The Haar measure in solid mechanics

Réunion thématique du GDR-GDM
Paris (IRCAM), 20–22 Novembre 2024.

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How to integrate over a group ?

- Let G be a finite or compact Lie group : $G = \{\mathrm{SO}(D), \mathrm{D}_{2n}(D)\}$;
- Let $f : G \to \mathbb{R}^+$ be a function

$$\left(\int_{g\in G} f(g)\mathrm{d}g =\right) \int_{g\in G} f(g)\mu_G(g), \quad \int_{g\in G} \mu_G(g) = 1;$$

- μ_G is the *"uniform"* measure on G.

How to use this formula ? What is the *"expression"* of μ_G ?

Practical computations for simple cases

G finite: $G = \{g_i\}_{i=1}^N$

$$\int_{g\in G} f(g)\mu_G(g) = \sum_{i=1}^{N} f(g_i)k(g_i)$$

- Constant function $\implies f(g) = 1$
- $k(g_i) =$ cte
- $k(g_i) = \frac{1}{N}$

$G = \mathrm{SO}(2)$:

$$R\colon \alpha \in [0; 2\pi[\mapsto \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$\int_{g\in G} f(g)\mu_G(g) = \int_{\alpha=0}^{2\pi} f(R(\alpha))k(\alpha)\mathrm{d}\alpha$$

- Constant function $\implies f(g) = 1$
- $k(\alpha) =$ cte
- $k(\alpha) = \frac{1}{2\pi}$

What about the group of 3D rotations ? $G = \mathrm{SO}(3)$

Since several parametrisations are possible (Euler angles, quaternions, etc), let R be

$$\begin{array}{rccc} R: & U \subset \mathbb{R}^3 & \to & \mathrm{SO}(3) \\ & u = (u^1, u^2, u^3) & \mapsto & R(u) \in \mathrm{M}_3(\mathbb{R}) \end{array}$$

Therefore,

$$\int_{g \in G} f(g) \mu_G(g) = \int_{u \in U} f(R(u))(R^* \mu_G)(u) = \int_{u \in U} \bar{f}(u) k(u) \mathrm{d}u$$

Problems

- For which parametrisation R does k is constant ? (if any)
- Only $\bar{f} = f \circ R$ is known for a given R, how can we find the k associated to it ? (without inverting complicated formulas linking two parametrisations)
- What is μ_G and what does *"uniformity"* mean in that context ?

The *Haar measure* in solid mechanics

The measure μ_G is called the *Haar measure* of a compact Lie group G and is necessary as soon as an integration has to be on G.

- Well-established in mathematical literature [Car40] :
  - Can lack introductory documents and practical examples [Stu08] ;
  - Introductory documentation is available in other fields (e.g. quantum physics) [Mel23].

- but less familiar among mechanicians (frequently not mentioned) :
  - Sampling of random rotations [Bra18] ;
  - In homogenisation [XAD24] with averaging tensors over a group ;
  - Except in invariant theory [Tau22].

Summary

Presentation based on [EK24] (pre-print, under review).

Goal of the presentation

- Introduce the Haar measure of a Lie group and provide a formula to compute its expression in any parametrisation ;
- Apply this formula to the group of 3D rotations for different parametrisations ;
- Present a tool, the Reynolds projector used in invariant theory to compute invariants.

Group integration

"Uniform" integration over a group

Definition : G Lie Group (finite dimension)

G is a manifold with a differentiable group structure : $(h \in G)$

$$R_h \text{ or } L_h : \begin{array}{ccc} G & \to & G \\ g & \mapsto & gh \text{ or } hg \end{array} \quad \text{and} \quad \begin{array}{ccc} G & \to & G \\ g & \mapsto & g^{-1} \end{array} \quad \text{are differentiable.}$$

Definition : Lie algebra $\mathfrak{g}$ of a Lie group G

$\mathfrak{g}$ is a vector space given by

$$\mathfrak{g} = T_{e_G} G$$

We recall the Lie algebras

$$\mathfrak{so}(D) = \left\{ A \in \mathrm{M}_D(\mathbb{R}) \;\middle|\; A^\top = -A \right\}.$$

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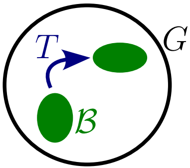
Intuitively, a uniform integration should not depend on the *"position"* but only on the *"size"* of the integrated set $\mathcal{B} \subset G$. A *"uniform"* integration measure μ_G is therefore invariant regarding any *"translation"* T



$$\int_{\mathcal{B}} \mu_G = \int_{\mathcal{B}} T_* \mu_G$$

- If $G = (\mathbb{R}^D, +)$: T are the usual translations and μ_G is the Lebesgue measure ;
- Generally, the *"translations"* are the L_h and R_h, usually different $(\forall h \in G)$.

Haar measure

Definition / Theorem [Car40] : Existence and Uniqueness of the *Haar measure*

On a compact Lie group G, it exists a unique, bi-invariant probability measure μ_G called the *Haar measure*. For any $h \in G$,

$$\forall \mathcal{B} \subset G, \quad \int_{\mathcal{B}} \mu_G = \int_{\mathcal{B}} (L_{h*}\mu_G) = \int_{\mathcal{B}} (R_{h*}\mu_G), \quad \int_G \mu_G = 1.$$

Expression of the *Haar measure* in a given parametrisation

Let G be a compact Lie group of dimension d and p a parametrisation

$$p: \begin{array}{rcl} U \subset \mathbb{R}^d & \to & G \\ u = (u^1, \ldots, u^d) & \mapsto & p(u) \end{array}$$

The integral of a dummy function $f : G \mapsto \mathbb{R}$ is given by

$$\int_{g \in G} f(g) \mu_G(g) = \int_{u \in U} f(p(u))(p^* \mu_G)(u) = \int_{u \in U} \bar{f}(u) k(u) \mathrm{d}u$$

Theorem [EK24] : local expression μ_G

Let (ξ_i) be a basis of the Lie algebra $\mathfrak{g}$ of G, and (θ^i) the components of the left-invariant Maurer-Cartan form on G. The local expression k is given by

$$u \in U, \quad k(u) = C^{-1} \det\left(\theta^i_{p(u)}\left(\frac{\partial p}{\partial u_j}\right)\right),$$

where C is a constant to be determined such that μ_G is a probability measure on G.

Expression of the *Haar measure* in a given parametrisation - Matrix group

Let G be a matrix group of dimension d and p a parametrisation

$$\begin{aligned} p: \quad U \subset \mathbb{R}^d &\to G \\ u = (u^1, \dots, u^d) &\mapsto p(u) \end{aligned}$$

The integral of a dummy function $f : G \mapsto \mathbb{R}$ is given by

$$\int_{g\in G} f(g)\mu_G(g) = \int_{u\in U} f(p(u))(p^*\mu_G)(u) = \int_{u\in U} \bar{f}(u)k(u)\mathrm{d}u$$

Theorem [EK24] : local expression μ_G for a matrix group

Let (ξ_i) be a basis of the Lie algebra $\mathfrak{g}$ of G. The local expression k is given by

$$k(u) = C^{-1} \det \left\langle p(u)^{-1} \frac{\partial p}{\partial u_j}, \xi_i \right\rangle,$$

where C is a constant to be determined such that μ_G is a probability measure on G.

Examples

Warm up - $G = \mathrm{SO}(2)$

$\mathrm{SO}(2)$ is a matrix group of dimension 1 and p the parametrisation

$$p: \begin{array}{ccc} [0;2\pi[\subset \mathbb{R} & \to & \mathrm{M}_2(\mathbb{R}) \\ \alpha & \mapsto & p(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \end{array}$$

The integral of a dummy function $f : \mathrm{SO}(2) \mapsto \mathbb{R}$ is given by

$$\int_{g\in \mathrm{SO}(2)} f(g)\mu_{\mathrm{SO}(2)}(g) = \int_{\alpha\in[0;2\pi[} f(p(u))(p^*\mu_{\mathrm{SO}(2)})(\alpha) = \int_{\alpha=0}^{2\pi} \bar{f}(\alpha)k(\alpha)\mathrm{d}\alpha$$

Warm up - $G = \mathrm{SO}(2)$

$\mathrm{SO}(2)$ is a matrix group of dimension 1 and p the parametrisation

$$p: \quad [0; 2\pi[\subset \mathbb{R} \to \mathrm{M}_2(\mathbb{R})$$
$$\alpha \mapsto p(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Application of the Theorem [EK24]

Choose $(\xi_1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ as a basis of the Lie algebra $\mathfrak{so}(2)$ of $\mathrm{SO}(2)$.

The local expression k is given by

$$k(\alpha) = C^{-1} \det \left\langle p(\alpha)^{-1} \frac{\partial p}{\partial \alpha}, \xi_1 \right\rangle,$$

where C is a constant to be determined such that $\mu_{\mathrm{SO}(2)}$ is a probability measure.

Warm up - $G = \mathrm{SO}(2)$

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Application of the Theorem [EK24]

The local expression k is given by

$$\begin{aligned} k(\alpha) &= \frac{C^{-1}}{2} \operatorname{Tr}\left(p(-\alpha) \frac{\partial p}{\partial \alpha} \xi_1 \right) \\ &= \frac{C^{-1}}{2} \operatorname{Tr}\left(\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} -\sin\alpha & -\cos\alpha \\ \cos\alpha & -\sin\alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \\ &= \frac{C^{-1}}{2} \operatorname{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = C^{-1} \end{aligned}$$

where C is a constant to be determined such that $\mu_{\mathrm{SO}(2)}$ is a probability measure.

Warm up - $G = \mathrm{SO}(2)$

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Application of the Theorem [EK24]

The local expression k is given by

$$k(\alpha) = \frac{1}{2\pi}$$

The integral of a dummy function $f : \mathrm{SO}(2) \mapsto \mathbb{R}$ is given by

$$\int_{g\in \mathrm{SO}(2)} f(g)\mu_{\mathrm{SO}(2)}(g) = \frac{1}{2\pi}\int_{\alpha=0}^{2\pi} \bar{f}(\alpha)\mathrm{d}\alpha$$

$G = \mathrm{SO}(3)$ - Euler angles

$\mathrm{SO}(3)$ is a matrix group of dimension 3 and p the parametrisation

$$\begin{array}{rrcl} p: & U = [-\pi;\pi] \times [0;\pi] \times [-\pi;\pi] & \to & \mathrm{M}_3(\mathbb{R}) \\ & u = (\alpha,\beta,\gamma) & \mapsto & R(u) \end{array}$$

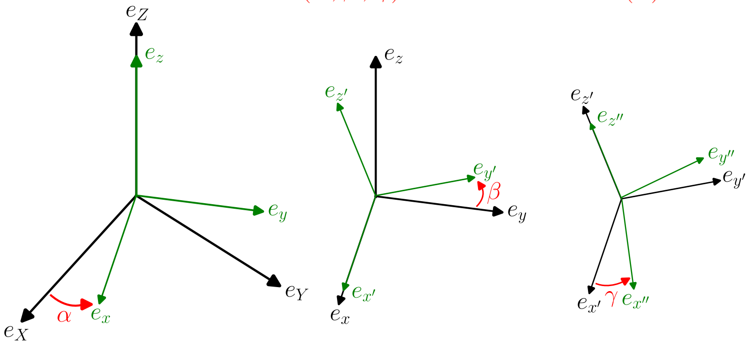

Figure: Representation of a rotation using Euler angles.

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$$R(u) = \begin{pmatrix} \cos\alpha\cos\gamma - \cos\beta\sin\alpha\sin\gamma & -\cos\alpha\sin\gamma - \cos\beta\cos\gamma\sin\alpha & \sin\alpha\sin\beta \\ \cos\gamma\sin\alpha + \cos\alpha\cos\beta\sin\gamma & \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\sin\beta \\ \sin\beta\sin\gamma & \cos\gamma\sin\beta & \cos\beta \end{pmatrix}$$

The integral of a dummy function $f : \mathrm{SO}(3) \mapsto \mathbb{R}$ is given by

$$\int_{g\in\mathrm{SO}(3)} f(g)\mu_{\mathrm{SO}(3)}(g) = \int_{\alpha=-\pi}^{\pi}\int_{\beta=0}^{\pi}\int_{\gamma=-\pi}^{\pi} \bar{f}(\alpha,\beta,\gamma)k(\alpha,\beta,\gamma)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma$$

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In 3D, there exists an accidental isomorphism j between $\mathbb{R}^3$ and $\mathfrak{so}(3)$ given by

$$j : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \to \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix} \in \mathfrak{so}(3),$$

$G = \mathrm{SO}(3)$ - Euler angles

$\mathrm{SO}(3)$ is a matrix group of dimension 3 and p the parametrisation

$$\begin{array}{rrcl} p: & U = [-\pi;\pi] \times [0;\pi] \times [-\pi;\pi] & \to & \mathrm{M}_3(\mathbb{R}) \\ & u = (\alpha,\beta,\gamma) & \mapsto & R(u) \end{array}$$

Application of the Theorem [EK24]

Choose $(\xi_i = j(\boldsymbol{e}_i))$ as a basis of the Lie algebra $\mathfrak{so}(3)$ of $\mathrm{SO}(3)$.
The local expression k is given by

$$k(\alpha,\beta,\gamma) = C^{-1} \det \left\langle R(\alpha,\beta,\gamma)^{-1} \frac{\partial R}{\partial u^j}, j(\boldsymbol{e}_i) \right\rangle,$$

where C is a constant to be determined such that $\mu_{\mathrm{SO}(3)}$ is a probability measure.

$G = \mathrm{SO}(3)$ - Euler angles

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Application of the Theorem [EK24]

The local expression k is given by

$$k(\alpha,\beta,\gamma) = C^{-1} \det \langle \tau_i(\alpha,\beta,\gamma), j(\boldsymbol{e}_j) \rangle \,, \quad \text{with } \tau_i = R(\alpha,\beta,\gamma)^{-1} \frac{\partial R}{\partial u^i}(\alpha,\beta,\gamma) \in \mathfrak{so}(3).$$

Let T be the components matrix of the vectors $j^{-1}(\tau_i) \in \mathbb{R}^3$ in the canonical basis of $\mathbb{R}^3$,

$$k(\alpha,\beta,\gamma) = C^{-1} \det T.$$

where C is a constant to be determined such that $\mu_{\mathrm{SO}(3)}$ is a probability measure.

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Application of the Theorem [EK24]

The local expression k is given by (using a symbolic computation software)

$$k(\alpha, \beta, \gamma) = \frac{1}{8\pi^2} \sin \beta$$

The integral of a dummy function $f : \mathrm{SO}(3) \mapsto \mathbb{R}$ is given by

$$\int_{g \in \mathrm{SO}(3)} f(g) \mu_{\mathrm{SO}(3)}(g) = \frac{1}{8\pi^2} \int_{\alpha=-\pi}^{\pi} \int_{\beta=0}^{\pi} \int_{\gamma=-\pi}^{\pi} \bar{f}(\alpha, \beta, \gamma) \sin \beta \, \mathrm{d}\alpha \, \mathrm{d}\beta \, \mathrm{d}\gamma$$

Usage in mechanics : Applied to the homogenisation of randomly oriented spheroidal voids in porous metals (Gurson model [XAD24]).

Group of unitary quaternions Q

Definition : Algebra of quaternions $\mathbb{H}$

$$\mathbb{H} = \left\{ q = w + xi + yj + zk \mid (w, x, y, z) \in \mathbb{R}^4 \right\}, \quad i^2 = j^2 = k^2 = ijk = -1.$$

It is isomorphic to the algebra $\mathrm{M}_2(\mathbb{C})$,

$$[q] = \begin{pmatrix} w + xi & -y - zi \\ y - zi & w - xi \end{pmatrix}$$

with

$$[\overline{q}] = [w - xi - yj - zk] = \overline{[q]}^t, \quad [q]^{-1} = \frac{[\overline{q}]}{\det[q]}$$

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Definition : Group of unitary quaternions Q

$$||q|| = w^2 + x^2 + y^2 + z^2 = \det[q] = 1, \quad [q]^{-1}[q] = [\bar{q}q] = \overline{[q]}^t[q] = [1] = \mathbf{I}$$

Therefore,

$$Q \simeq S^3 \simeq \mathsf{SU}(2) = \left\{ [q] \in \mathrm{M}_2(\mathbb{C}) \;\middle|\; \overline{[q]}^t[q] = \mathbf{I},\ \det[q] = 1 \right\}$$

Group of unitary quaternions Q

Lie algebra morphisms

$$\mathfrak{q} = \mathrm{Im}(\mathbb{H}) = \{v = v_x i + v_y j + v_z k \mid \mathbf{v} \in \mathbb{R}^3\} \simeq \mathfrak{su}(2) = \{[v] \mid v \in \mathfrak{q}\}$$

Looking at the application (*adjoint representation of* $SU(2)$ *on* $\mathfrak{su}(2)$)

$$\begin{array}{rccl} \rho: & \mathrm{SU}(2) & \to & \mathrm{GL}(\mathfrak{su}(2)) \simeq \mathrm{GL}_3(\mathbb{R}) \\ & [q] & \mapsto & \left([v] \mapsto [q][v][q]^{-1} = [qv\bar{q}] = [\rho(q)v]\right) \end{array}$$

Group of unitary quaternions Q

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Properties

- $||\rho(q)v|| = \det([q][v][q]^{-1}) = \det([v]) = ||v||$ (isometry)
- $S^3 \simeq \mathrm{SU}(2)$ is connected and $v \mapsto [1][v][1]^{-1} = [v]$ $(\rho(1) = \mathbf{I})$

Group of unitary quaternions Q

Lie algebra morphisms

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We have a (non-injective) morphism between SU(2) and $\mathrm{SO}(3)$.

$G = SO(3)$ - Unitary Quaternions

A unitary quaternion $q = w + xi + yj + zk \in Q$ can be associated to a rotation $R(\boldsymbol{n}, \alpha) \in \mathsf{GL}_3(\mathbb{R})$ (morphism between $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$) :

- The rotation axis $\boldsymbol{n}$ is given by $\boldsymbol{n} = (x, y, z)$;
  - $\mathsf{Vect}(\boldsymbol{n}) = \ker(\rho(q) - \mathbb{I})$;
- The angle of rotation α is given by $\alpha = 2\arccos w = 2\arcsin\sqrt{x^2 + y^2 + z^2}$;
  - $1 + 2\cos\alpha = \mathsf{Tr}(R(\boldsymbol{n}, \alpha))$;
- The rotated vector of a vector $\mathbf{v} \in \mathbb{R}^3$ is given by the multiplications $qv\bar{q}$;
  - *adjoint representation of $SU(2)$ on $\mathfrak{su}(2)$* ;
  - $v = v_x i + v_y j + v_z k$ ($\mathsf{Im}(\mathbf{H}) \simeq \mathfrak{su}(2) \simeq \mathbb{R}^3$) ;

$G = SO(3)$ - Unitary Quaternions

Hyperpolar parametrisation → Application of Theorem [EK24]

$$\begin{array}{rccc} p: & U & \to & Q \subset \mathbb{H} \\ & (\theta,\psi,\phi) & \mapsto & q = w + xi + yj + zk \end{array}$$

$$(\theta,\psi,\phi) \in U = [0;\pi] \times [0;\pi] \times [0;2\pi], \quad \begin{cases} w & = \cos\theta \\ x & = \sin\theta\cos\psi \\ y & = \sin\theta\sin\psi\cos\phi \\ z & = \sin\theta\sin\psi\sin\phi \end{cases}$$

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$$(\theta, \psi, \phi) \in U = [0; \pi] \times [0; \pi] \times [0; 2\pi], \quad \begin{cases} w & = \cos\theta \\ x & = \sin\theta\cos\psi \\ y & = \sin\theta\sin\psi\cos\phi \\ z & = \sin\theta\sin\psi\sin\phi \end{cases}$$

Is there a $p\colon U \subset \mathbb{R}^3 \to \mathrm{SO}(3)$ such as $k(u) = 1$?

Usage in mechanics : For numerical computations (random sampling) with rotations [Bra18].

Overview : Integration over the group of 3D rotations

Problems solved ?

- What is μ_G and what does *"uniformity"* mean in that context ?
  - It is the unique, bi-invariant *Haar measure* over the group.
- Only $\bar{f} = f \circ R$ is known for a given R, how can we find the k associated to it ?
  - Applications of Theorem [EK24].
- For which parametrisation $R : U \to \mathrm{SO}(3)$ does k is constant ?
  - I don't know (probably not).

What are the applications of the *Haar measure* in solid mechanics ?

The Reynolds Projector

Averaged tensor

Let $\mathbb{T}$ be a tensor of order n on $\mathbb{R}^D$ and G a group acting on $\mathbb{R}^D$.
G acts on $\mathbb{T}$ through $\star$,

$$(g \star \mathbb{T})_{i_1 \ldots i_n} = g_{i_1 j_1} \ldots g_{i_n j_n} \mathbb{T}_{j_1 \ldots j_n}, \quad g \in G.$$

If $G = \{g_i\}_{i=1}^N$, an averaged tensor $\tilde{\mathbb{T}}$ can be computed,

$$\tilde{\mathbb{T}} = \frac{1}{N} \sum_{i=1}^{N} g_i \star \mathbb{T} = \sum_{i=1}^{N} (g_i \star \mathbb{T}) \mu_G(g_i).$$

Extension to a Lie group G using the *Haar measure*

$$\tilde{\mathbb{T}} = \int_{g \in G} (g \star \mathbb{T}) \mu_G(g).$$

The Reynolds projector

Let V a vector space on which G acts threw a linear representation $\rho_V : G \to \mathrm{GL}(V)$,

$$v \in V, \quad g \star v := \rho_V(g)v.$$

Definition : The *Reynolds projector/operator* R_G^V of G on V [Stu08]

$$\begin{array}{rccc} \mathrm{R}_G^V : & V & \to & V \\ & v & \mapsto & \displaystyle\int_{g\in G} (\rho_V(g)v)\mu_G(g) \end{array}$$

Properties

- It is a projector : linear and $(\mathrm{R}_G^V)^2 = \mathrm{R}_G^V$;
- It is invariant under the action of G : $\rho_V(g)\,\mathrm{R}_G^V(v) = \mathrm{R}_G^V(v)$.

Usage in mechanics : In homogenisation ([XAD24]) the Reynolds projector appears as an averaging operator.

Dimension of invariant elements of V

Let V a vector space on which G acts threw a linear representation $\rho_V : G \to \mathrm{GL}(V)$

$$V^G := \{v \in V \mid v \text{ is invariant under } G\} = \{v \in V \mid \rho_V(g)v = v, \forall g \in G\}.$$

Trace formula (since R_G^V is a projector)

$$\dim V^G = \dim \operatorname{Im}(\mathrm{R}_G^V) = \int_{g \in G} \mathrm{Tr}(\rho_V(g))\mu_G(g).$$

If $V = \otimes^n \mathbb{R}^D$, tensors of order n over $\mathbb{R}^D$ without indicial symmetries, and $\rho \colon G \to \mathrm{GL}_D(\mathbb{R})$ the canonical representation

$$\dim V^G = \int_{g \in G} \mathrm{Tr}(\rho(g))^n \mu_G(g).$$

Usage in mechanics : In invariant theory [And04] this dimension is used to compute the number of coefficients needed to represent higher order tensors for anisotropic generalised continua.

Overview : Usage of the *Haar measure* in solid mechanics

Key points

- The *Haar measure* is the basic ingredient for integrating over a Lie group ;
- It defines the uniform probability space $(G, \sigma(G), \mu_G)$ on the group ;
- It allows to formulate the *Reynolds projector* R_G^V of a compact Lie group on a vector space V :
  - It is an averaging operator : it is a projector on the space V^G of invariant elements ;
  - Used to determine the dimension of invariant spaces.

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