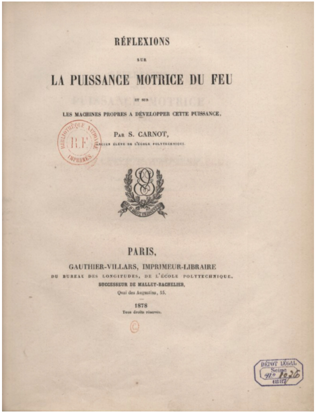
Thermodynamique pentadimensionnelle des milieux continus

Géry de Saxcé

LaMcube UMR CNRS 9013 Université Lille

GDR-GDM 2024 Paris IRCAM

Bicentenary of the Thermodynamics



RÉFLEXIONS

SUR

LA PUISSANCE MOTRICE DU FEU

ET SUR

LES MACHINES PROPRES A DÉVELOPPER CETTE PUISSANCE,

PAR S. CARNOT,

ANCIEN ÉLÈVE DE L'ÉCOLE POLYTECHNIQUE.

PARIS,

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE

DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE,

SUCCESSEUR DE MALLET-BACHELIER,

Quai des Augustins, 55.

1878

Tous droits réservés.

Relativity and Thermodynamics

Geometrization process

General Relativity

a **model** for the mechanics and physics of continua

- Temperature $\longrightarrow$ vector W
- Entropy $\longrightarrow$ vector S
- Energy $\longrightarrow$ tensor T
- Gravitation $\longrightarrow$ covariant derivative ∇

State of the Art

- Background
  C Eckart, Phys. Rev. (1940) $div\ T = 0$ (1^{st} principle)

  Landau-Lifshitz, Fluid Mech. (1960) $T = T_R + T_I, \quad S = T\,W$

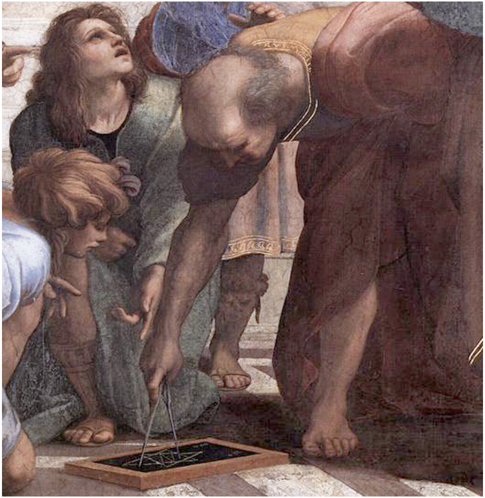
- Inspiration sources
  JM Souriau, Lect. Notes Math. (1976) $div\ S \geq 0$ (2^{nd} principle)

  C Vallée, IJES (1981) constitutive laws

- The present contribution
  de Saxcé-Vallée, IJES (2012) Galilean version

  de Saxcé-Vallée, Galilean Mechanics and Thermodynamics of Continua (2016) revisiting the relativistic version of the 2^{nd} principle

Geometric approach



Galilean transformations

- Event occuring at position x and at time t

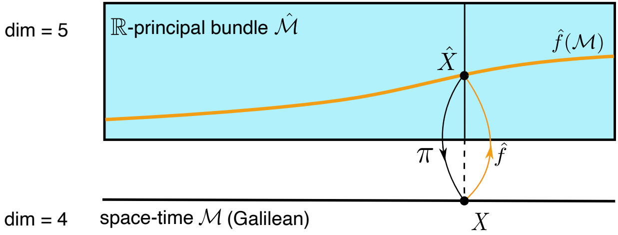
$$X = \begin{pmatrix} t \\ x \end{pmatrix} \in \textbf{space-time } \mathcal{M}$$

- **Symmetry group :** Lorentz-Poincaré group $\longrightarrow$ Galileo's group
- The **Galilean transformations** are space-time transformations preserving **inertial motions**, **durations** and **distances**, then affine of the form $X = P\,X' + C$ with :

$$P = \begin{pmatrix} 1 & 0 \\ v_t & R \end{pmatrix}, \qquad C = \begin{pmatrix} \tau_0 \\ k \end{pmatrix}$$

  where $v_t \in \mathbb{R}^3$ is the **Galilean boost** and R is a rotation
- Their set is **Galileo's group**, a Lie group of dimension 10
- **Dimension 4 or 5 ?**

Bargmannian transformations



- We introduce a $\mathbb{R}$-**principal bundle** $\pi : \hat{\mathcal{M}} \rightarrow \mathcal{M}$ and we consider a **section** $\hat{f} : \mathcal{M} \rightarrow \hat{\mathcal{M}} \; : \; \boldsymbol{X} \mapsto \hat{\boldsymbol{X}} = \hat{f}\,(\boldsymbol{X})$
- We built a group of affine transformations $\hat{X}' \mapsto \hat{X} = \hat{P}\,\hat{X}' + \hat{C}$ of $\mathbb{R}^5$ which are Galilean when acting onto the space-time hence :

$$\hat{P} = \begin{pmatrix} P & 0 \\ \Phi & \alpha \end{pmatrix} ,$$

where P is Galilean, Φ, α have a physical meaning linked to the energy

Bargmannian transformations

- Thus we know that, under the action of a boost v_t and a rotation R, the kinetic energy is transformed according to :

$$e = \frac{1}{2}\, m \parallel v_t + R\, v' \parallel^2 = \frac{1}{2}\, m \parallel v_t \parallel^2 + m\, v_t \cdot (R\, v') + \frac{1}{2}\, m \parallel v' \parallel^2 \ .$$

- We claim that the fifth dimension is linked to the energy by :

$$dz = \frac{e}{m}\, dt = \frac{1}{2} \parallel v_t \parallel^2 dt' + v_t^T R\, dx' + dz'$$

that leads to consider the **Bargmannian transformations** of $\mathbb{R}^5$

of which the linear part is : $\hat{P} = \begin{pmatrix} 1 & 0 & 0 \\ v_t & R & 0 \\ \frac{1}{2} \parallel v_t \parallel^2 & v_t^T R & 1 \end{pmatrix}$

Their set is the **Bargmann's group**,
introduced in quantum mechanics for cohomologic reasons
but which turns out very useful in Thermodynamics !

Temperature, friction and momentum tensor

Temperature 5-vector

The reciprocal temperature $\beta = \frac{1}{\theta} = \frac{1}{k_B T}$ is generalized as a Bargmannian 5-vector :

$$\hat{W} = \begin{pmatrix} W \\ \zeta \end{pmatrix} = \begin{pmatrix} \beta \\ w \\ \zeta \end{pmatrix} ,$$

Step 1 :

- The transformation law $\hat{W}' = \hat{P}^{-1} \hat{W}$ leads to :

$$\beta' = \beta \ , \quad w' = R^T (w - \beta v_t), \quad \zeta' = \zeta - w \cdot v_t + \frac{\beta}{2} \parallel v_t \parallel^2$$

- Picking up $v_t = w \,/\, \beta$, we obtain the **reduced form**

$$\hat{W}' = \begin{pmatrix} \beta \\ 0 \\ \zeta_{int} \end{pmatrix}$$

interpreted as the temperature vector of a volume element **at rest**

Temperature 5-vector

Step 2 : Starting from the reduced form, we apply the Galilean transformation of boost v, that gives :

$$\hat{W} = \begin{pmatrix} \beta \\ w \\ \zeta \end{pmatrix} = \begin{pmatrix} \beta \\ \beta\, v \\ \zeta_{int} + \frac{\beta}{2} \parallel v \parallel^2 \end{pmatrix} .$$

where ζ is **Planck's potential**

This is the covariant form of the temperature vector,
i.e. **remaining the same under all Galilean transformation**

Boost method :

- **Step 1 :** symmetry group action $\longrightarrow$ reduced form
- **Step 2 :** boost $\longrightarrow$ covariant form

Friction tensor

Friction tensor

The **friction tensor** is a mixed 1-covariant and 1-contravariant tensor :

$$\boldsymbol{f} = \nabla\, \vec{\boldsymbol{W}}$$

represented by the 4×4 matrix $f = \nabla\, W$

- This object introduced by Souriau merges the temperature gradient and the strain velocity
- In dimension 5, we can also introduce

  $$\hat{\boldsymbol{f}} = \nabla\, \hat{\vec{\boldsymbol{W}}}$$

  represented by a 5×4 matrix

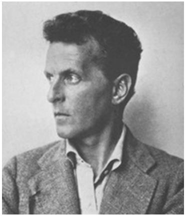
  $$\hat{f} = \nabla\, \hat{W} = \begin{pmatrix} f \\ \nabla\, \zeta \end{pmatrix}$$

Momentum tensor

Method

Taking care **to walk up and down the rough ground of the reality** (Wittgenstein),

we want to work, in dimension 4 ou 5, with tensors of which the transformation law respects the physics



The meaning of the components is not given *a priori* but results, through the transformation law, from the choice of the symmetry group

Momentum tensor

Momentum tensor

Linear map from the tangent space to $\hat{\mathcal{M}}$ at $\hat{\boldsymbol{X}} = \hat{f}(\boldsymbol{X})$ into the tangent space to $\mathcal{M}$ at $\boldsymbol{X}$, hence a **mixed tensor** $\hat{\boldsymbol{T}}$ of rank 2

- **Galilean momentum tensors :** represented by a 4×5 matrix

$$\hat{T} = \begin{pmatrix} \mathcal{H} & -p^T & \rho \\ k & \sigma_\star & p \end{pmatrix}$$

In matrix form, the transformation law is $\hat{T}' = P\ \hat{T}\ \hat{P}^{-1}$

- **We let the symmetry group act,** that leads to the **reduced form** :

$$\hat{T}' = \begin{pmatrix} \rho\, e_{int} & 0 & \rho \\ h & \sigma & 0 \end{pmatrix} ,$$

interpreted as the momentum of a volume element **at rest**,
where occur the **density** ρ, the **internal energy** e_{int},
the **heat flux** h, and **Cauchy's stresses** σ

Momentum tensor

Hence the **boost method** reveals its **covariant form**

Galilean momentum tensor

Object where occurs

- **Hamiltonian** (by volume unit) $\mathcal{H} = \rho\left(e_{int} + \frac{1}{2} \parallel v \parallel^2\right)$
- **linear momentum** $p = \rho\, v$,

represented by the matrix :

$$\hat{T} = \begin{pmatrix} \mathcal{H} & -p^T & \rho \\ h + \mathcal{H}\, v - \sigma\, v & \sigma - \rho\, v\, v^T & p \end{pmatrix}$$

These particular quantities are gathered here in big tensors

First and second principles

First principle

Momentum divergence

5-row $div\ \hat{T}$ such that, for all smooth 5-vector field $\hat{W}$:

$$Div\ (\hat{T}\ \hat{W}) = (Div\ \hat{T})\ \hat{W} + Tr\ \left(\hat{T}\ \nabla\ \hat{W}\right)$$

Covariant form of the 1^{st} principle

$$Div\ \hat{T} = 0$$

First principle

In absence of gravity, we recover the balance equations of :

- mass : $\dfrac{\partial \rho}{\partial t} + div\,(\rho\, v) = 0$
- linear momentum : $\rho \left[\partial_t v + \dfrac{\partial v}{\partial x}\, v \right] = (div\ \sigma)^T$
- energy : $\partial_t \mathcal{H} + div\ \ (h + \mathcal{H} v - \sigma\, v) = 0$

First principle

4-velocity $U = \frac{dX}{dt} = \frac{d}{dt}\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ v \end{pmatrix}$

Reversible medium

if ζ is a function of
- the other components of $\hat{W}$ through W,
- the right Cauchy strain $C = F^T F$
- and the Lagrangean coordinates,

then the 4×4 matrix $T_R = U \otimes \Pi_R + \begin{pmatrix} 0 & 0 \\ -\sigma_R v & \sigma_R \end{pmatrix}$

with $\Pi_R = -\rho \, \frac{\partial \zeta}{\partial W}$ $\qquad \sigma_R = -\frac{2\rho}{\beta} F \, \frac{\partial \zeta}{\partial C} \, F^T$

is such that :

- ♡ $\hat{T}_R = \begin{pmatrix} T_R & N \end{pmatrix}$ with $N = \rho \, U$ represents a momentum tensor $\hat{\boldsymbol{T}}_R$
- ♢ $Tr\left(\hat{T}_R \nabla \hat{W}\right) = 0$

First principle

ζ is the prototype of **thermodynamic potentials** :

- the **internal energy** $e_{int} = -\dfrac{\partial \zeta_{int}}{\partial \beta}$
- the **specific entropy** $s = \zeta_{int} - \beta\, \dfrac{\partial \zeta_{int}}{\partial \beta}$

  of which the 4-flux $\vec{\boldsymbol{S}} = s\, \vec{\boldsymbol{N}}$ is the Galilean 4-vector

$$\vec{\boldsymbol{S}} = \hat{\mathrm{T}}_R\, \hat{\vec{\boldsymbol{W}}}$$

- the **free energy** $\psi = -\dfrac{1}{\beta}\, \zeta_{int} = -\theta\, \zeta_{int}$ allows to recover

  $-e_{int} = \theta\, \frac{\partial \psi}{\partial \theta} - \psi, \qquad -s = \frac{\partial \psi}{\partial \theta}$

The interest of Planck's potential ζ is that it generates all the other ones

Second principle

Additive decomposition of the momentum tensor

$\hat{\mathrm{T}} = \hat{\mathrm{T}}_R + \hat{\mathrm{T}}_I$ with

- the reversible part $\hat{\mathrm{T}}_R$ represented by :

$$\hat{T}_R = \begin{pmatrix} \mathcal{H}_R & -p^T & \rho \\ \mathcal{H}_R v - \sigma_R v & \sigma_R - v p^T & \rho v \end{pmatrix}$$

- the irreversible one $\hat{\mathrm{T}}_I$ represented by :

$$\hat{T}_I = \begin{pmatrix} \mathcal{H}_I & 0 & 0 \\ h + \mathcal{H}_I v - \sigma_I v & \sigma_I & 0 \end{pmatrix}$$

where σ_I are the **dissipative stresses** and $\mathcal{H}_I = -\rho q_I$ is the dissipative part of the energy due to the **irreversible heat sources** q_I (for instance of electrical, chemical or nuclear origin)

Second principle

Clock form

Linear form $\boldsymbol{\tau} = dt$ represented by an invariant row under Galilean transformation :

$$\tau = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$

Entropy 4-vector $\vec{\boldsymbol{S}} = \hat{\boldsymbol{T}}\ \hat{\vec{\boldsymbol{W}}} = \hat{\boldsymbol{T}}_R \hat{\vec{\boldsymbol{W}}} + \hat{\boldsymbol{T}}_I\ \hat{\vec{\boldsymbol{W}}} = \vec{\boldsymbol{S}}_R + \vec{\boldsymbol{S}}_I$

Covariant form of the second principle

The **local production of entropy** of a medium caracterized by a temperature vector $\hat{\vec{\boldsymbol{W}}}$ and a momentum tensor $\hat{\mathrm{T}}$ is non negative :

$$\Phi = \boldsymbol{Div}\, \vec{\boldsymbol{S}} - \left(\boldsymbol{\tau}(\boldsymbol{f}(\vec{\boldsymbol{U}}))\right) \left(\boldsymbol{\tau}(\boldsymbol{T}_I(\vec{\boldsymbol{U}}))\right) \geq 0$$

and vanishes if and only if the process is reversible

Second principle

- The local production of entropy

$$\Phi = \boldsymbol{Div}\,\vec{\boldsymbol{S}} - \left(\boldsymbol{\tau}(\boldsymbol{f}(\vec{\boldsymbol{U}}))\right)\,\left(\boldsymbol{\tau}(\boldsymbol{T}_I(\vec{\boldsymbol{U}}))\right)$$

is a **Galilean invariant !**

- After some manipulations, it can be put in the classical form of **Clausius-Duhem inequality**

$$\Phi = \rho\,\frac{ds}{dt} - \frac{\rho}{\theta}\,\frac{dq_I}{dt} + div\,\left(\frac{h}{\theta}\right) \geq 0$$

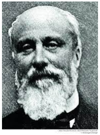
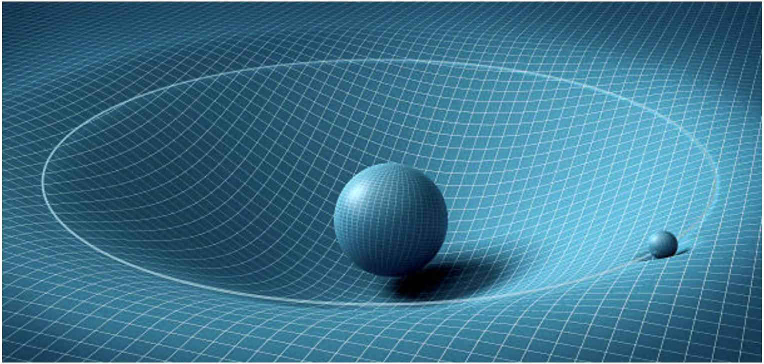
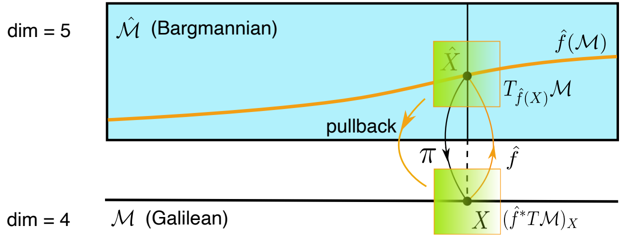


FIGURE – Pierre Duhem

Thermodynamics and Galilean gravitation



Thermodynamics and Galilean gravitation



- We consider the pullback bundle $\hat{f}^* T\hat{\mathcal{M}}$
- The space-time $\mathcal{M}$ is endowed with the pullback connection $(\hat{f}^*\hat{\nabla})_{\boldsymbol{U}}(\hat{f}^*\hat{\boldsymbol{W}}) = \hat{f}^*(\hat{\nabla}_{(Tf)\boldsymbol{U}}\hat{\boldsymbol{W}}) \qquad \boldsymbol{U} \in T\mathcal{M}, \qquad \hat{\boldsymbol{W}} \in T\hat{\mathcal{U}}$
- Galileo's group does not preserve space-time metrics
- Bargmann's group preserves the metrics $ds^2 = \| dx \|^2 - 2\, dz\, dt$, then the space $\hat{\mathcal{M}}$ is a riemannian manifold

Thermodynamics and Galilean gravitation

- With the **potentials of the Galilean gravitation** ϕ, A generating the gravity $g = -grad\,\phi - \partial_t A$ and Coriolis effect $\Omega = \frac{1}{2}\,curl\,A$, the Lagrangian is $\mathcal{L}(t,x,v) = \frac{1}{2}\,m \parallel v \parallel^2 - m\,\phi + m\,A\cdot v$
- that suggests to introduce a base change $dz' = \frac{\mathcal{L}}{m}\,dt = dz - \phi\,dt + A\cdot dx\ ,\quad dt' = dt,\quad dx' = dx$
- In the new coordinates, the **Bargmannian connection** is

$$\hat{\Gamma}(d\hat{X}) = \begin{pmatrix} 0 & 0 & 0 \\ j(\Omega)\,dx - g\,dt & j(\Omega)\,dt & 0 \\ \begin{array}{c}((\partial_t\phi - A\cdot g)\ dt \\ +(grad\,\phi - \Omega\times A)\cdot dx)\end{array} & \begin{array}{c}((grad\,\phi - \Omega\times A)\ dt \\ -grad_s A\,dx)^T\end{array} & 0 \end{pmatrix}$$

- As in electromagnetism, the potentials ϕ, A are defined modulo a gauge transformation. It can be shown that it corresponds to a change of section $\hat{f}$ then **the choice of the section does not modify the equations of Thermodynamics**.

Thermodynamics and Galilean gravitation

The developments are similar to the ones in absence of gravitation but with some exceptions :

- Planck's potential becomes $\zeta = \zeta_{int} + \frac{\beta}{2} \parallel v \parallel^2 - \beta\,\phi + A \cdot w$
- the Hamiltonian becomes $\mathcal{H} = \rho\, \left(e_{int} + \frac{1}{2} \parallel v \parallel^2 + \phi - q_I \right)$,
- the linear momentum becomes $p = \rho\,(v + A)$.

In presence of gravitation, the first principle provides **in covariant form** the balance equations of the mass and of

- the linear momentum : $\rho\, \dfrac{dv}{dt} = (div\ \sigma)^T + \rho\, (g - 2\,\Omega \times v)$
- the energy : $\partial_t \mathcal{H} + div\ \ (h + \mathcal{H} v - \sigma v) = \rho\,(\partial_t \phi - \partial_t A \cdot v)$

A smidgen of relativistic Thermodynamics

- **Espistemological reversal** : we come back to the relativity with **Lorentz-Poincaré** symmetry group

- In this approach, the temperature is transformed according to

$$\theta' = \frac{\theta}{\gamma} = \theta \sqrt{1 - \frac{\parallel v \parallel^2}{c^2}}$$

This the **temperature contraction** !

- thanks to the space-time Minkowski's metrics $ds^2 = c^2 dt^2 - \parallel dx \parallel^2$, we can associate to the 4-velocity $\vec{\boldsymbol{U}}$ a unique linear form $\boldsymbol{U}^*$ represented by

$$U^T G = \left(\gamma, \gamma\, v^T\right) \begin{pmatrix} c^2 & 0 \\ 0 & -1_{\mathbb{R}^3} \end{pmatrix} = c^2 \left(\gamma, -\frac{1}{c^2} \gamma\, v^T\right) \ ,$$

which approaches $c^2 \boldsymbol{\tau}$ when c approaches $+\infty$

A smidgen of relativistic Thermodynamics

On this ground, we replace $\boldsymbol{\tau}$ by $\boldsymbol{U}^* / \, c^2$ in the Galilean expression of Clausius-Duhem inequality, that lead to

Relativistic form of the 2^{nd} principle

The **local production of entropy** of a medium characterized by a temperature vector $\vec{\mathrm{W}}$ and a momentum tensor $\hat{\mathrm{T}}$ is non negative :

$$\Phi = \boldsymbol{Div}\,\vec{\boldsymbol{S}} - \frac{1}{c^2}\left(\boldsymbol{U}^*(\boldsymbol{f}(\vec{\boldsymbol{U}}))\right)\frac{1}{c^2}\left(\boldsymbol{U}^*(\boldsymbol{T}_I(\vec{\boldsymbol{U}}))\right) \geq 0 \; ,$$

and vanishes if and only if the process is reversible

$$\phi = \nabla_\alpha S^\alpha - \frac{1}{c^2} f_{\alpha\beta} U^\alpha U^\beta \cdot \frac{1}{c^2} T_{I\alpha\beta} U^\alpha U^\beta \geqslant 0$$

The scheme of General Relativity is still valid in classical Mechanics

$$\phi = \nabla_\alpha S^\alpha - \frac{1}{c^2} f_{\alpha\beta} U^\alpha U^\beta , \frac{1}{c^2} T_{I\alpha\beta} U^\alpha U^\beta \geq 0$$

Thank you !