Une introduction à la géométrie multisymplectique

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- the Euler–Lagrange system of equations is
$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\left(t, \mathbf{u}(t), \frac{d\mathbf{u}}{dt}(t)\right)\right) = \frac{\partial L}{\partial y^i}\left(x, \mathbf{u}(x), \frac{d\mathbf{u}}{dt}(t)\right).$$

The multisymplectic formalism

... in a few words.

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where dvol is a volume n-form on X^n

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  where dvol is a volume n-form on X^n
- the Euler–Lagrange system of equations is

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial v^i_\mu}(x, \mathbf{u}(x), d\mathbf{u}_x)\right) = \frac{\partial L}{\partial y^i}(x, \mathbf{u}(x), d\mathbf{u}_x). \qquad (1)$$

Legendre hypothesis

Set

$$p_i^\mu := \frac{\partial L}{\partial v_\mu^i}(x^\mu, y^i, v_\mu^i)$$

and assume that $(x^\mu, y^i, v_\mu^i) \longmapsto (x^\mu, y^i, p_i^\mu)$ is invertible

- consider the Hamiltonian function:

$$H(x^\mu, y^i, p_i^\mu) := p_i^\mu v_\mu^i - L(x^\mu, y^i, v_\mu^i)$$

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- then the Euler–Lagrange equations reads

$$\left\{ \begin{array}{rcl} \dfrac{\partial \mathbf{u}^i}{\partial x^\mu}(x) & = & \dfrac{\partial H}{\partial p_i^\mu}(x^\mu, \mathbf{u}^i(x), \mathbf{p}_i^\mu(x)) \\ \dfrac{\partial \mathbf{p}_i^\mu}{\partial x^\mu}(x) & = & -\dfrac{\partial H}{\partial y^i}(x^\mu, \mathbf{u}^i(x), \mathbf{p}_i^\mu(x)), \end{array} \right. \tag{2}$$

We call them the **first Volterra** equations (1890).

Historical remarks

Very often called the **De Donder–Weyl** equations.

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- Each of these two filiations were motivated by different questions.
- a geometrization of these ideas was initiated by P. Dedecker (1953)
- and fully accomplished by the Polish school starting with an (unpublished) seminar in Warsaw by W. Tulczyjew after 1965 and followed by publications by J. Kijowski, K. Gawedski, W. Sczcyrba in the seventies.

Geometrization

In $\mathrm{X}^n \times \mathrm{Y}^k \times \mathrm{End}(\mathrm{X}^n, \mathrm{Y}^k)^* \ni (x^\mu, y^i, p_i^\mu)$ consider the n-dimensional submanifold Γ defined by:

$$y^i = \mathbf{u}^i(x) \quad \text{and} \quad p_i^\mu = \frac{\partial L}{\partial v_\mu^i}(x^\mu, \mathbf{u}^i(x), \partial_\mu \mathbf{u}^i(x)).$$

and the pre-**multisymplectic** $(n+1)$-form

$$\omega := \mathrm{d}p_i^\mu \wedge \mathrm{d}y^i \wedge \mathrm{dvol}(\partial_\mu, \cdots) - \mathrm{d}(H(x, y, p)) \wedge \mathrm{dvol}.$$

Then the Hamilton-Volterra–De Donder–Weyl system of equations reads:

$$\forall \xi \text{ (vector field)}, \quad \omega(\xi, \cdots)|_\Gamma = 0.$$

Geometric generalizations 1

We can replace maps $\mathbf{u} : \mathrm{X}^n \longrightarrow \mathrm{Y}^k$ by sections $\mathbf{u} : \mathcal{X}^n \longrightarrow \mathcal{Z}^{n+k}$ of a vector bundle

$$\pi_{\mathcal{X}} : \mathcal{Z}^{n+k} \longrightarrow \mathcal{X}^n.$$

Consider the vector bundle $\Lambda^n T^*\mathcal{Z}^{n+k}$ of n-forms over $\mathcal{Z}^{n+k}$.
There is a canonical n-form θ on $\Lambda^n T^*\mathcal{Z}^{n+k}$ defined by:
$\forall z \in \mathcal{Z}^{n+k}$, $\forall p \in \Lambda^n T^*_z\mathcal{Z}^{n+k}$,

$$\theta_{(z,p)}(X_1, \cdots, X_n) := p(\mathrm{d}\pi_{\mathcal{X}}(X_1), \cdots, \mathrm{d}\pi_{\mathcal{X}}(X_n)).$$

Then

$$\omega := \mathrm{d}\theta$$

is a **multisymplectic** $(n+1)$-form.

Geometric generalizations 2

The multisymplectic manifold $(\Lambda^n T^*\mathcal{Z}^{n+k}, \omega)$ is too large. We restrict ourself to a submanifold of $\Lambda^n T^*\mathcal{Z}^{n+k}$.

- Call a tangent vector $v \in T_z\mathcal{Z}$ s.t. $d\pi_{\mathcal{X}}(v) = 0$ a **vertical** vector field;

This gives us the right generalization of:

$$\omega := dp_i^\mu \wedge dy^i \wedge d\mathrm{vol}(\partial_\mu, \cdots) - d(H(x,y,p)) \wedge d\mathrm{vol}.$$

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- define $\Lambda^n_2 T^*\mathcal{Z}$ to be the set of $(z,p) \in \Lambda^n T^*\mathcal{Z}$ s.t.

$$\text{if } v \text{ and } w \text{ are vertical, then } p(v, w, \cdots) = 0;$$

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- set θ_2 : the restriction of θ to $\Lambda^n_2 T^*\mathcal{Z}$.
- (optional) if $\mathcal{H} : \Lambda^n_2 T^*\mathcal{Z} \longrightarrow \mathbb{R}$ is a Hamiltonian function, restrict further to the submanifold $\mathcal{H} = 0$ (pre-n-plectic).

This gives us the right generalization of:

$$\omega := dp_i^\mu \wedge dy^i \wedge d\text{vol}(\partial_\mu, \cdots) - d(H(x,y,p)) \wedge d\text{vol}.$$

Multisymplectic manifolds : general framework

An *n*-plectic manifold $\mathcal{N}$ is a manifold equipped with an $(n+1)$-form ω s.t.

- ω is closed: $d\omega = 0$;

A Hamiltonian function on an *n*-plectic manifold $(\mathcal{N}, \omega)$ is a function $\mathcal{H} : \mathcal{N} \longrightarrow \mathbb{R}$ s.t. $d\mathcal{H} \neq 0$.

An *n*-dimensional submanifold $\Gamma \subset \mathcal{N}$ is a solution of the generalized Hamilton equations if:

$$\text{for any vector field } \xi, \quad d\mathcal{H}(\xi) = 0 \Longrightarrow \omega(\xi, \cdots)|_\Gamma = 0$$

and Γ is then called a Hamiltonian *n*-curve.

Multisymplectic manifolds : general framework

An n-plectic manifold $\mathcal{N}$ is a manifold equipped with an $(n+1)$-form ω s.t.

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- ω is non degenerate: for any vector ξ, if $\omega(\xi, \cdots) = 0$, then $\xi = 0$.

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An example

The scalar fields $\mathbf{u} : \mathbb{R}^n \longrightarrow \mathbb{R}$ which are solutions of the Klein–Gordon equation

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - \Delta \mathbf{u} + V'(\mathbf{u}) = 0.$$

On $\mathcal{N} := \{(x^\mu, y, p^\mu, e) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}\}$ set

$$\omega = \mathrm{d}e \wedge \beta + \mathrm{d}p^\mu \wedge \mathrm{d}y \wedge \beta_\mu,$$

where $\beta := \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{n-1}$ and $\beta_\mu := \beta(\partial_\mu, \cdots)$; and

$$\mathcal{H}(x^\mu, y, p^\mu, e) := e + \frac{1}{2}\eta_{\mu\nu} p^\mu p^\nu + V(y);$$

Then $\mathbf{u}$ is a solution of the Klein–Gordon equation iff the submanifold Γ of equation $y = \mathbf{u}(x)$, $\mathbf{p}^\mu(x) = \eta^{\mu\nu}\partial_\nu \mathbf{u}(x)$ and $\mathcal{H}(x^\mu, \mathbf{u}(x), \mathbf{p}^\mu(x), \mathbf{e}(x)) = 0$ is a solution of:

$$\text{for any vector field } \xi, \quad \mathrm{d}\mathcal{H}(\xi) = 0 \Longrightarrow \omega(\xi, \cdots)|_\Gamma = 0.$$

Pre-multisymplectic manifolds : general framework

A pre-n-plectic manifold $\mathcal{N}$ is a manifold equipped with an $(n+1)$-form ω and a 'volume' n-form β s.t. ω is closed: $d\omega = 0$ and β does not vanish.

Fundamental example: let $(\mathcal{N}, \omega)$ be an n-plectic manifold and $\mathcal{H} : \mathcal{N} \longrightarrow \mathbb{R}$ is a Hamiltonian function. Then the submanifold

$$\mathcal{N}^0 := \mathcal{H}^{-1}(0) := \{z \in \mathcal{N} |\ \mathcal{H}(z) = 0\}$$

with the pre-$(n+1)$-plectic form $\omega|_{\mathcal{N}^0}$ and the 'volume' n-form $\beta = \omega(\tau, \cdots)|_{\mathcal{N}^0}$ (where τ is a vector field s.t. $d\mathcal{H}(\tau) = 1$) is a pre-n-plectic manifold.

Then the dynamical equations read

$$\forall \xi, \quad \omega(\xi, \cdots)|_\Gamma = 0 \quad \text{and} \quad \beta|_\Gamma \neq 0.$$

and Γ is then called a Hamiltonian n-curve.

Basic results in a pre-multisymplectic manifold $\mathcal{N}$, 1

Variational principle

Any n-dimensional submanifold Γ on which β does not vanish is a critical point of

$$\mathcal{A}[\Gamma] := \int_\Gamma \theta$$

iff it is a Hamiltonian n-curve.

Basic results in a pre-multisymplectic manifold $\mathcal{N}$, 2

Observable $(n-1)$-forms

An $(n-1)$-form F on $\mathcal{N}$ is observable if there exists a vector field ξ_F on $\mathcal{N}$ s.t.

$$\mathrm{d}F + \omega(\xi_F, \cdots) = 0.$$

Brackets

Given two observable $(n-1)$-forms F and G their bracket is the observable $(n-1)$-form defined by

$$\{F, G\} := \omega(\xi_F, \xi_G, \cdots) = \mathrm{d}G(\xi_F, \cdots) = -\mathrm{d}F(\xi_G, \cdots).$$

Then $\{F, G\} + \{G, F\} = 0$ and

$$\{\{F, G\}, H\} + \{\{H, F\}, G\} + \{\{G, H\}, F\} = \mathrm{d}(\omega(\xi_F, \xi_G, \xi_G, \cdots)).$$

Basic results in a pre-multisymplectic manifold $\mathcal{N}$, 3

Observable functionals

Given a (codimension 1) hypersurface Σ in $\mathcal{N}$ and an observable $(n-1)$-form F, one defines the functional on the set of n-dimensional submanifolds Γ

$$\begin{array}{rccc} \displaystyle\int_\Sigma F : & \{\Gamma\} & \longrightarrow & \mathbb{R} \\ & \Gamma & \longmapsto & \displaystyle\int_{\Sigma\cap\Gamma} F. \end{array}$$

If Γ is a Hamiltonian n-curve, then this functional depends only on the homology class of Σ. We also define the Poisson bracket

$$\left\{\int_\Sigma F, \int_\Sigma G\right\} := \int_\Sigma \{F, G\}.$$

This bracket coincides with the standard bracket.

The covariant phase

Main principle:

The set of solutions $\mathcal{E}$ of a Euler–Lagrange is endowed with a canonical pre-symplectic structure

In Mechanics:

The Cauchy data at two different times provides two charts on $\mathcal{E}$ which carries the same symplectic structure, because the flow is Hamiltonian (Lagrange, Hamilton, Jacobi, Poincaré, Cartan, Dedecker, Souriau,...).

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- general, synthetic and geometric presentation by J. Kijiowski–W. Szczyrba in 1976, by using the multisymplectic formalism;
- rediscovered by C. Crnkovic–E. Witten in and G. Zuckerman in 1987, being inspired by the variational complex of F. Takens (1979), A. Vinogradov (1984);

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The covariant phase through the multisymplectic formalism

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- multisymplectic approach...

A short presentation using multisymplectic formalism

Recall that $\mathcal{E}$ is the set of Hamiltonian n-curves Γ for some $\mathcal{H}$.
Assume that $\omega = \mathrm{d}\theta$, for some n-form θ. For any $\Gamma \in \mathcal{E}$,

$$\text{an infinitesimal deformation of } \Gamma \quad \longleftrightarrow \quad \text{a vector } \delta\Gamma \in T_\Gamma\mathcal{E}$$
$$\longleftrightarrow \quad \text{a Jacobi vector field } \xi \text{ along } \Gamma$$

where a Jacobi vector field ξ is a tangent vector field along Γ which is a solution of:

$$\forall \zeta, \quad (\mathrm{L}_\xi \omega)(\zeta, \cdots)|_\Gamma = 0.$$

We then write

$$\delta\Gamma = \int_\Gamma \xi \quad \text{and} \quad \Theta^\Sigma_\Gamma(\delta\Gamma) := \int_{\Gamma\cap\Sigma} \theta(\xi, \cdots).$$

A short presentation using multisymplectic formalism

A general formula

Assume that Γ is an n-dimensional submanifold and that Σ_1 and Σ_2 are two 'slices' (hypersurfaces) in the same homology class. Let $\mathcal{D}_1^2$ be the portion of $\mathcal{N}$ between Σ_1 and Σ_2, i.e.

$$\partial\mathcal{D}_1^2 = \Sigma_2 - \Sigma_1.$$

Then, if $\delta\Gamma = \int_\Gamma \xi$,

$$\delta(S_{\Sigma_1}^{\Sigma_2})_\Gamma(\delta\Gamma) - \Theta_\Gamma^{\Sigma_2}(\delta\Gamma) + \Theta_\Gamma^{\Sigma_1}(\delta\Gamma) = \int_{\Gamma\cap\mathcal{D}_1^2} \omega(\xi, \cdots)$$

where

$$S_{\Sigma_1}^{\Sigma_2}(\Gamma) := \int_{\Gamma\cap\mathcal{D}_1^2} \theta = \text{action between } \Sigma_1 \text{ and } \Sigma_2.$$

A general formula: picture

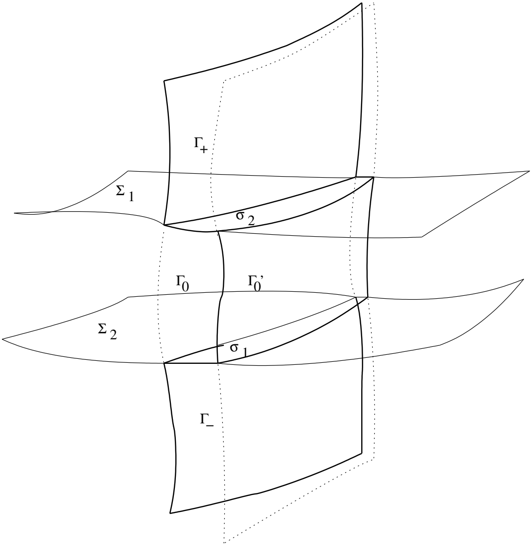

Figure:

First consequence of the general formula

The right hand term

$$\int_{\Gamma\cap\mathcal{D}_1^2}\omega(\xi,\cdots)$$

vanishes iff Γ is a Hamiltonian n-curve. If so

$$\Theta^{\Sigma_2}-\Theta^{\Sigma_1}=\delta(S_{\Sigma_1}^{\Sigma_2})$$

is exact on $\mathcal{E}$ and so its differential $\Omega := \delta\Theta^{\Sigma}$ does not depend on Σ and is hence a 2-form on $\mathcal{E}$: the canonical pre-symplectic form.

Second consequence of the general formula

Conversely if the left hand side

$$\delta(S_{\Sigma_1}^{\Sigma_2})_\Gamma(\delta\Gamma) - \Theta_\Gamma^{\Sigma_2}(\delta\Gamma) + \Theta_\Gamma^{\Sigma_1}(\delta\Gamma)$$

vanishes for all Σ_2 and Σ_1 (and for 'all' ξ's), then Γ is a solution of the Hamilton equations: this is a generalization (see De Donder) of the Cartan principle of dynamics by using the invariant integrals of Poincaré and Cartan.

To summarize

There are three main formulations of the classical dynamics of fields and dynamics:

- the Lagrangian one;
- the generalized Hamilton equations (multisymplectic version);
- the Poincaré–Cartan integral invariants.

The multisymplectic formalism provides an easy way to compare these formulations.

The secondary calculus of A. Vinogradov is an alternative way, tehcnically more complex, but more precise.

Multisymplectic formulation of Yang–Mills theory (F.H., 2015)

$$\begin{array}{ccc} \mathcal{X}, \quad x = (x^\mu) & \longrightarrow & \mathcal{P} \simeq \mathcal{X} \times \mathfrak{G}, \quad (x, g) \\ \mathbf{A} & \longrightarrow & \theta = g^{-1}\mathrm{d}g + g^{-1}\mathbf{A}g \\ \mathbf{F} = \mathrm{d}\mathbf{A} + \frac{1}{2}[\mathbf{A} \wedge \mathbf{A}] & \longrightarrow & \Theta = \mathrm{d}\theta + \frac{1}{2}[\theta \wedge \theta] \\ \int_{\mathcal{X}} \|\mathbf{F}\|^2 & = & \frac{1}{|\mathfrak{G}|} \int_{\mathcal{P}} \|\Theta\|^2 \end{array}$$

Equivariance & normalization of θ: $z \cdot \xi \lrcorner \theta = \xi$ and $z \cdot \xi \lrcorner \Theta = 0$

Legendre transform **by assuming** $z \cdot \xi \lrcorner \theta = \xi$ and $z \cdot \xi \lrcorner \Theta = 0$:
$\longrightarrow (\vartheta, p, x, y) \in (\mathfrak{g} \otimes T^*\mathcal{P}) \oplus_{\mathcal{P}} (\mathfrak{g}^* \otimes \Lambda^{N-2} T^*\mathcal{P})$, $\mathrm{d}x^{(n)} \wedge \vartheta^{(r)} \neq 0$

$$\omega = \mathrm{d}\left(\frac{1}{2} \|p_i{}^{\mu\nu}\|^2 \mathrm{d}x^{(n)} \wedge \vartheta^{(r)} + p_i \wedge \left(\mathrm{d}\vartheta + \frac{1}{2}[\vartheta \wedge \vartheta] \right)^i \right)$$

Application to gauge theories, II

Hamiltonian 4-curves are critical points of the action:

$$\mathcal{A}[\theta^i, \pi_i] := \int_{\mathcal{F}} \frac{1}{4}|\pi_i{}^{\lambda\mu}|^2 \beta^{(n)} \wedge \theta^{(r)} + \pi_i \wedge (\mathrm{d}\theta + \frac{1}{2}[\theta \wedge \theta])^i$$

Then critical points satisfy

$$\begin{cases} (\mathrm{d}\theta + \frac{1}{2}[\theta \wedge \theta])^i{}_{\lambda\mu} & = & -\mathrm{k}^{ij}\eta_{\lambda\nu}\eta_{\mu\sigma}\pi_j{}^{\nu\sigma} + 0 + 0 \\ (d\pi + \mathrm{ad}^*_\theta \wedge \pi)_i & = & \frac{1}{2}|\pi_i{}^{\lambda\mu}|^2 \beta^{(n)} \wedge \theta_i^{(r-1)} \end{cases}$$

$\longrightarrow$ **local principal** bundle structure **and** θ^i defines a **Yang–Mills** connection on it

Application to gauge theories, III

Gravity: a similar theory (F.H.–D. Vey, 2016)
$\mathcal{Y}$ is an oriented 10-dimensional manifold and $\mathfrak{p} := \mathbb{R}^4 \oplus so(1,3)$ is the Poincaré Lie algebra
Fields are pairs
$(\varphi^A, \pi_A) = (\varphi^\lambda, \varphi^i, \pi_\lambda, \pi_i) \in (\mathfrak{p} \otimes \Omega^1(\mathcal{Y})) \oplus (\mathfrak{p}^* \otimes \Omega^8(\mathcal{Y}))$, with the **constraints** $\varphi^\lambda \wedge \varphi^\mu \wedge \pi_i = 0$ and $\varphi^\lambda \wedge \varphi^\mu \wedge \pi_i = \kappa_i{}^{\lambda\mu} \varphi^{(10)}$
The action is

$$\mathcal{A}[\theta^A, \pi_A] := \int_{\mathcal{F}} \pi_A \wedge (\mathrm{d}\theta + \frac{1}{2}[\theta \wedge \theta])^A$$

Then critical points endow $\mathcal{Y}$ with a local principal bundle structure over a 4-dimensional manifold $\mathcal{X}$ and φ^A corresponds to a vierbein of $\mathcal{X}$ plus a connection 1-form which are solutions of the Einstein–Cartan system of equations.
One can then incorporate fermions ($\longrightarrow$ Einstein–Cartan–Dirac system of equations) : work by Jérémie **Pierard de Maujouy**

Application to gauge theories, III

Construction of an improved Kaluza–Klein theory (a gift from multisymplectic geometry):

- Let $\mathfrak{G}$ be a compact, unimodular Lie group, of dimension r (examples: $U(1)$, $SU(2)$, $SU(2) \times SU(3)$), with Lie algebra $\mathfrak{g}$

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- Let $\mathfrak{G}$ be a compact, unimodular Lie group, of dimension r (examples: $U(1)$, $SU(2)$, $SU(2) \times SU(3)$), with Lie algebra $\mathfrak{g}$
- Consider a connected, oriented manifold $\mathcal{F}$ of dimension

$$\mathrm{dim}\mathcal{F} = 4 + r = 4 + \mathrm{dim}\mathfrak{G} = N,$$

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- with the constraint

$$\theta^\mu \wedge \theta^\nu \wedge \pi = 0, \qquad \text{for } 1 \leq \mu, \nu \leq 4$$

The action functional

- It is the sum of $\int_{\mathcal{F}} \pi_A \wedge \left(\mathrm{d}\theta + \frac{1}{2}[\theta\wedge\theta]_{\mathfrak{g}}\right)^A$ and of the higher dimensional **Palatini action for** (θ,φ) :

$$\begin{aligned}\mathscr{A}[\theta,\varphi,\pi] \;=\; & \int_{\mathcal{F}} \pi_A \wedge \left(\mathrm{d}\theta + \frac{1}{2}[\theta\wedge\theta]_{\mathbb{R}^4\oplus\mathfrak{g}}\right)^A \\ & + \theta^{(N-2)}_{AB} \wedge \left(\mathrm{d}\varphi + \frac{1}{2}[\varphi\wedge\varphi]_{so(\mathbb{R}^4\oplus\mathfrak{g})}\right)^{AB}\end{aligned}$$

with the constraint $\theta^{\mu}\wedge\theta^{\nu}\wedge\pi = 0$, for $1 \leq \mu,\nu \leq 4$

Application to gauge theories, IV

The action functional

- It is the sum of $\int_{\mathcal{F}} \pi_A \wedge \left(\mathrm{d}\theta + \frac{1}{2}[\theta\wedge\theta]_{\mathfrak{g}}\right)^A$ and of the higher dimensional **Palatini action for** (θ, φ) :

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- Let $\widehat{\mathfrak{G}}$ be the universal cover of $\mathfrak{G}$.
- **Theorem** (H 2020-22-23) — Let (θ,φ,π) be a critical point of $\mathscr{A}$ such that $\mathrm{rank}\,\theta = N+1$. Assume that $\widehat{\mathfrak{G}}$ is simply connected and **compact** (e.g. $SU(2), SU(3)$). Then (θ,φ,π) endows $\mathcal{F}$ with a structure of principal bundle over a 4-dimensional manifold $\mathcal{X}$ and gives rise to a solution to the Einstein–Yang–Mills system on $\mathcal{X}$.

Perspectives

Some direction to develop:

- the search for expressions for first integrals in terms of (convergent) series (F.H.– R.D. Harrivel, arXiv:0704.2674; F.H., Trans. AMS 368 (2016));
- gauge theories
- examples of L_∞-algebras (C. Rogers 2011, Frégier-Rogers-Zambon 2013; Ryvkin-Wurzbacher 2014)

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