aspects.

Kinematics

Internal motion

$$\begin{aligned} g : [0, L_0] \times \mathbb{R}^+ &\longrightarrow \mathbb{R} \times \mathbb{R}^+ \\ (S_0, t) &\longrightarrow (S(t), t) \end{aligned}$$

$S_0 \in [0, L_0]$ initial configuration and $S \in [0, L(t)]$ deformed configuration.

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Configuration

$$p(S, t) = D(S_0, t) \bullet p_0(S_0),$$

$D(S_0, t)$ displacement.
$p_0(S_0)$ reference configuration.
$\bullet$ free and transitive left-action of $\mathbb{D}$ on the configuration space.

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Strain

$$\begin{array}{rcl} \mathbf{e}_0 : [0, L_0] \times \mathbb{R}^+ & \longrightarrow & \mathfrak{D}(\mathcal{E}) \\ (S_0, t) & \longrightarrow & \mathbf{e}_0(S_0, t) = \vartheta(\frac{\partial D(S_0,t)}{\partial S_0}) = \mathbf{D}^{-1}(S_0, t) \circ \frac{\partial D(S_0,t)}{\partial S_0} \end{array}$$

Where $\mathbf{D}$ linear part of D.

Dynamics

Deformed configuration (S, t)

$$\rho \mathcal{T} + \frac{\partial \Theta}{\partial S} = \frac{\partial}{\partial t} \left(\rho \, H(V) \right)$$

- $\rho \mathcal{T}$ force distribution.
- V velocity
- Θ Internal actions
- ρH inertia

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- $\rho(S,t)dS = \rho(S_0)dS_0$
- $U(S,t) = Ad(D(S_0,t))U(S_0,t)\ \forall\, U \in \mathfrak{D}(\mathcal{E})$
- $[AdDX, AdDY] = AdD[X,Y]$

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- $U(S,t) = Ad(D(S_0,t))U(S_0,t) \;\; \forall\, U \in \mathfrak{D}(\mathcal{E})$
- $[AdDX, AdDY] = AdD[X,Y]$

Initial configuration (S_0, t)

$$\rho_0 \mathcal{T}_0 + \frac{\partial \Theta_0}{\partial S_0} + [\mathbf{e}_0, \Theta_0] = \rho_0 \Big(H_0(\frac{\partial V_0}{\partial t}) + [V_0, H_0(V_0)] \Big)$$

Kinematics

Displacement decomposition

$$\mathbb{D}(\mathcal{E}) = \mathbb{T}(\mathcal{E}) \times_C \mathbb{R}(\mathcal{E})_C$$

- $\times_C$ semi-direct product
- C center of mass depends on S_0
- $\mathbb{T}(\mathcal{E})$ translation group
- $\mathbb{R}(\mathcal{E})$ rotation group

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Lie algebra

$$\mathfrak{D}(\mathcal{E}) = \mathfrak{T} \oplus \mathfrak{R}_C$$

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- $\mathfrak{T} = \{X \in \mathfrak{D}(\mathcal{E}) | \omega_X = 0\}$
- $\mathfrak{R}_C = \{X \in \mathfrak{D}(\mathcal{E}) | X(C) = 0\}$

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Strain

$$\mathbf{e}_0(S_0, t) = \underbrace{\varepsilon_0(S_0, t)}_{\in \mathfrak{T}} + \underbrace{\boldsymbol{\kappa}_0(S_0, t)}_{\in \mathfrak{R}_C}$$

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Statics

Forces and moments

$$\Theta_0(S_0, t) = \underbrace{\mathbf{M}_0(S_0, t)}_{\in \mathfrak{T}} + \underbrace{\mathbf{F}_0(S_0, t)}_{\in \mathfrak{R}_C}$$

Statics

Forces and moments

$$\Theta_0(S_0,t) = \underbrace{\mathbf{M}_0(S_0,t)}_{\in\mathfrak{T}} + \underbrace{\mathbf{F}_0(S_0,t)}_{\in\mathfrak{R}_C}$$

Statics-level one

$$\rho_0\mathcal{T}_0 + \frac{\partial\Theta_0}{\partial S_0} + [\mathbf{e}_0, \Theta_0] = 0$$

$$\Downarrow \quad \rho_0\mathcal{T}_0 = \underbrace{\mathbf{m}}_{\in\mathfrak{T}} + \underbrace{\mathbf{q}}_{\in\mathfrak{R}_C}$$

Statics-level two

$$\begin{cases} \dfrac{d_{C(S_0)}\mathbf{F}_0}{dS_0} + [\boldsymbol{\kappa}_0, \mathbf{F}_0] + \mathbf{q} = 0 \\ \dfrac{d_{C(S_0)}\mathbf{M}_0}{dS_0} + [\boldsymbol{\varepsilon}_0, \mathbf{F}_0] + [\boldsymbol{\kappa}_0, \mathbf{M}_0] + \mathbf{m} = 0 \end{cases}$$

Statics

$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ basis in $\mathfrak{T}$

(ξ_1, ξ_2, ξ_3) basis in $\mathfrak{R}_C$

Statics

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Deformation|Force

$$\boldsymbol{\varepsilon}_0 = \sum_{i=1}^{3} \varepsilon_i \mathbf{e}_i \quad \Big| \quad \mathbf{F}_0 = \sum_{i=1}^{3} F_i \xi_i$$

Curvature | Moment

$$\boldsymbol{\kappa}_0 = \sum_{i=1}^{3} \kappa_i \xi_i \quad \Big| \quad \mathbf{M}_0 = \sum_{i=1}^{3} M_i \mathbf{e}_i$$

Statics

$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ basis in $\mathfrak{T}$

(ξ_1, ξ_2, ξ_3) basis in $\mathfrak{R}_C$

Deformation|Force

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Curvature | Moment

$$\boldsymbol{\kappa}_0 = \sum_{i=1}^{3} \kappa_i \xi_i \quad \Big| \quad \mathbf{M}_0 = \sum_{i=1}^{3} M_i \mathbf{e}_i$$

$$\frac{d_{C(S_0)}\mathbf{F}_0}{dS_0} + [\boldsymbol{\kappa}_0, \mathbf{F}_0] + \mathbf{q} = 0$$

$$\frac{d_{C(S_0)}\mathbf{M}_0}{dS_0} + [\boldsymbol{\varepsilon}_0, \mathbf{F}_0] + [\boldsymbol{\kappa}_0, \mathbf{M}_0] + \mathbf{m} = 0$$

⇓ Straight beam

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$$\begin{aligned} F_1' - F_2\kappa_3 + F_3\kappa_2 + q_1 &= 0 \\ F_2' + F_1\kappa_3 - F_3\kappa_1 + q_2 &= 0 \\ F_3' - F_1\kappa_2 + F_2\kappa_1 + q_3 &= 0 \end{aligned}$$

$$\begin{aligned} M_1' - M_2\kappa_3 + M_3\kappa_2 - F_2(\varepsilon_3 + 1) + F_3\varepsilon_2 + m_1 &= 0 \\ M_2' - M_3\kappa_1 + M_1\kappa_3 - F_3\varepsilon_1 + F_1(\varepsilon_3 + 1) + m_2 &= 0 \\ M_3' - M_1\kappa_2 + M_2\kappa_1 - F_1\varepsilon_2 + F_2\varepsilon_1 + m_3 &= 0 \end{aligned}$$

Constitutive laws

Linear constitutive law|Level one

$$\Theta_0 = \mathfrak{E}(\mathbf{e}_0), \qquad \mathfrak{E} : \mathfrak{D}(\mathcal{E}) \to \mathfrak{D}(\mathcal{E}), \quad \text{linear}$$

Elastic energy|Level one

$$\mathsf{E}_{el} = \frac{1}{2}[[\mathfrak{E}(\mathbf{e}_0)|\mathbf{e}_0]]$$

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Assume that $\mathfrak{E}$ is symmetric with respect to the Klein form:

$$[[\mathfrak{E}(u)|v]] = [[u|\mathfrak{E}(v)]], \quad \forall u, v \in \mathfrak{D}.$$

Elastic energy|Level two

$$\mathrm{E}_{el} = \frac{1}{2}\Big([[\mathfrak{E}(\boldsymbol{\varepsilon}_0)|\boldsymbol{\varepsilon}_0]] + 2[[\mathfrak{E}(\boldsymbol{\varepsilon}_0)|\boldsymbol{\kappa}_0]] + [[\mathfrak{E}(\boldsymbol{\kappa}_0)|\boldsymbol{\kappa}_0]]\Big)$$

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Using a normalized basis $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \xi_1, \xi_2, \xi_3)$ that diagonalizes the symmetrical operator

Elastic energy|Level three

$$\mathrm{E}_{el} = \frac{1}{2}\Big(EI_1\varepsilon_1^2 + EI_2\varepsilon_2^2 + GI_3\varepsilon_3^3 + EI_1\kappa_1^2 + EI_2\kappa_2^2 + GI_3\kappa_3^2\Big)$$

Problem statement

- Beam at an equilibrium and subjected to linear perturbation.
- Level one equation.
- $D(S_0, t) = \exp X(S_0, t) \circ D_0(S_0)$ where
  - $D_0(S_0)$ solution of the equilibrium equation
  - $X(S_0, t) \in \mathfrak{D}(\mathcal{E})$ the unknown kinematic of the vibration problem.

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equilibrium

$$\rho_0 \mathcal{T}_0(S_0) + \frac{\partial \Theta_{e,0}}{\partial S_0}(S_0) + [\mathbf{e}_{e,0}(S_0), \Theta_{e,0}(S_0)] = 0$$

- $\mathbf{e}_{e,0}(S_0)$ Lagrangian deformation at equilibrium.
- $\Theta_{e,0}(S_0)$ Lagrangian internal actions at equilibrium
- $\mathcal{T}_0(S_0)$ Lagrangian external actions at the equilibrium

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- $\mathcal{T}_0(S_0)$ Lagrangian external actions at the equilibrium

Note that

No additional perturbation is added to $\mathcal{T}_0(S_0)$.

Linear approximation calculation

$$\rho_0 \mathcal{T}_0 + \frac{\partial \Theta_0}{\partial S_0} + [\mathbf{e}_0, \Theta_0] = \rho_0 \Big(H_0 (\frac{\partial V_0}{\partial t}) + [V_0, H_0(V_0)] \Big)$$

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$$\begin{aligned} \mathbf{e}_0(S_0, t) &\approx \mathrm{Ad}\, D_0^{-1}(S_0) \frac{\partial X(S_0, t)}{\partial S_0} + \mathbf{e}_{e,0}(S_0) \\ \Theta_0(S_0, t) &= \Theta_{e,0}(S_0) + \chi_0(S_0, t) \end{aligned}$$

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$$\begin{aligned} V_0(S_0, t) &\approx \text{Ad}\, D_0^{-1}(S_0) \frac{\partial X(S_0, t)}{\partial t} \\ \frac{\partial V_0(S_0, t)}{\partial t} &\approx \text{Ad}\, D_0^{-1}(S_0) \frac{\partial^2 X(S_0, t)}{\partial t^2} \end{aligned}$$

Linear approximation calculation

$$\rho_0 \mathcal{T}_0 + \frac{\partial \Theta_0}{\partial S_0} + [\mathbf{e}_0, \Theta_0] = \rho_0 \Big(H_0(\frac{\partial V_0}{\partial t}) + [V_0, H_0(V_0)] \Big)$$

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Perturbated equation

$$\frac{\partial \chi_0}{\partial S_0} + [\text{Ad}\, D_0^{-1}(S_0)(\frac{\partial X}{\partial S_0}), \Theta_{e,0}(S_0)] + [\mathbf{e}_{e,0}(S_0), \chi_0] = \rho_0 \Big(H_0(\text{Ad}\, D_0^{-1}(S_0) \frac{\partial^2 X}{\partial t^2}) \Big)$$

Dynamic equation

Linear elastic contitutive law

$$\chi_0(S_0, t) = \mathfrak{E}_0 \left(\mathrm{Ad}\, D_0^{-1}(S_0) \frac{\partial X(S_0, t)}{\partial S_0} \right)$$

Dynamic equation

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$$\chi_0(S_0,t) = \mathfrak{E}_0\left(\mathrm{Ad}\, D_0^{-1}(S_0)\frac{\partial X(S_0,t)}{\partial S_0}\right)$$

- Applying the left action of $\mathrm{Ad}\, D_0(S_0)$ on perturbated equation.
- For all $Y \in \mathfrak{D}(\mathcal{E})$, we let

$$\widetilde{\mathbf{e}}_{e,0}(S_0) = \mathrm{Ad}\, D_0(S_0)(\mathbf{e}_{e,0}(S_0))$$

$$\widetilde{H}_0(Y) = \mathrm{Ad}\, D_0(S_0)\circ H_0 \circ \mathrm{Ad}\, D_0^{-1}(S_0)(Y)$$

$$\widetilde{\Theta}_{e,0}(S_0) = \mathrm{Ad}\, D_0(S_0)(\Theta_{e,0}(S_0))$$

$$\widetilde{\mathfrak{E}}_0(Y) = \mathrm{Ad}\, D_0(S_0)\circ \mathfrak{E}_0 \circ \mathrm{Ad}\, D_0^{-1}(S_0)(Y)$$

$$\widetilde{R}_0(Y) = [Y, \widetilde{\Theta}_{e,0}(S_0)] + [\widetilde{\mathbf{e}}_{e,0}(S_0), \widetilde{\mathfrak{E}}_0(Y)] + \widetilde{\mathfrak{E}}_0([Y, \widetilde{\mathbf{e}}_{e,0}(S_0)])$$

Dynamic equation

Linear elastic contitutive law

$$\chi_0(S_0,t) = \mathfrak{E}_0 \left(\text{Ad } D_0^{-1}(S_0) \frac{\partial X(S_0,t)}{\partial S_0} \right)$$

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Perturbated equation

$$\widetilde{\mathfrak{E}}_0(\frac{\partial^2 X(S_0,t)}{\partial S_0^2}) + \widetilde{R}_0(\frac{\partial X(S_0,t)}{\partial S_0}) = \rho_0 \widetilde{H}_0(\frac{\partial^2 X(S_0,t)}{\partial t^2})$$

Variational formulation

- Boundary conditions should be given to define a well-posed problem.
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- Dynamic properties imply that $\widetilde{H}_0$ is a symmetric operator with respect to the Klein form.

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$$\widetilde{\mathfrak{E}}_0\left(\frac{\partial^2 X(S_0,t)}{\partial S_0^2}\right) + \widetilde{R}_0\left(\frac{\partial X(S_0,t)}{\partial S_0}\right)$$

Replace $X(S_0,t)$ by $U(S_0)$

$$\widetilde{\mathfrak{E}}_0(U''(S_0)) + \widetilde{R}_0(U'(S_0))$$

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- Suppose that $\widetilde{\mathfrak{E}}_0$ is a symmetric operator w.r.t to the Klein form $[[\cdot|\cdot]]$.
- Dynamic properties imply that $\widetilde{H}_0$ is a symmetric operator with respect to the Klein form.

$$\widetilde{\mathfrak{E}}_0(\frac{\partial^2 X(S_0,t)}{\partial S_0^2}) + \widetilde{R}_0(\frac{\partial X(S_0,t)}{\partial S_0})$$

Replace $X(S_0,t)$ by $U(S_0)$

$$\widetilde{\mathfrak{E}}_0(U''(S_0)) + \widetilde{R}_0(U'(S_0))$$

Integrate along the beam

$$\Psi_0(U,V) = \int_0^{L_0} [[\widetilde{\mathfrak{E}}_0(U''(S_0)) + \widetilde{R}_0(U'(S_0)) \mid V(S_0)]] dS_0$$

Variational formulation

Integration by parts

$$\Psi_0(U,V) = \underbrace{-\int_0^{L_0} [[(\widetilde{\mathfrak{E}}_0(U')) \mid V']] dS_0}_{\text{Symmetrical operator}} +$$

$$\underbrace{\int_0^{L_0} \Big[\Big[[U', \widetilde{\mathfrak{E}}_0(\widetilde{\mathbf{e}}_{e,0}))] \mid V\Big]\Big] dS_0}_{\mathfrak{B}(U,V)} + \underbrace{\Big[[[(\widetilde{\mathfrak{E}}_0(U')) \mid V]]\Big]_0^{L_0}}_{\text{Boundary condition}}$$

- Symmetry of $\mathfrak{B}(U,V) \Longrightarrow$ symmetry of $\Psi_0(U,V)$.

Variational formulation

Integration by parts

$$\boldsymbol{\Psi}_0(U,V) = \underbrace{-\int_0^{L_0} [[(\widetilde{\mathfrak{E}}_0(U')) \mid V']] dS_0}_{\text{Symmetrical operator}} +$$

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- Symmetry of $\mathfrak{B}(U,V) \Longrightarrow$ symmetry of $\Psi_0(U,V)$.

$$\mathfrak{B}(U,V) - \mathfrak{B}(V,U) = -\int_0^{L_0} \left[\left[[U,V] \mid \widetilde{\mathfrak{E}}_0(\widetilde{\mathbf{e}}_{e,0})'\right]\right] dS_0 + \left[\left[\left[[U,V] \mid \widetilde{\mathfrak{E}}_0(\widetilde{\mathbf{e}}_{e,0})\right)\right]\right]\right]_0^{L_0}$$

Variational formulation

Integration by parts

$$\Psi_0(U,V) = \underbrace{-\int_0^{L_0} [[(\widetilde{\mathfrak{E}}_0(U')) \mid V']] dS_0}_{\text{Symmetrical operator}} +$$

$$\underbrace{\int_0^{L_0} \left[\left[[U', \widetilde{\mathfrak{E}}_0(\widetilde{\mathbf{e}}_{e,0}))] \mid V\right]\right] dS_0}_{\mathfrak{B}(U,V)} + \underbrace{\Big[[[(\widetilde{\mathfrak{E}}_0(U')) \mid V]] \Big]_0^{L_0}}_{\text{Boundary condition}}$$

- Symmetry of $\mathfrak{B}(U,V) \Longrightarrow$ symmetry of $\Psi_0(U,V)$.

$$\mathfrak{B}(U,V) - \mathfrak{B}(V,U) = -\int_0^{L_0} \left[\left[[U,V] \mid \widetilde{\mathfrak{E}}_0(\widetilde{\mathbf{e}}_{e,0})'\right]\right] dS_0 + \left[\left[\left[[U,V] \mid \widetilde{\mathfrak{E}}_0(\widetilde{\mathbf{e}}_{e,0}))\right]\right]\right]_0^{L_0}$$

Examples : $\mathfrak{B}(U,V)$ is symmetric if

$\widetilde{\mathfrak{E}}_0(\widetilde{\mathbf{e}}_{e,0}) = 0$

Clamped-Clamped beam and a uniform $\widetilde{\mathfrak{E}}_0(\widetilde{\mathbf{e}}_{e,0})$

Conclusion and perspectives

- Exploration of one-dimensional structures through the application of Lie group structures.

- Three levels of equation setting
  - First level $\Longrightarrow$ a single equation within the Lie algebra of the displacement group $\mathbb{D}(\mathcal{E})$.
  - Second level $\Longrightarrow$ semi-direct product $\mathbb{D}(\mathcal{E}) = \mathbb{T}(\mathcal{E}) \times_C \mathbb{R}(\mathcal{E})_C$.
  - The third level $\Longrightarrow$ introduction of an appropriate basis.

- Perturbation of linear dynamics around an arbitrary equilibrium position :strong and week formulations.

- Perspectives :
  - Boundary conditions investigations.
  - Deformed cross sections.