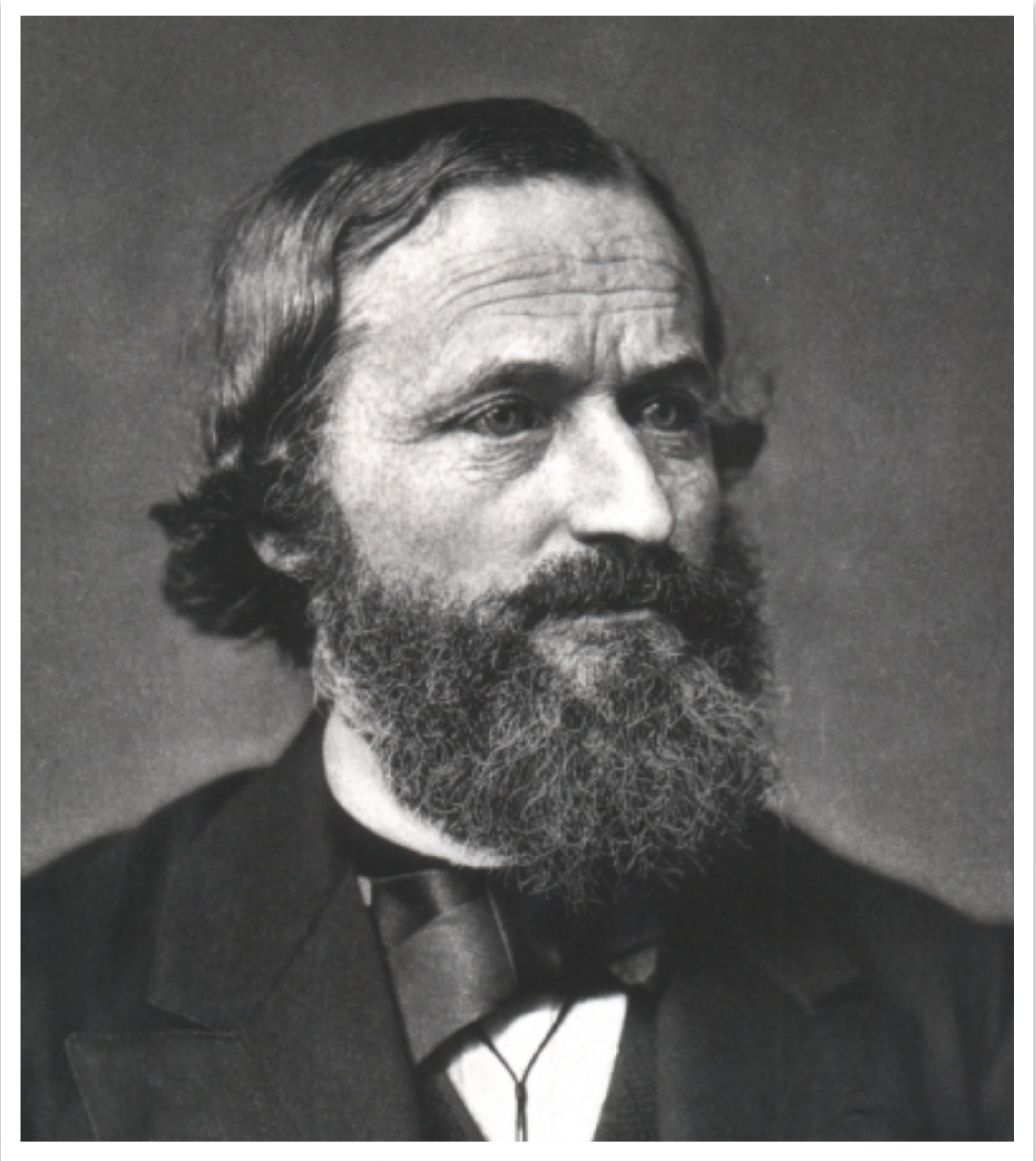
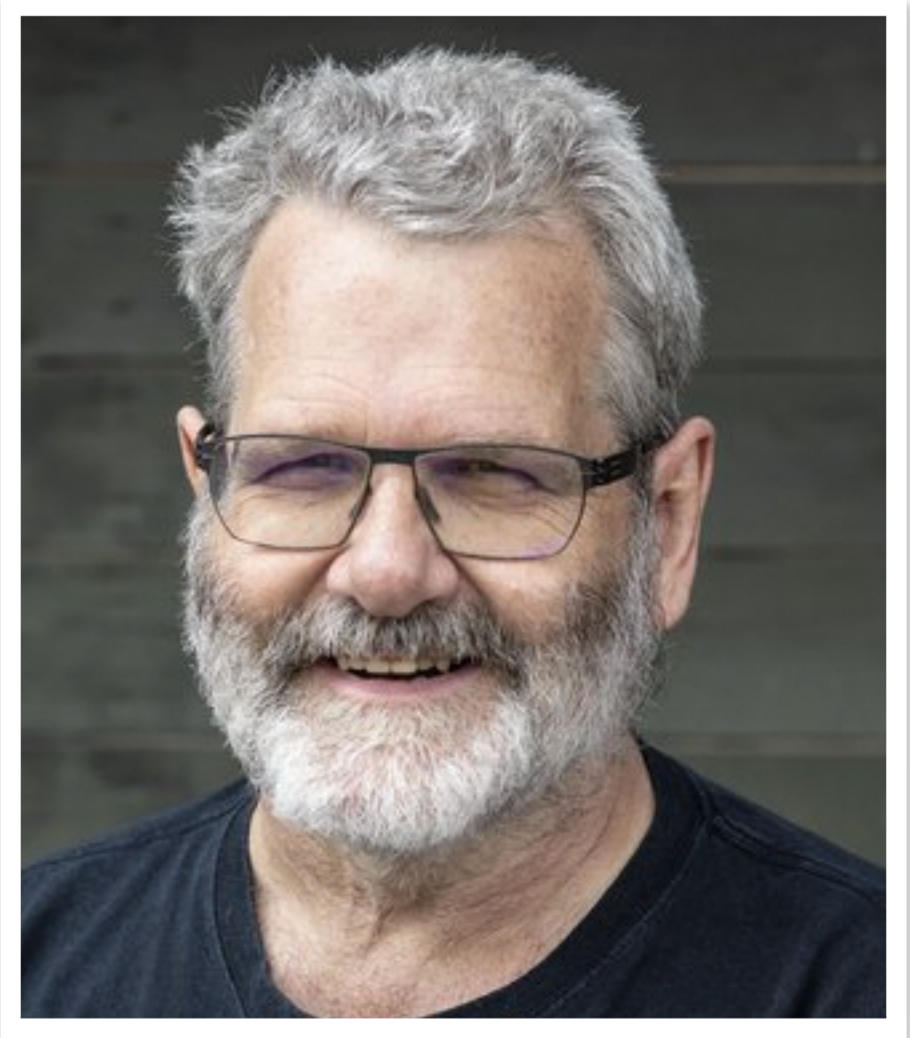
KIRCHHOFF, NOETHER, AND MADDOCKS

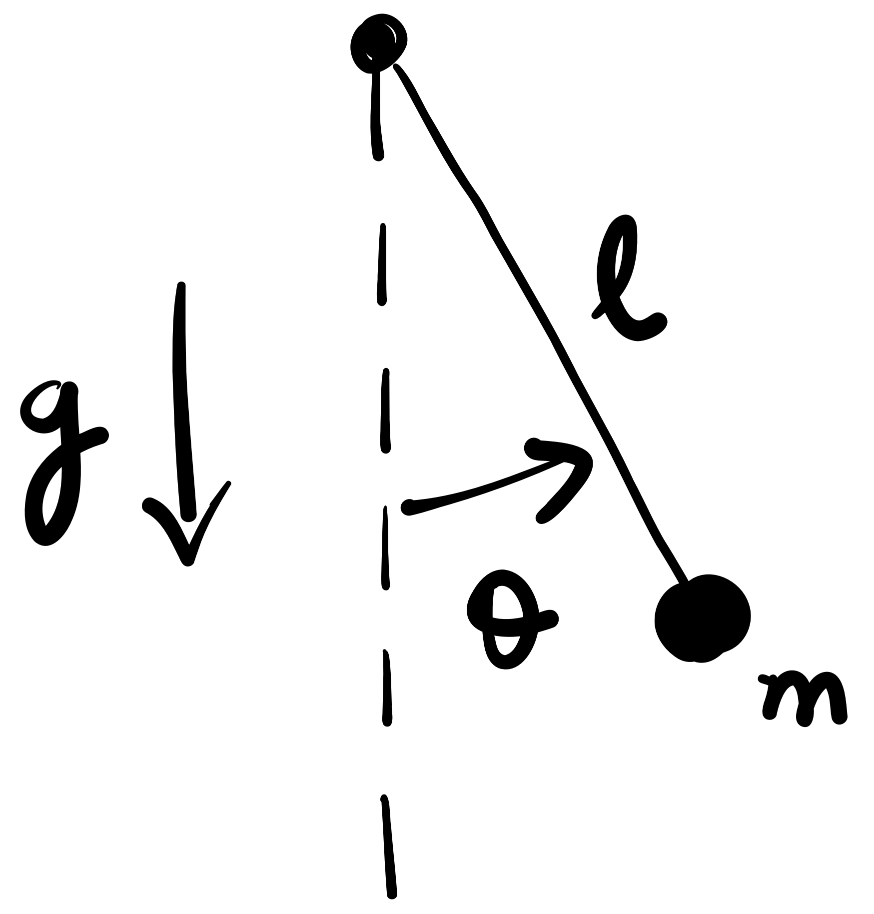
Florence Bertails-Descoubes (INRIA Grenoble, France)
Sébastien Neukirch (CNRS & Sorbonne University, Paris, France)





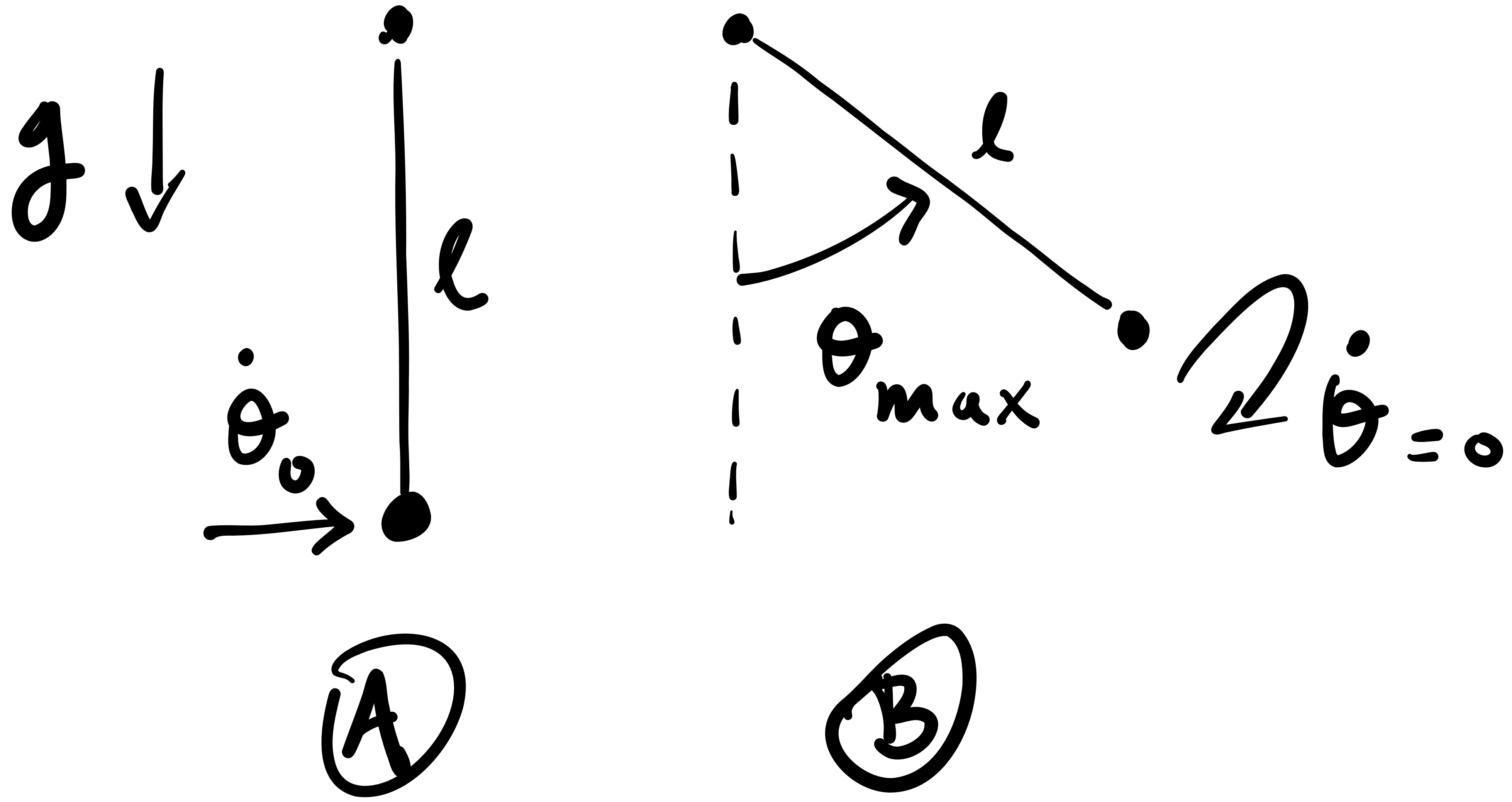
CONSERVATION OF MECHANICAL ENERGY

Dynamics of the nonlinear pendulum



$$\ddot{\theta}(t) = -\frac{g}{\ell}\sin\theta(t)$$

$\theta(t)$ using Jacobi Elliptic functions



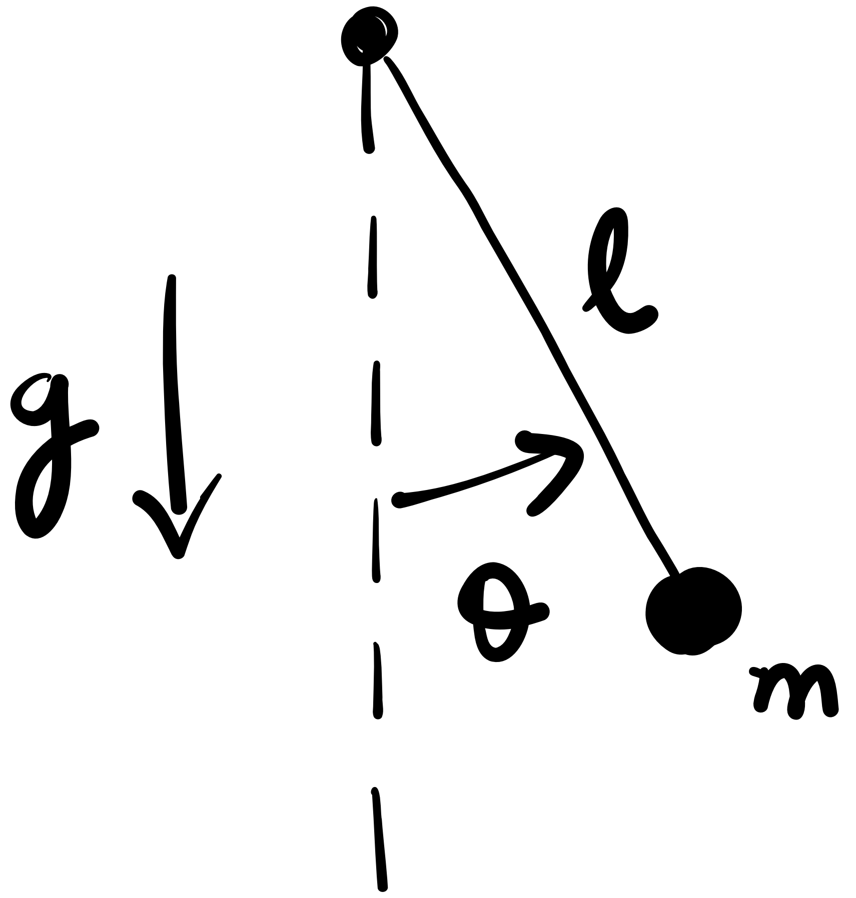
$$H = \frac{1}{2}\dot{\theta}^2 - \frac{g}{\ell}\cos\theta$$

(conserved) mechanical energy, i.e. $\dot{H} = 0$

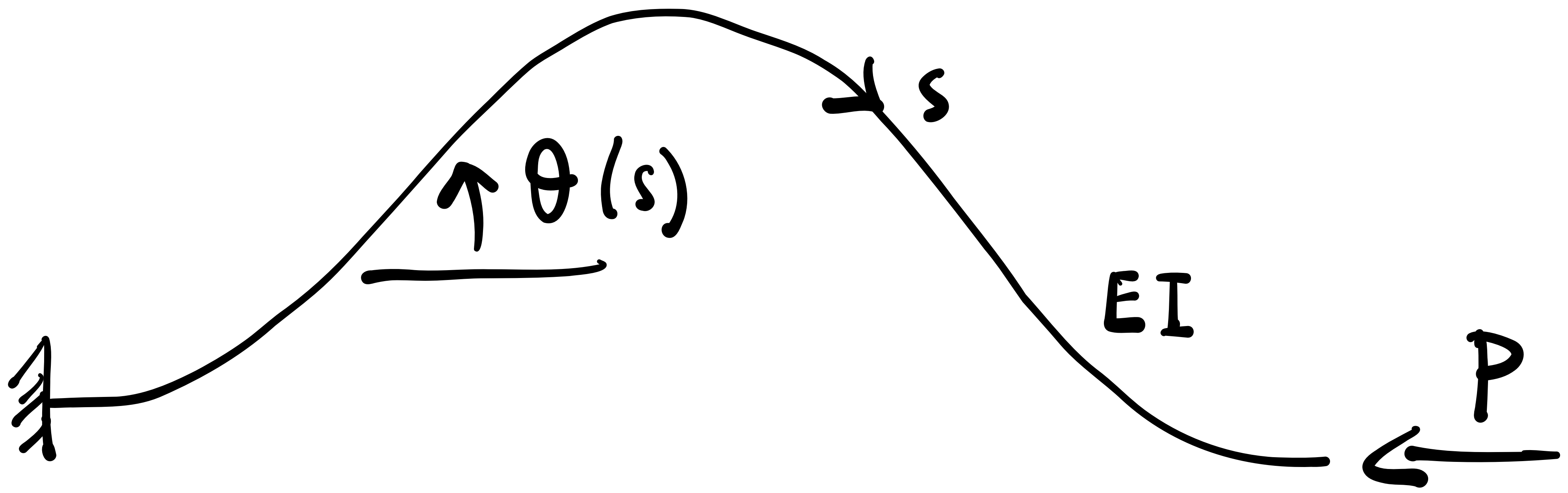
$$H_A = H_B \Rightarrow \cos\theta_{max} = 1 - \frac{\ell}{2g}\dot{\theta}_0^{\,2}$$

KIRCHHOFF ANALOGY

nonlinear pendulum



planar elastica



dynamics $$\ddot{\theta}(t) = -\frac{g}{\ell}\sin\theta(t)$$

$$\theta''(s) = -\frac{P}{EI}\sin\theta(s)$$ statics

mechanical energy $$H = \frac{1}{2}\dot{\theta}^2 - \frac{g}{\ell}\cos\theta$$

$$H = \frac{1}{2}EI\,\theta'^2 - P\cos\theta$$

Maddocks invariant: $H' = 0$

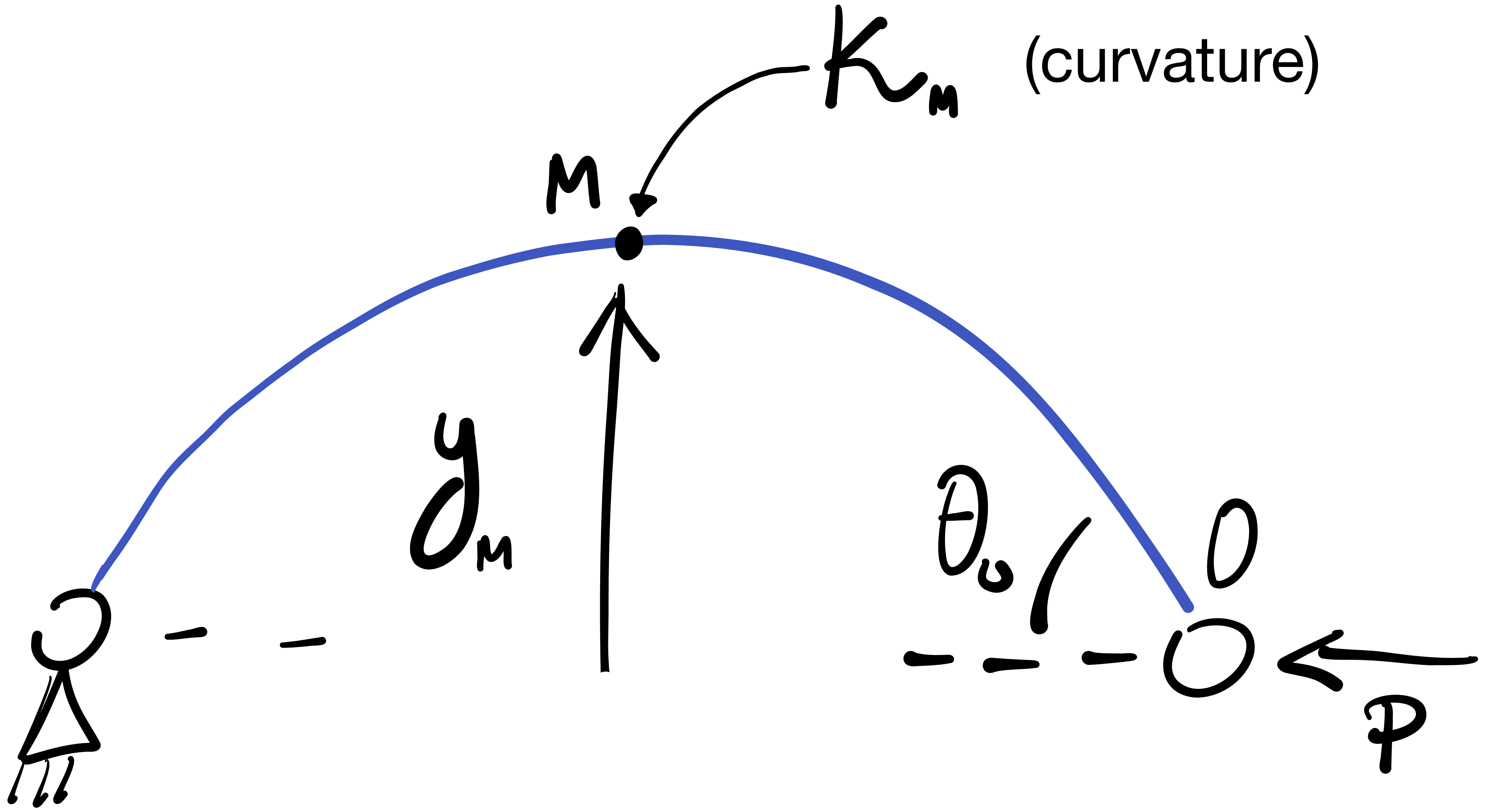
curvature energy

tension

A.E.H Love 1944 (first occurrence?)
Kehrbaum & Maddocks 1997
O'Reilly 2007
Jung 2011 (discrete version)
Singh 2021 (pseudo-momentum)

MAKING USE OF THE INVARIANT

pinned-pinned planar elastica



We impose P, we obtain the deflection θ_0

Q: values of y_M and κ_M ?

$$H_0 = -P\cos\theta_0$$

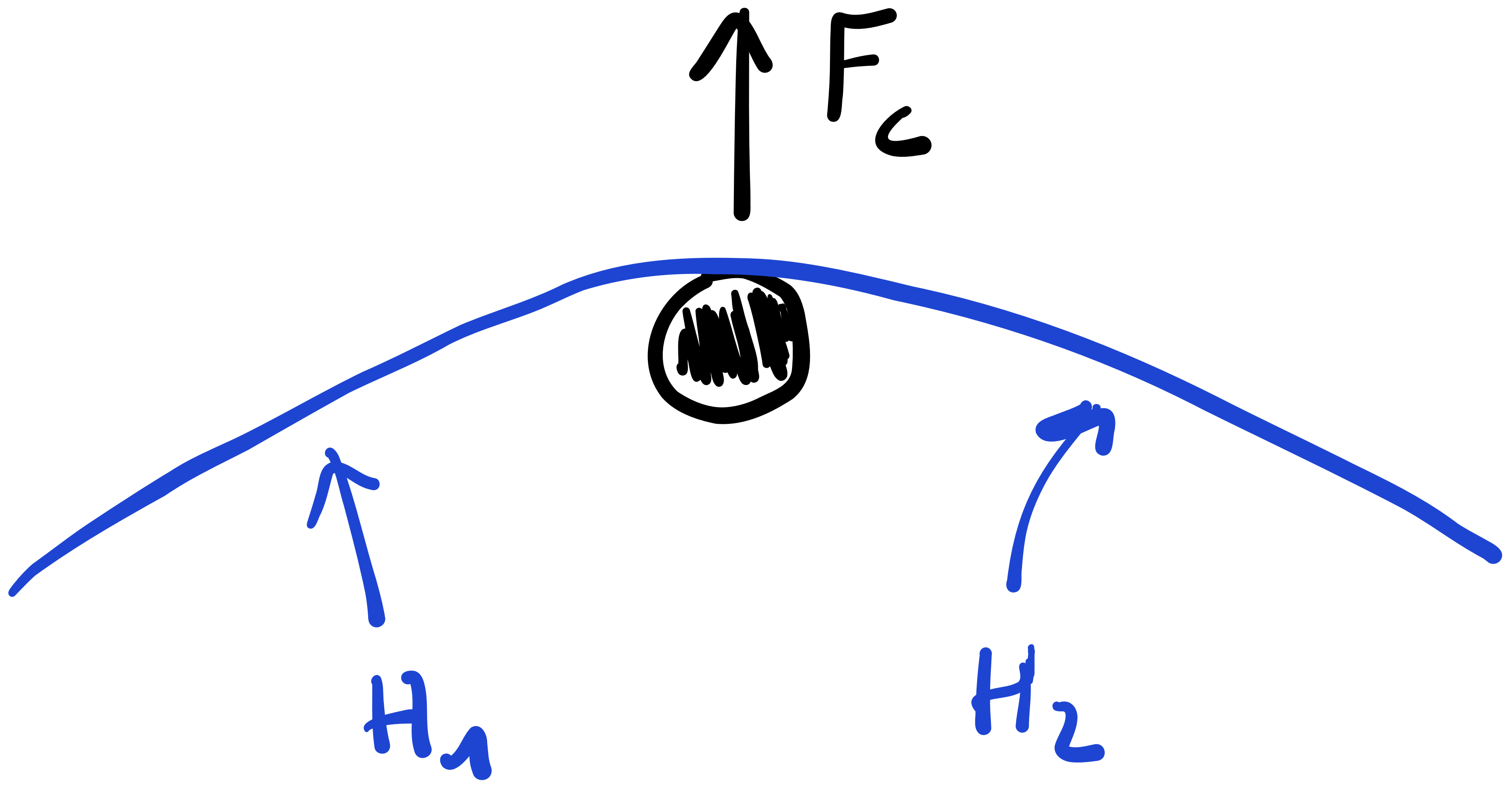
$$H_M = \frac{1}{2}EI\kappa_M^2 - P$$

$$H_0 = H_M \Rightarrow \begin{cases} \kappa_M = \sqrt{\frac{2P}{EI}(1-\cos\theta_0)} \\ y_M = \sqrt{\frac{2EI}{P}(1-\cos\theta_0)} \end{cases}$$

=> could be used for validation of numerics

HOW UNIVERSAL IS IT?

frictionless contact



curvature energy

$$H = \frac{1}{2} EI\,\theta'^2 + \text{tension}$$

- the curvature does not jump
- the tension does not jump

$$\Downarrow$$

$$H_1 = H_2$$

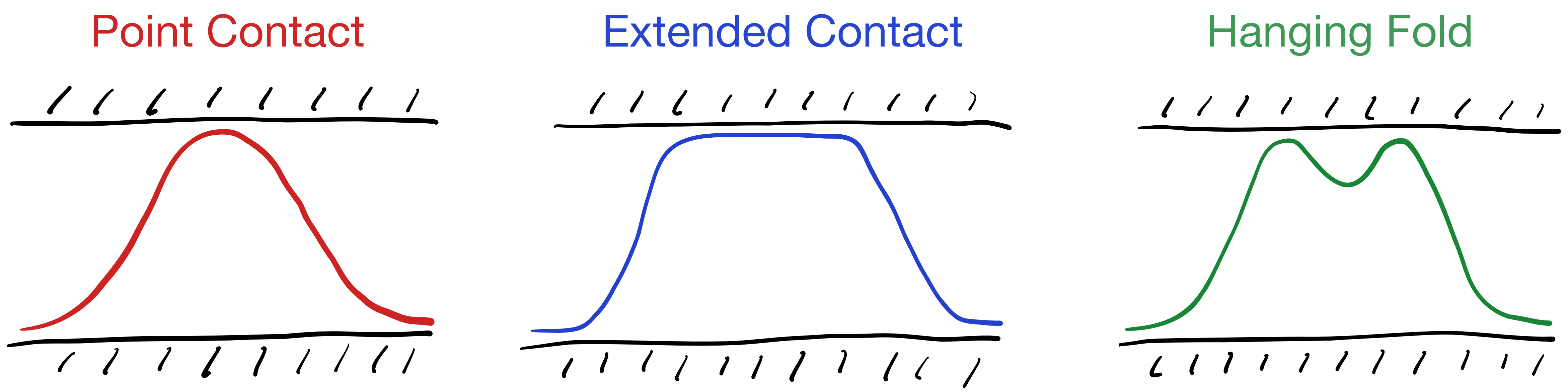
=> works in presence of **contact**

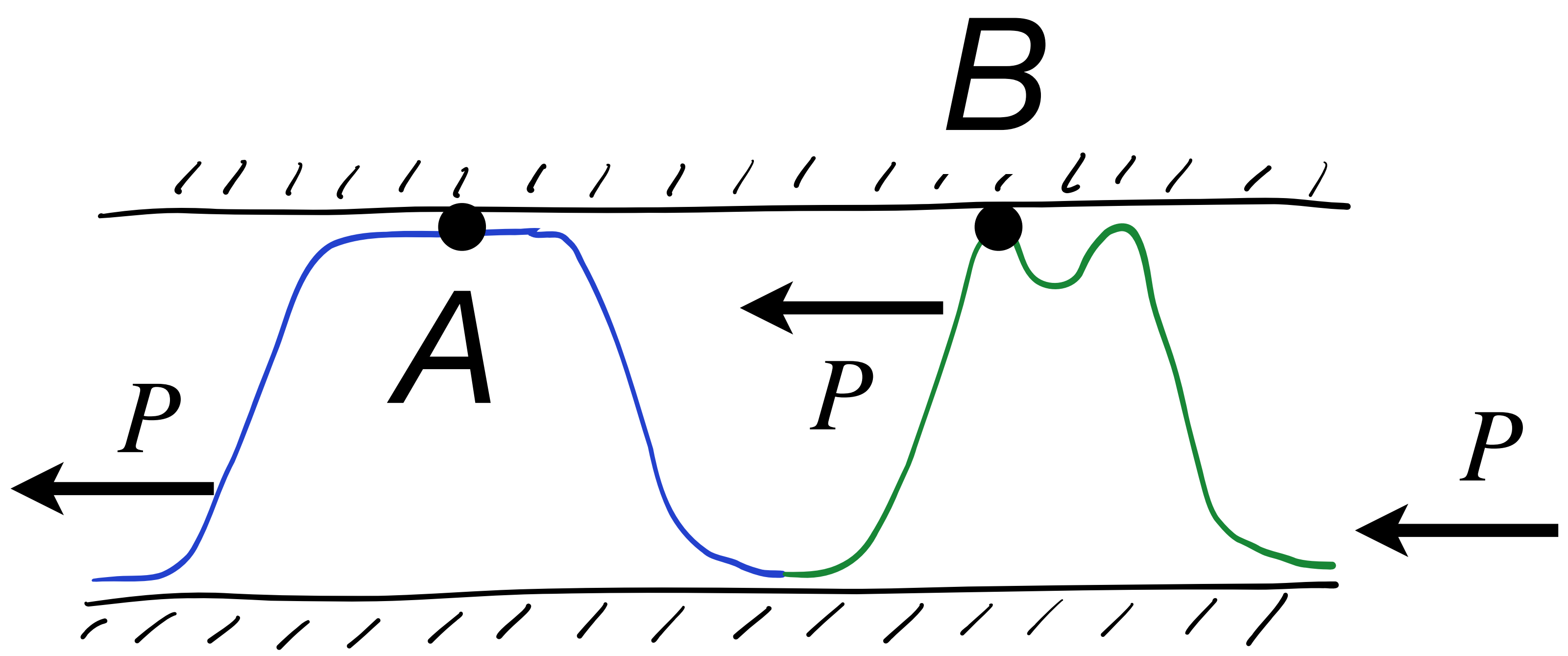
see also
Clauvelin 2009 (knot)
O'Reilly 2017 (jump cond.)
Singh 2022 (capstan)

MAKING USE OF THE INVARIANT (CONTACT)

Euler buckling in cavity

Lubinski 1962, Domokos 1996, Roman 1999, Deboeuf 2024, etc.



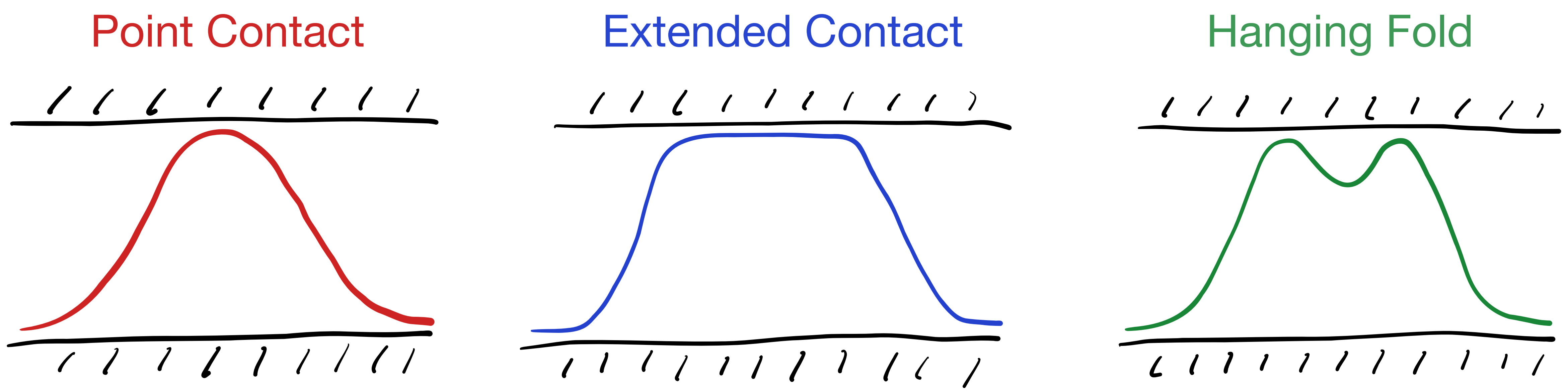


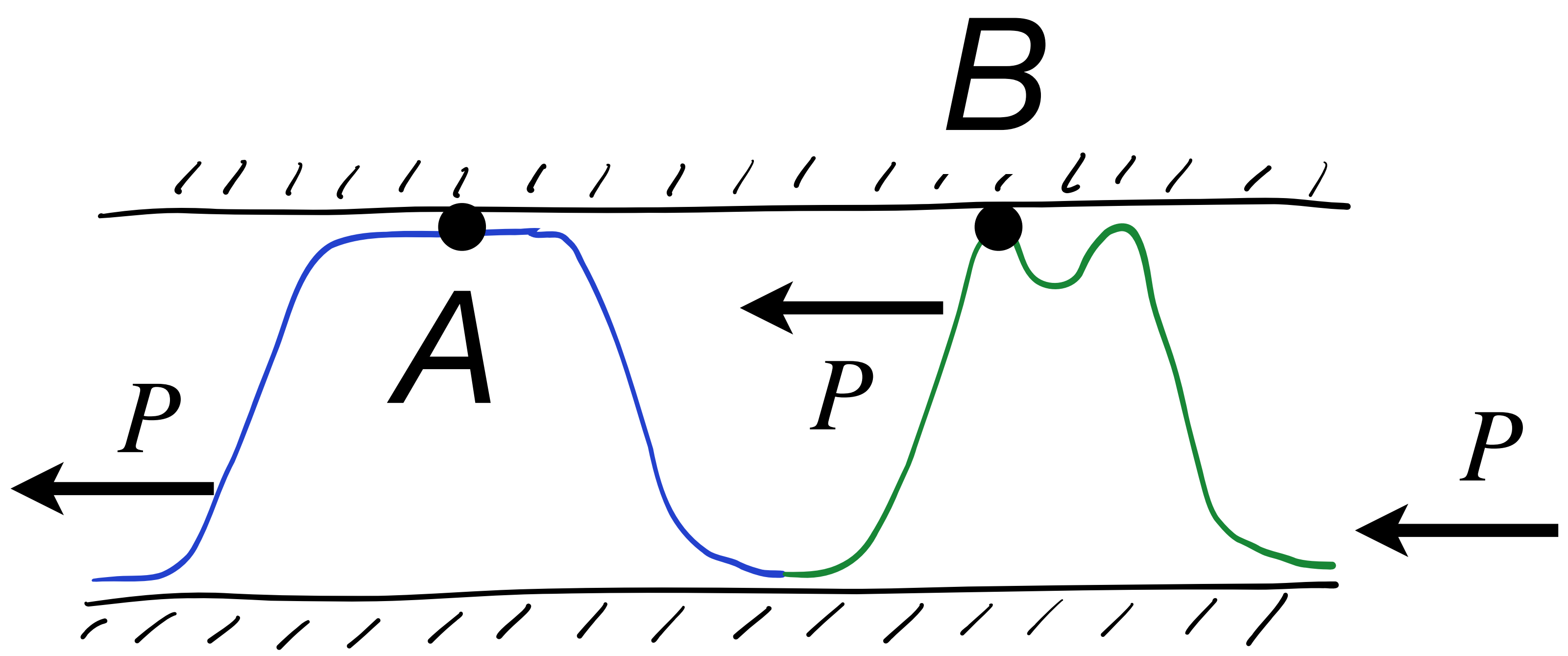
Question: coexistence?

MAKING USE OF THE INVARIANT (CONTACT)

Euler buckling in cavity

Lubinski 1962, Domokos 1996, Roman 1999, Deboeuf 2024, etc.





Question: coexistence?

Answer:

no friction $\Rightarrow P$ uniform

$$H_A = 0 - P$$

$$H_B = \frac{1}{2} EI\,\kappa_B^2 - P \text{ with } \kappa_B > 0$$

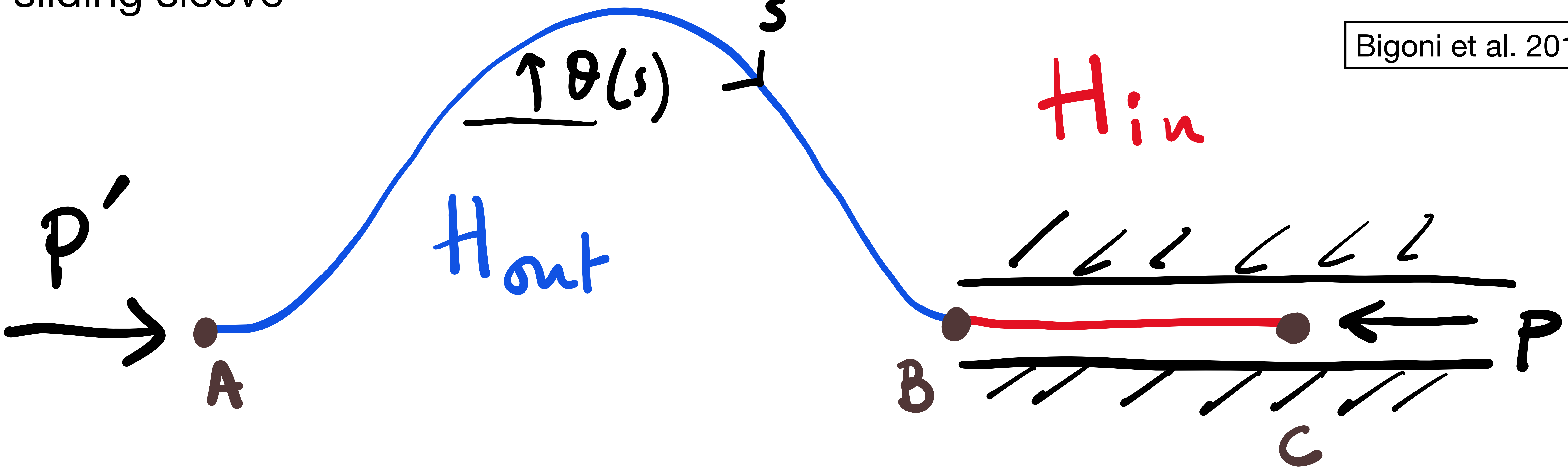
$\Rightarrow H_A = H_B$ impossible

=> coexistence is imposssible

MAKING USE OF THE INVARIANT (3)

rigid sliding sleeve

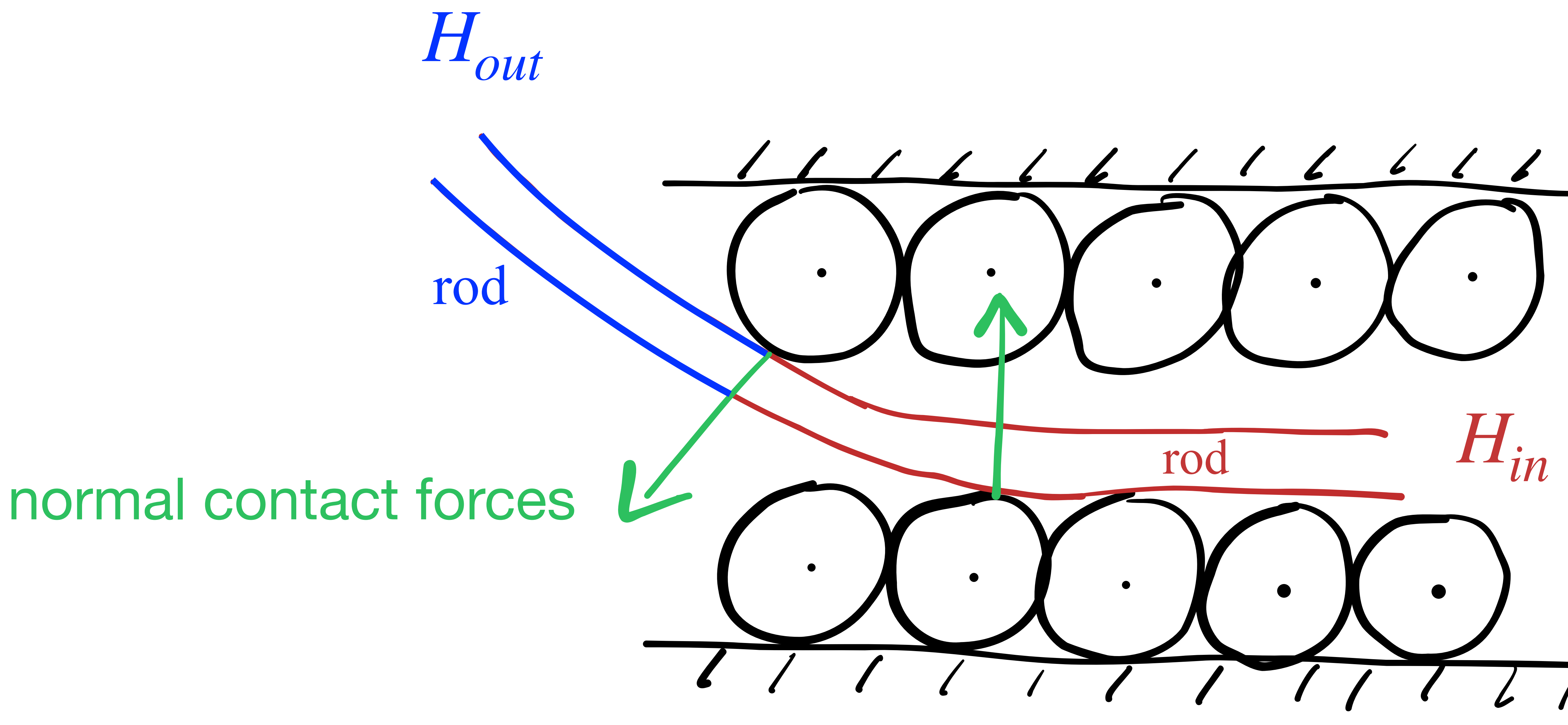
Bigoni et al. 2015



$$\left.\begin{array}{l} H_{out} = H_A = H_B = \dfrac{1}{2} EI\,\kappa_B^2 - P' \\[2ex] H_{in} = H_C = 0 - P \end{array}\right\} \quad H_{out} = H_{in} \Rightarrow P' = P + \frac{1}{2} EI\,\kappa_B^2$$

CONSERVATION OF THE INVARIANT AT THE EXIT OF A SLIDING SLEEVE

Bigoni et al. 2015



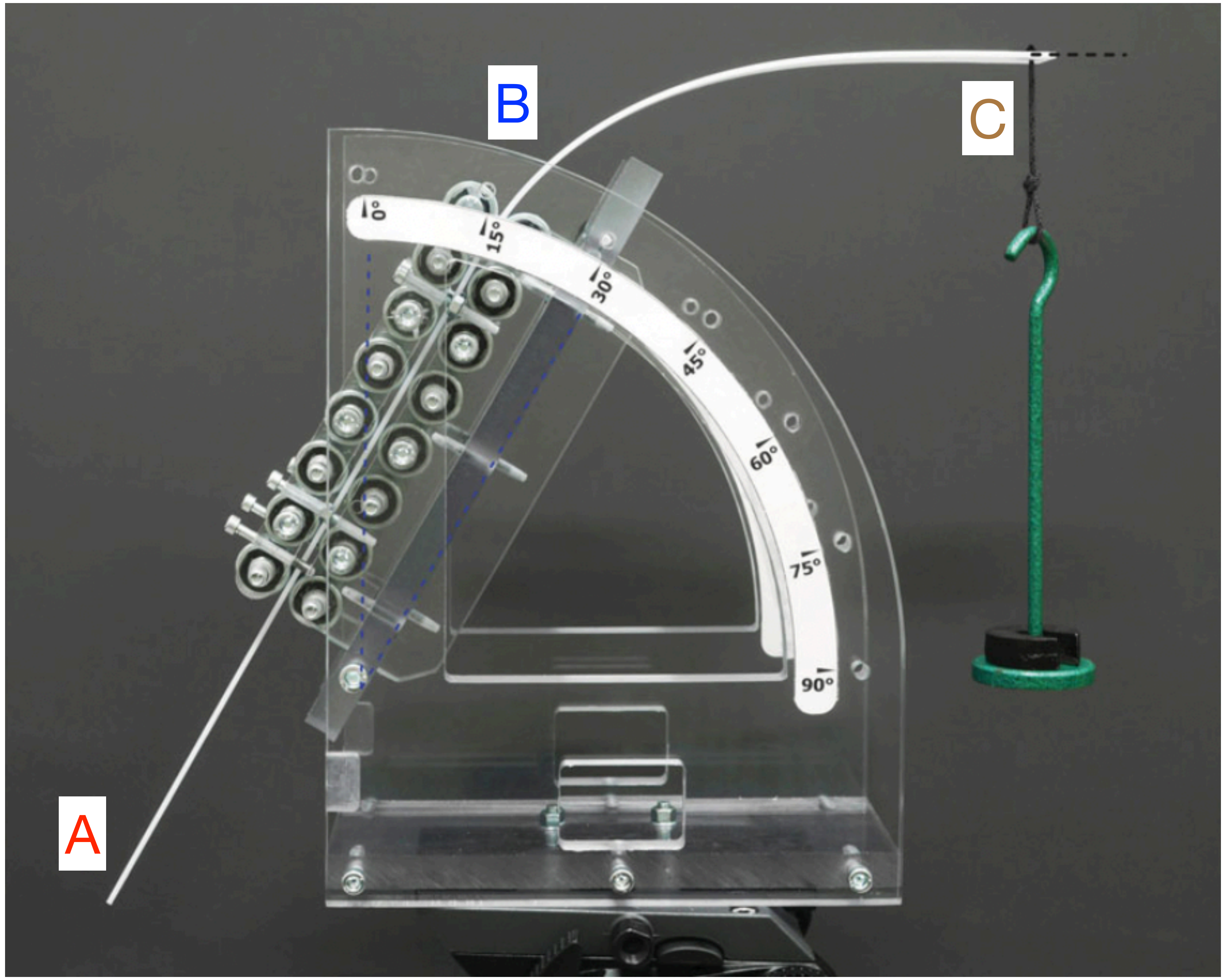
roller bearings

only normal contact forces $\Rightarrow H_{out} = H_{in}$

MAKING USE OF THE INVARIANT (4)

The Elastica Arm Scale

Bosi et al. 2014



$$H_A = 0 + 0$$

$$H_B = \frac{1}{2} EI\,\kappa^2 + tension = 0$$

=> the beam is under compression at point B

$$H_C = \frac{1}{2} EI\,\kappa_C^2 - Mg\,\mathbf{e_y} \cdot tangente_C = 0$$

bound. cond. => $\kappa_C = 0$

=> $tangente_C \perp \mathbf{e_y}$

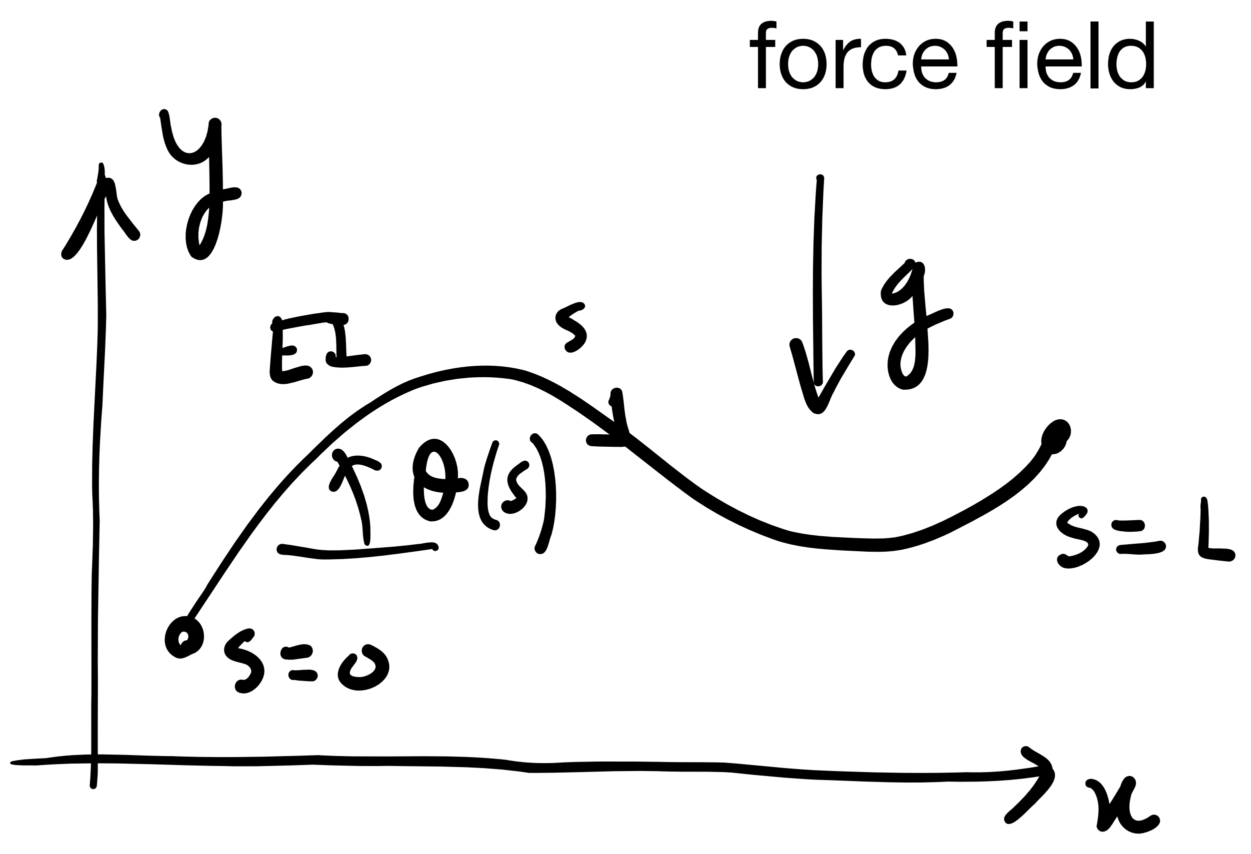
THE INVARIANT: WHERE DOES IT COME FROM?

Variational approach $A = \int_0^L \mathscr{L}[\theta(s), \dots]\, ds$

$V(x, y)$

$$W = \frac{1}{2} EI\,(\kappa(s) - \hat{\kappa})^2 + \frac{1}{2} EA\,(e(s) - 1)^2 + \rho A g\, y(s)$$

curvature extension gravity



bound. cond.

$x(0) = \dots$ $x(L) = \dots$

$y(0) = \dots$ $y(L) = \dots$

$\theta(0) = \dots$ $\theta(L) = \dots$

Under the kinematic constraints between $x(s)$, $y(s)$, $\theta(s)$, $\kappa(s)$, $e(s)$

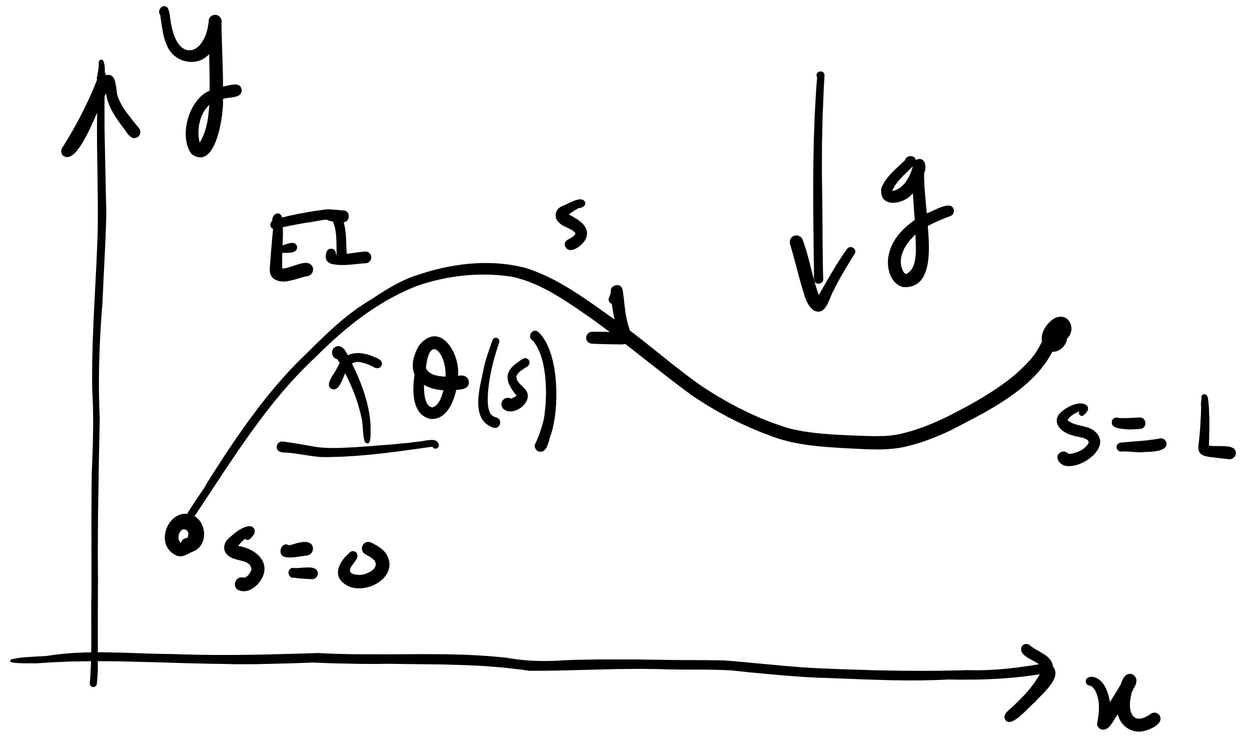
$$\mathscr{L} = W + \mathit{constraints}$$

$$\mathscr{L} = \frac{1}{2} EI\,(\kappa(s) - \hat{\kappa})^2 + \frac{1}{2} EA\,(e(s) - 1)^2 + \rho A g\, y(s) + \lambda_1(x' - e\ \cos\theta) + \lambda_2(y' - e\ \sin\theta) + \lambda_3(\theta' - \kappa)$$

study $\mathscr{L}(\mathbf{q}, \mathbf{q}')$ with $\mathbf{q} = \{x, y, e, \theta, \kappa\}$ => Euler-Lagrange: $\dfrac{\partial \mathscr{L}}{\partial \mathbf{q}} = \dfrac{d}{ds}\dfrac{\partial \mathscr{L}}{\partial \mathbf{q}'}$

THE INVARIANT: WHERE DOES IT COME FROM?

force field: $-\nabla V$



$$\frac{\partial \mathscr{L}}{\partial \kappa} - \frac{d}{ds}\frac{\partial \mathscr{L}}{\partial \kappa'} = 0 \Rightarrow \overset{\substack{\text{bending moment}\\ \swarrow}}{\lambda_3} = EI\,(\kappa(s) - \hat{\kappa})$$

$$\frac{\partial \mathscr{L}}{\partial e} - \frac{d}{ds}\frac{\partial \mathscr{L}}{\partial e'} = 0 \Rightarrow \underbrace{\lambda_1 \cos\theta + \lambda_2 \sin\theta}_{\text{tension (axial force)}} = EA\,(e(s) - 1)$$

constitutive relations

$$\left.\begin{aligned}
\frac{\partial \mathscr{L}}{\partial x} - \frac{d}{ds}\frac{\partial \mathscr{L}}{\partial x'} = 0 \Rightarrow \lambda_1'(s) = -\frac{\partial V(x,y)}{\partial x}\\
\frac{\partial \mathscr{L}}{\partial y} - \frac{d}{ds}\frac{\partial \mathscr{L}}{\partial y'} = 0 \Rightarrow \lambda_2'(s) = -\frac{\partial V(x,y)}{\partial y}
\end{aligned}\right\} \Rightarrow \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} \text{ force vector}$$

$$\frac{\partial \mathscr{L}}{\partial \theta} - \frac{d}{ds}\frac{\partial \mathscr{L}}{\partial \theta'} = 0 \Rightarrow \underset{\downarrow}{\lambda_3'(s)} = \underbrace{\lambda_1 \sin\theta - \lambda_2 \cos\theta}_{\swarrow}$$

$$EI\,\theta''(s) = \textit{shear force}$$

force and moment equilibrium

bound. cond.

$x(0) = \dots$ $\quad x(L) = \dots$

$y(0) = \dots$ $\quad y(L) = \dots$

$\theta(0) = \dots$ $\quad \theta(L) = \dots$

THE INVARIANT: WHERE DOES IT COME FROM?

$$\frac{\partial \mathscr{L}}{\partial \kappa} - \frac{d}{ds}\frac{\partial \mathscr{L}}{\partial \kappa'} = 0 \Rightarrow \overset{\substack{\text{bending moment}\\ \swarrow}}{\lambda_3} = EI\,(\kappa(s) - \hat{\kappa})$$

$$\frac{\partial \mathscr{L}}{\partial e} - \frac{d}{ds}\frac{\partial \mathscr{L}}{\partial e'} = 0 \Rightarrow \underbrace{\lambda_1 \cos\theta + \lambda_2 \sin\theta}_{\text{tension (axial force)}} = EA\,(e(s) - 1)$$

constitutive relations

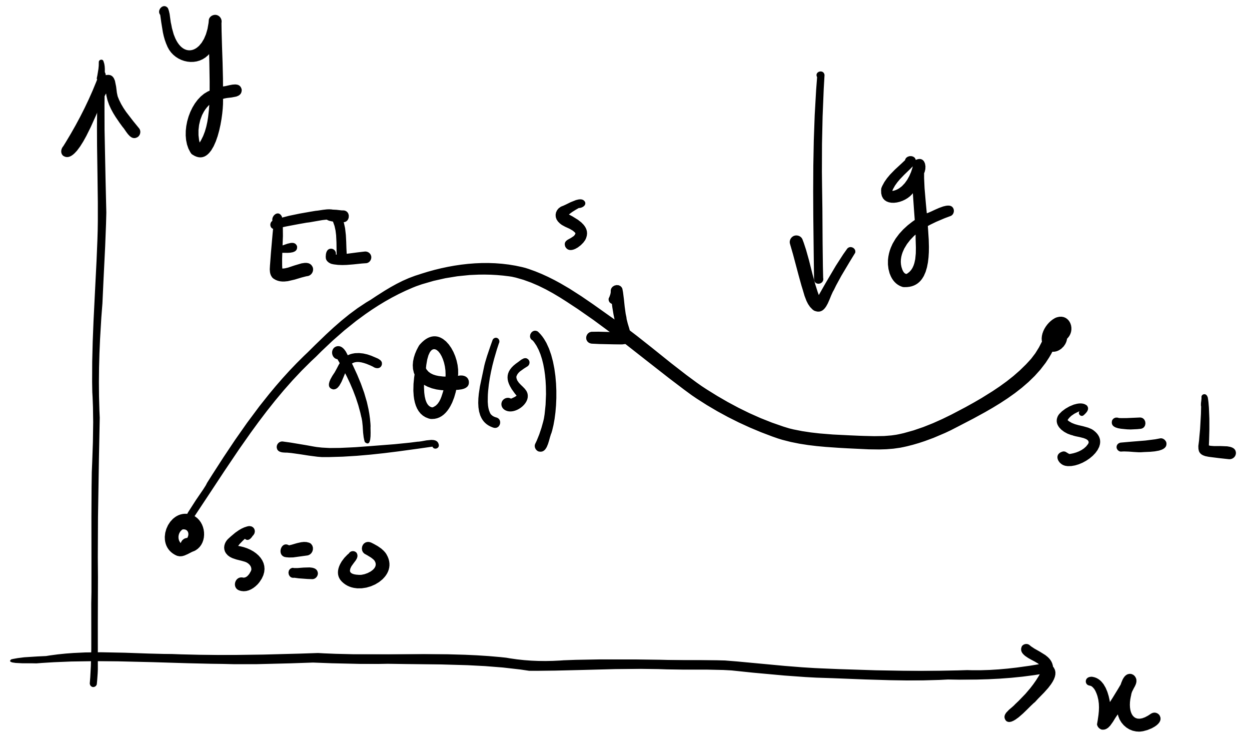
$$\left.\begin{aligned} \frac{\partial \mathscr{L}}{\partial x} - \frac{d}{ds}\frac{\partial \mathscr{L}}{\partial x'} = 0 &\Rightarrow \lambda_1'(s) = -\frac{\partial V(x,y)}{\partial x} \\ \frac{\partial \mathscr{L}}{\partial y} - \frac{d}{ds}\frac{\partial \mathscr{L}}{\partial y'} = 0 &\Rightarrow \lambda_2'(s) = -\frac{\partial V(x,y)}{\partial y} \end{aligned}\right\} \Rightarrow \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} \text{ force vector}$$

$$\frac{\partial \mathscr{L}}{\partial \theta} - \frac{d}{ds}\frac{\partial \mathscr{L}}{\partial \theta'} = 0 \Rightarrow \underset{\downarrow}{\lambda_3'(s)} = \underbrace{\lambda_1 \sin\theta - \lambda_2 \cos\theta}_{\swarrow}$$

$$EI\,\theta''(s) = \mathit{shear\ force}$$

force and moment equilibrium

force field: $-\nabla V$



bound. cond.

$$\begin{aligned} x(0) &= \ldots & x(L) &= \ldots \\ y(0) &= \ldots & y(L) &= \ldots \\ \theta(0) &= \ldots & \theta(L) &= \ldots \end{aligned}$$

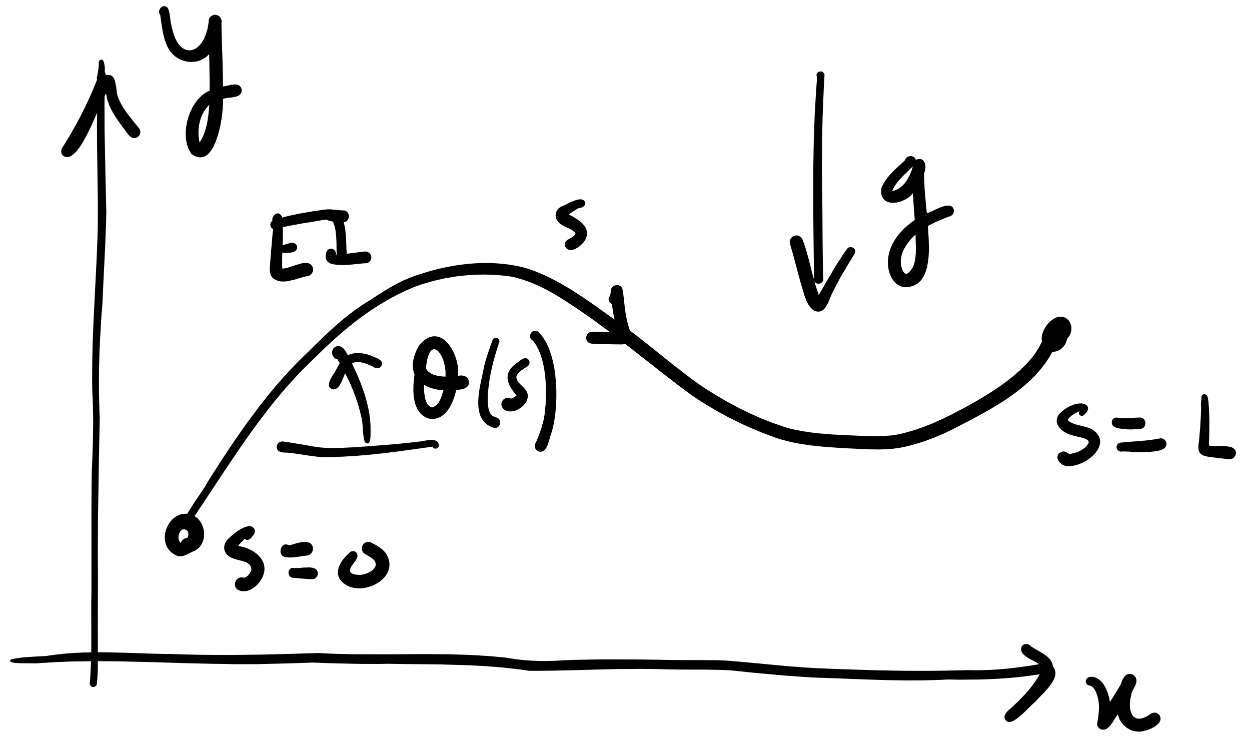
THE INVARIANT: WHERE DOES IT COME FROM?

in the case where $EI(\cancel{s})$ and $EA(\cancel{s})$ and $\hat{\kappa}(\cancel{s})$ (no explicit s dependence)

Noether's theorem: $H = \dfrac{\partial \mathscr{L}}{\partial \mathbf{q}'} \cdot \mathbf{q}' - \mathscr{L}$ with $H'(s) \equiv 0$ at equilibrium $\quad \mathbf{q} = \{x, y, e, \theta, \kappa\}$

Noether, 'Invariante Variationsprobleme'
Gottinger Nachrichten of 1918, pp. 235-257

force field: $-\nabla V$



bound. cond.

$x(0) = \dots \qquad x(L) = \dots$
$y(0) = \dots \qquad y(L) = \dots$
$\theta(0) = \dots \qquad \theta(L) = \dots$

$$H = \frac{\partial \mathscr{L}}{\partial x'} x' + \frac{\partial \mathscr{L}}{\partial y'} y' + \frac{\partial \mathscr{L}}{\partial e'} e' + \frac{\partial \mathscr{L}}{\partial \theta'} \theta' + \frac{\partial \mathscr{L}}{\partial \kappa'} \kappa' - \mathscr{L}$$

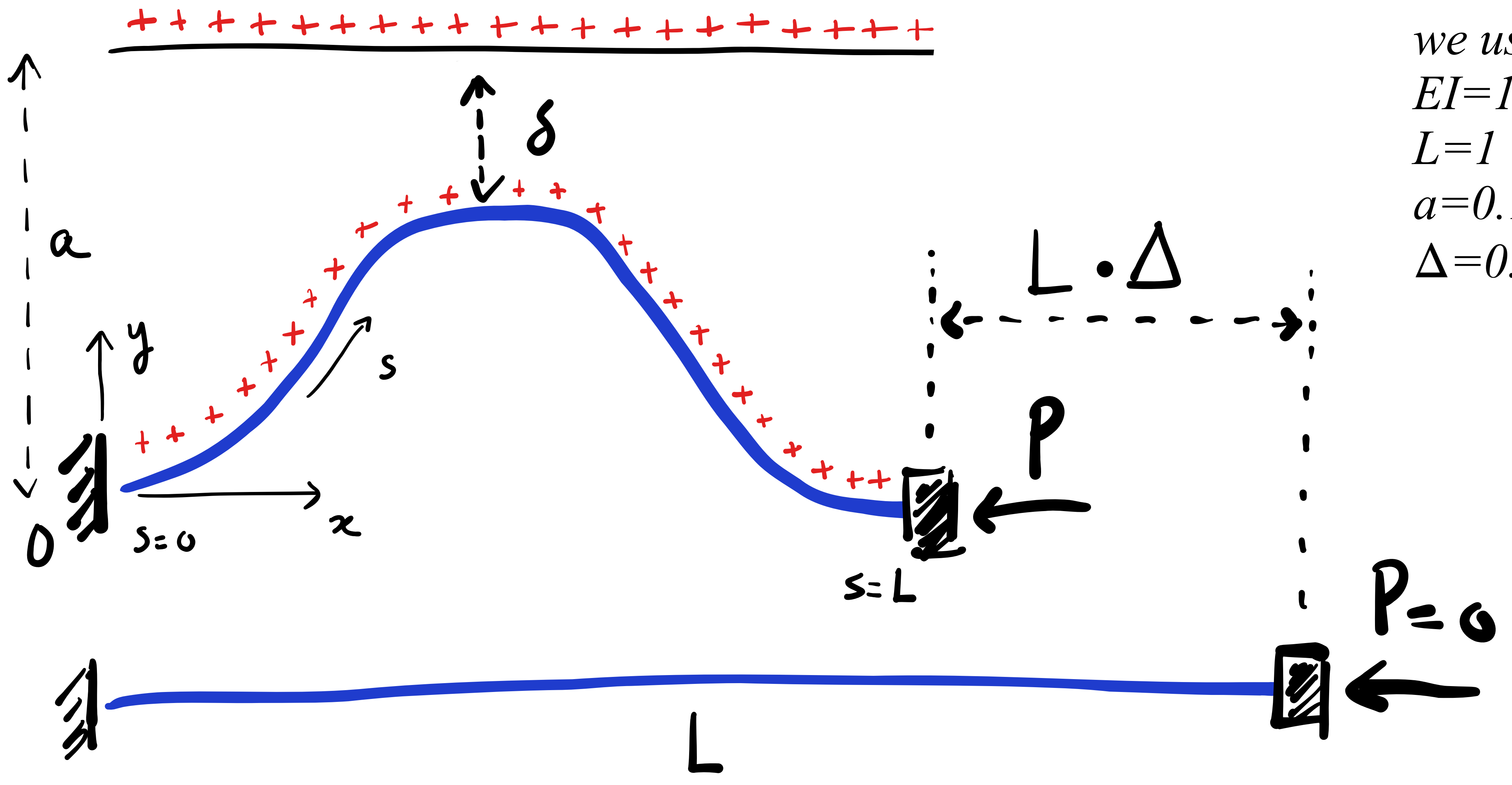
$$H = \frac{1}{2} EI\,[\kappa^2(s) - \hat{\kappa}^2] + \frac{1}{2} EA\,[e^2(s) - 1] - V(x, y)$$

not strain energies!

$$H = \underbrace{\frac{1}{2} EI\,\kappa^2(s) + tension}_{\substack{\text{inextensible} \\ \text{naturally flat}}} - V(x, y)$$

force field: $-\nabla V$ $\left\{ \begin{array}{c} \text{Gravity} \\ \text{Electrostatics} \\ \text{Contact} \end{array} \right.$

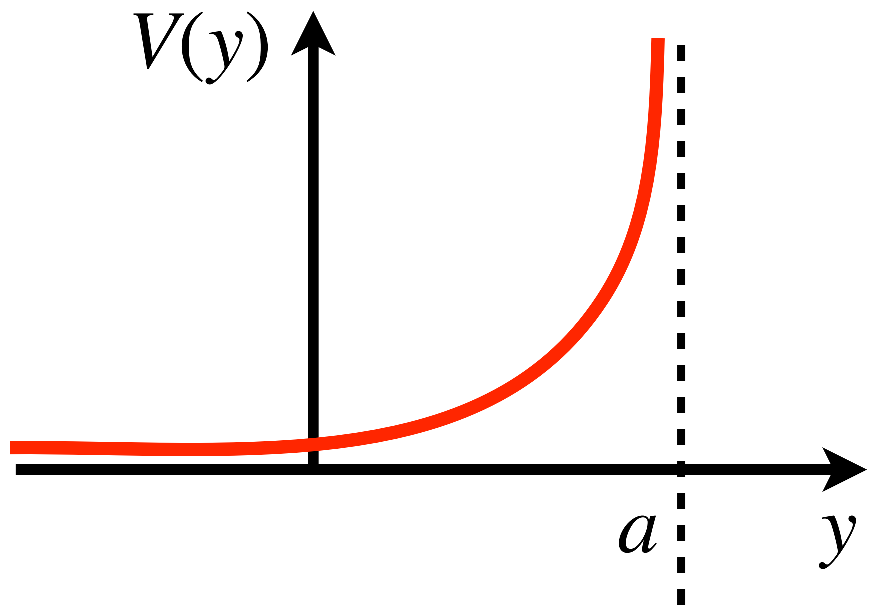
CONSERVATIVE EXTERNAL POTENTIAL



we use
EI=1
L=1
a=0.1
Δ=0.026

soft-wall (or barrier) potential

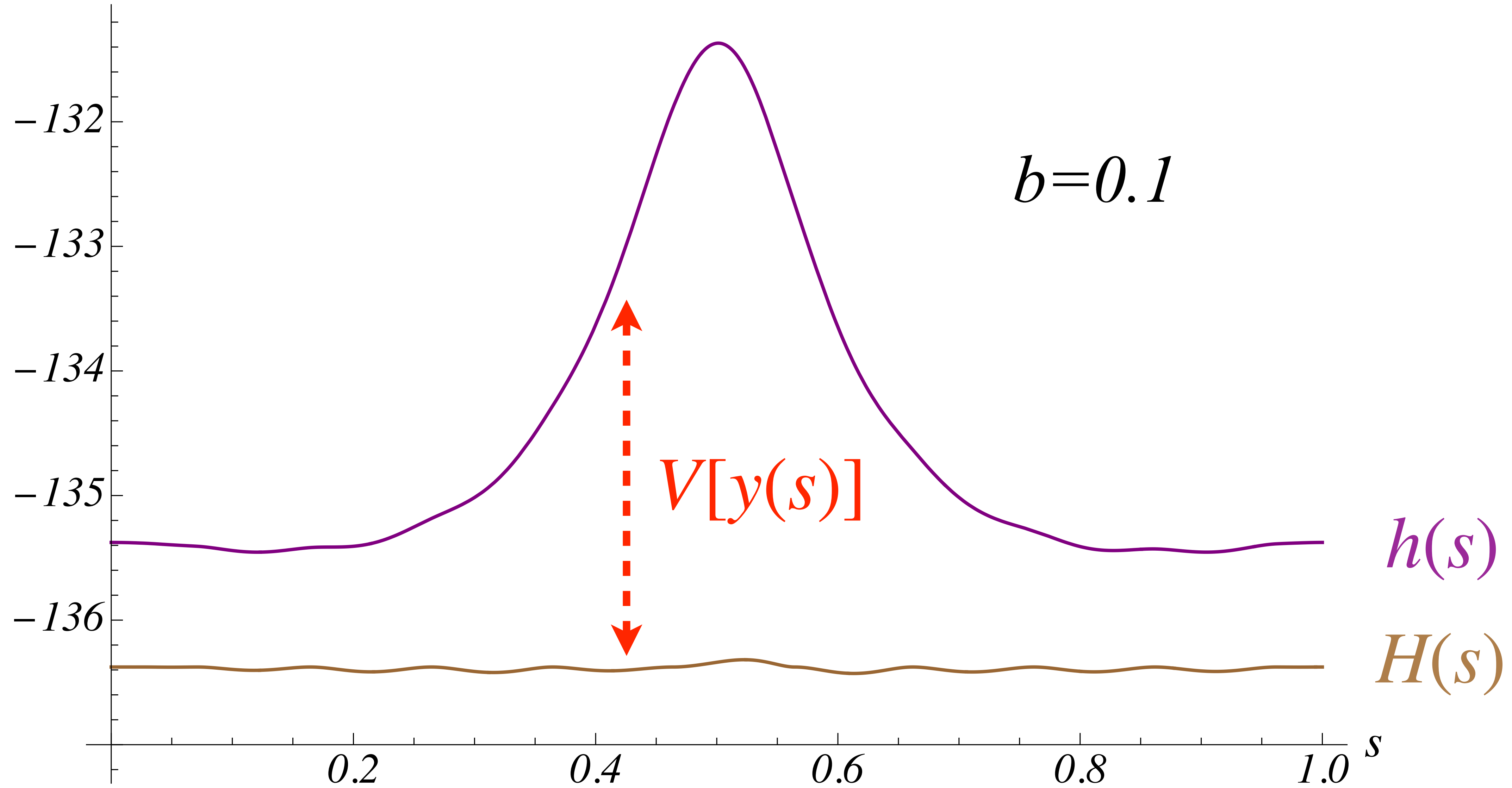
$$V(y) = \frac{b}{a - y(s)}$$



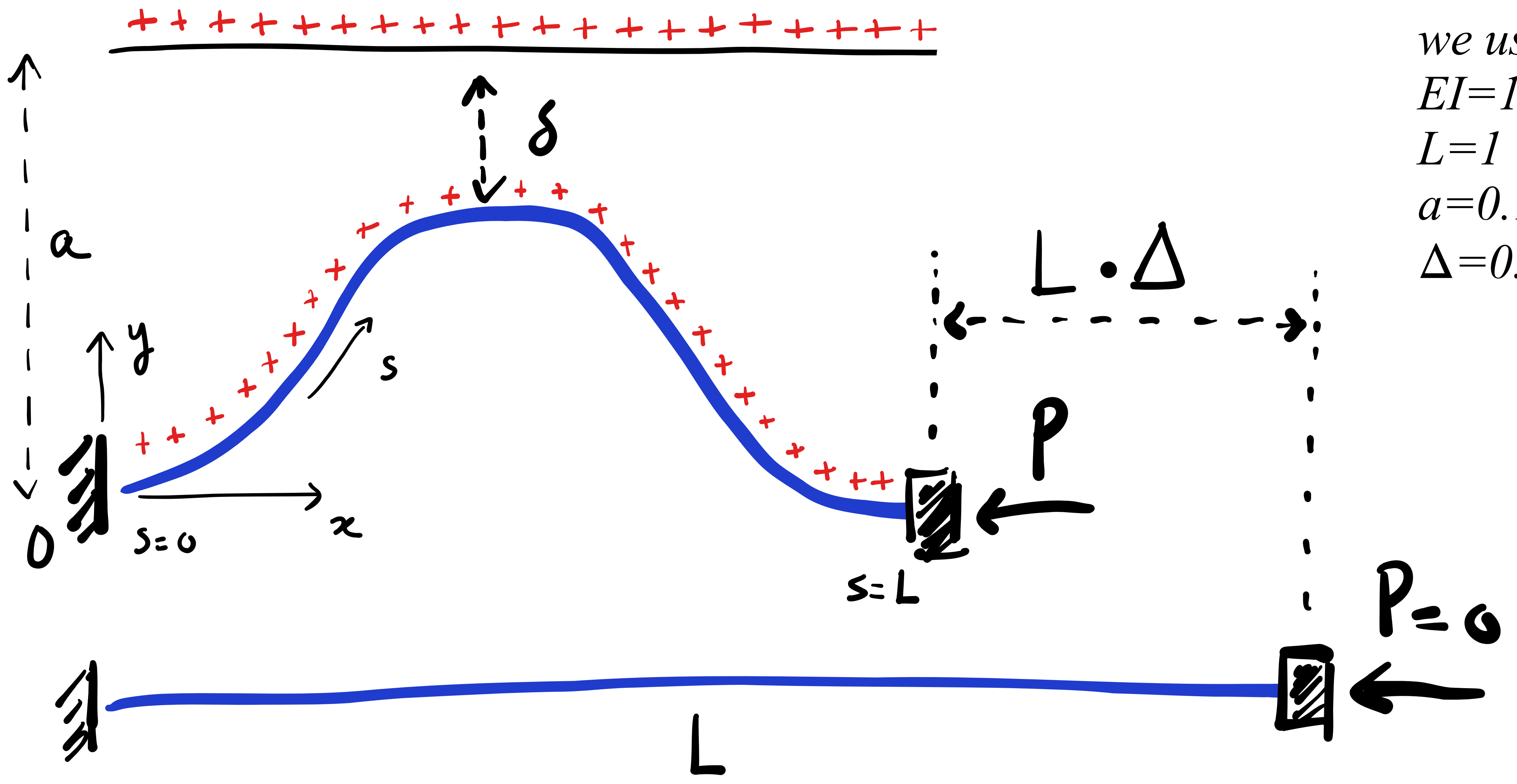
$$H = \underbrace{\frac{1}{2} EI\,\kappa^2(s) + tension}_{h(s)} - V[y(s)]$$

$$H'(s) = 0 \ \forall s$$

$$h'(s) \neq 0$$



CONSERVATIVE EXTERNAL POTENTIAL



we use
EI=1
L=1
a=0.1
Δ=0.026

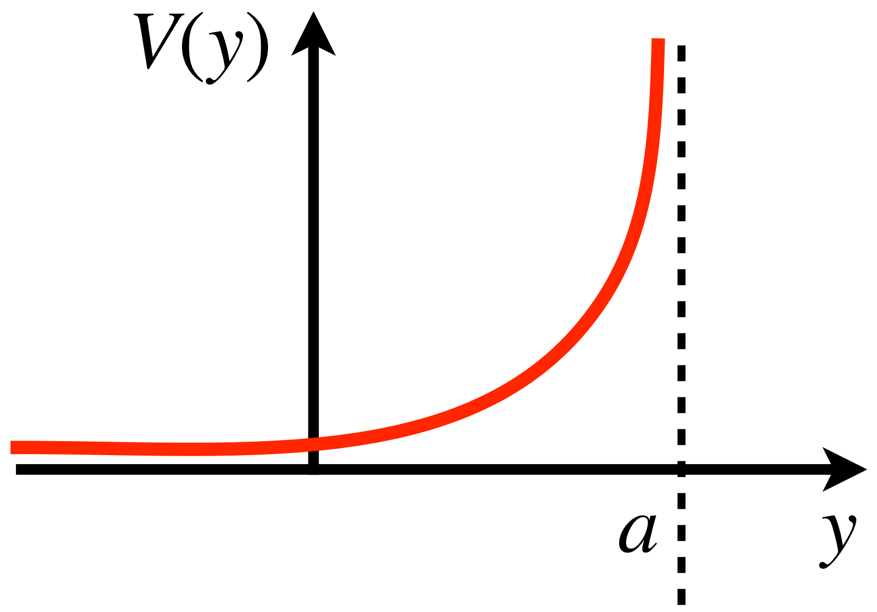
$$H = \underbrace{\frac{1}{2} EI\,\kappa^2(s) + tension}_{h(s)} - V[y(s)]$$

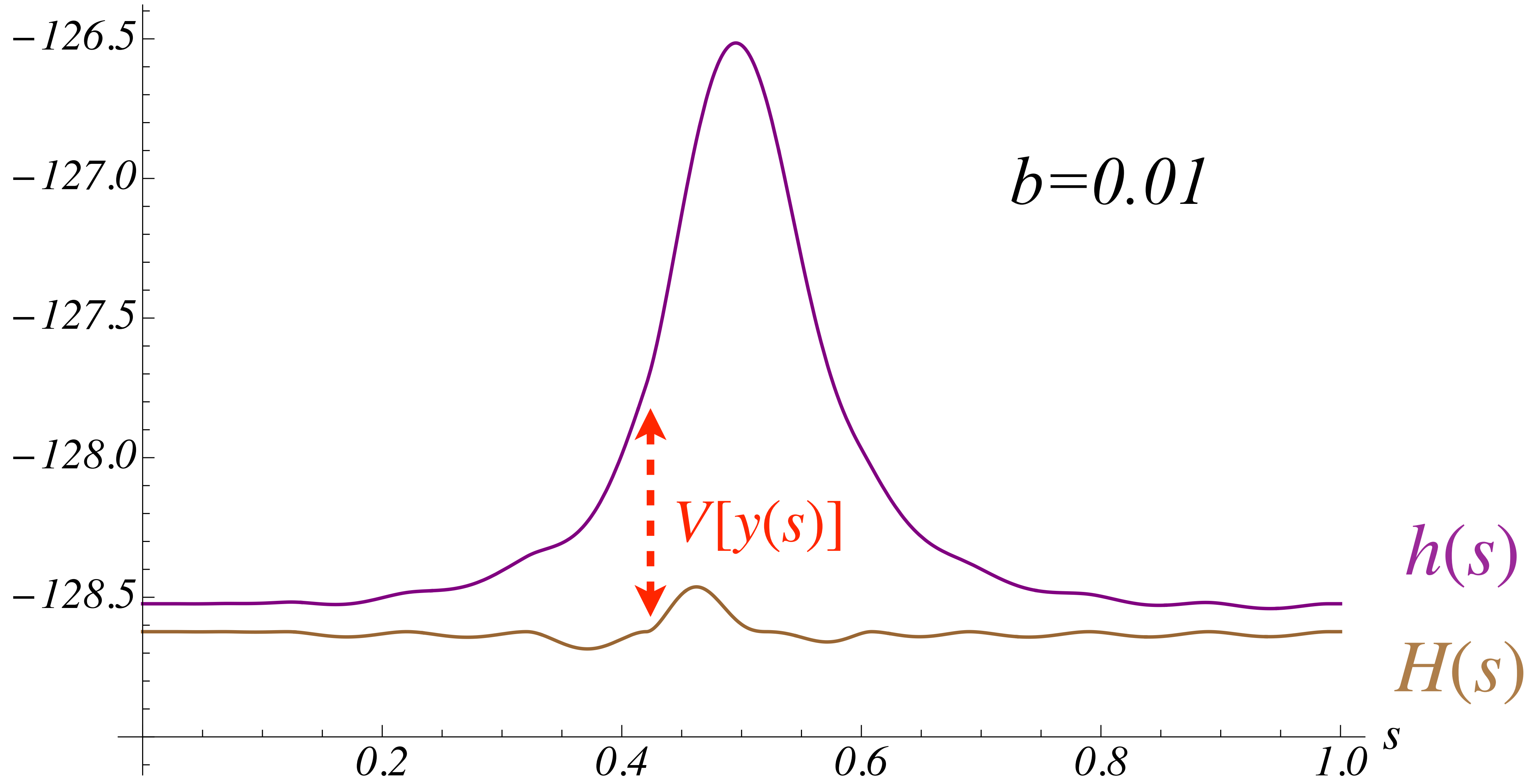
$$H'(s) = 0 \; \forall s$$

$$h'(s) \neq 0$$

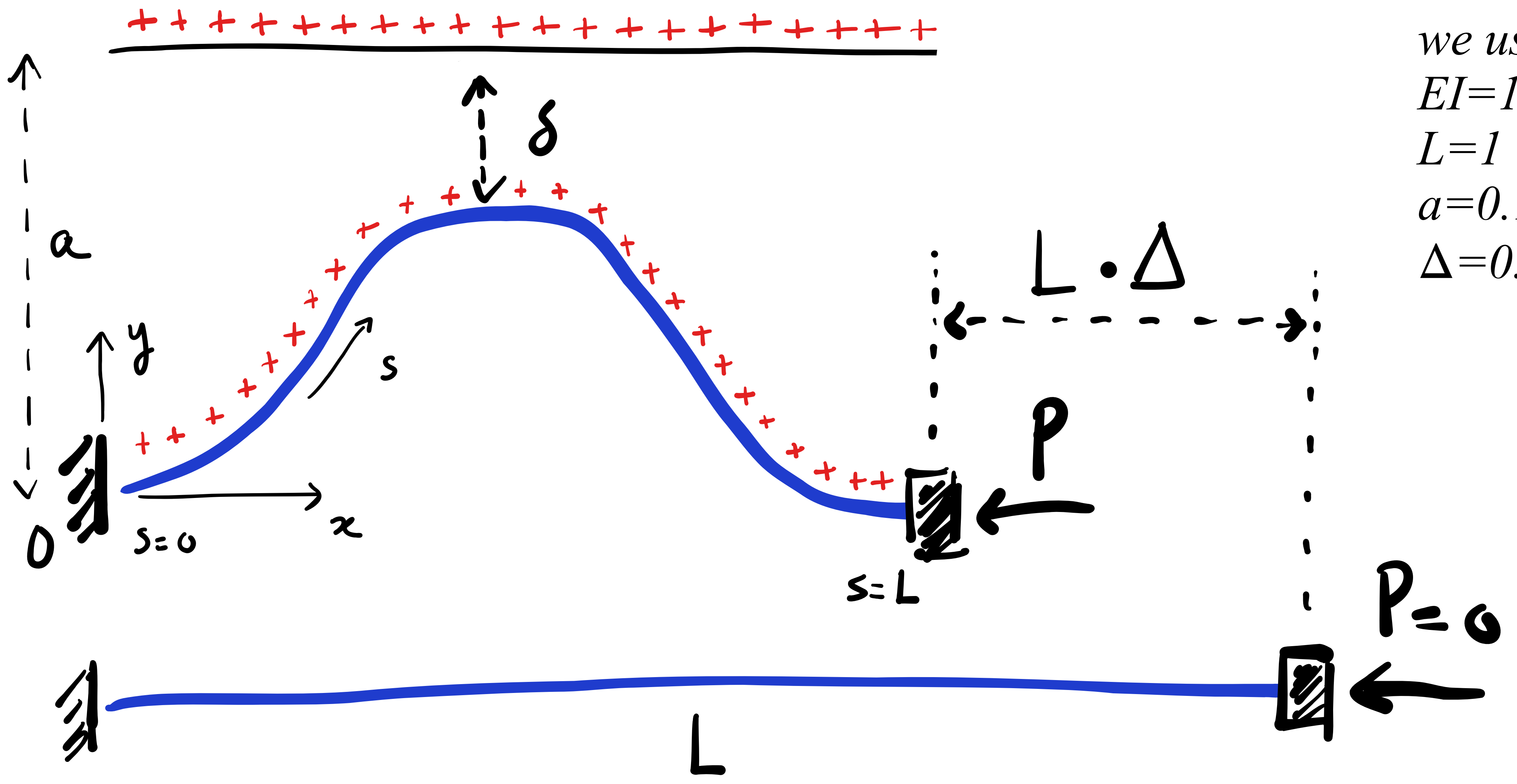
soft-wall (or barrier) potential

$$V(y) = \frac{b}{a - y(s)}$$





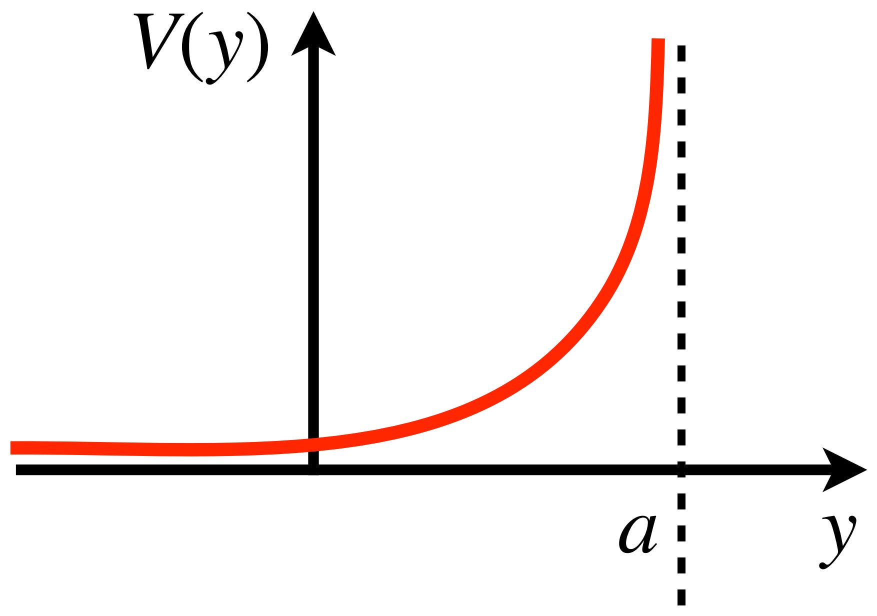
CONSERVATIVE EXTERNAL POTENTIAL



we use
EI=1
L=1
a=0.1
Δ=0.026

soft-wall (or barrier) potential

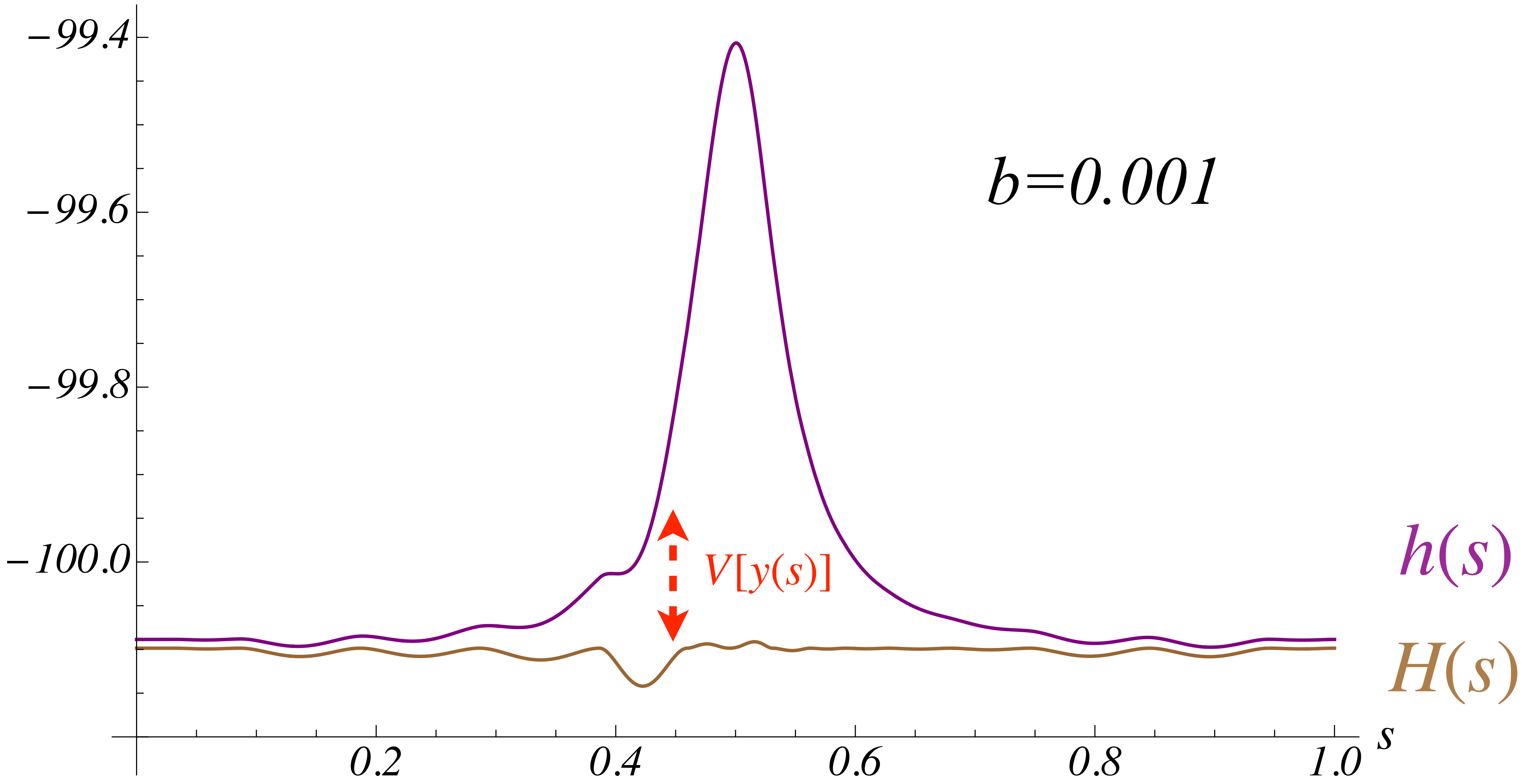
$$V(y) = \frac{b}{a - y(s)}$$



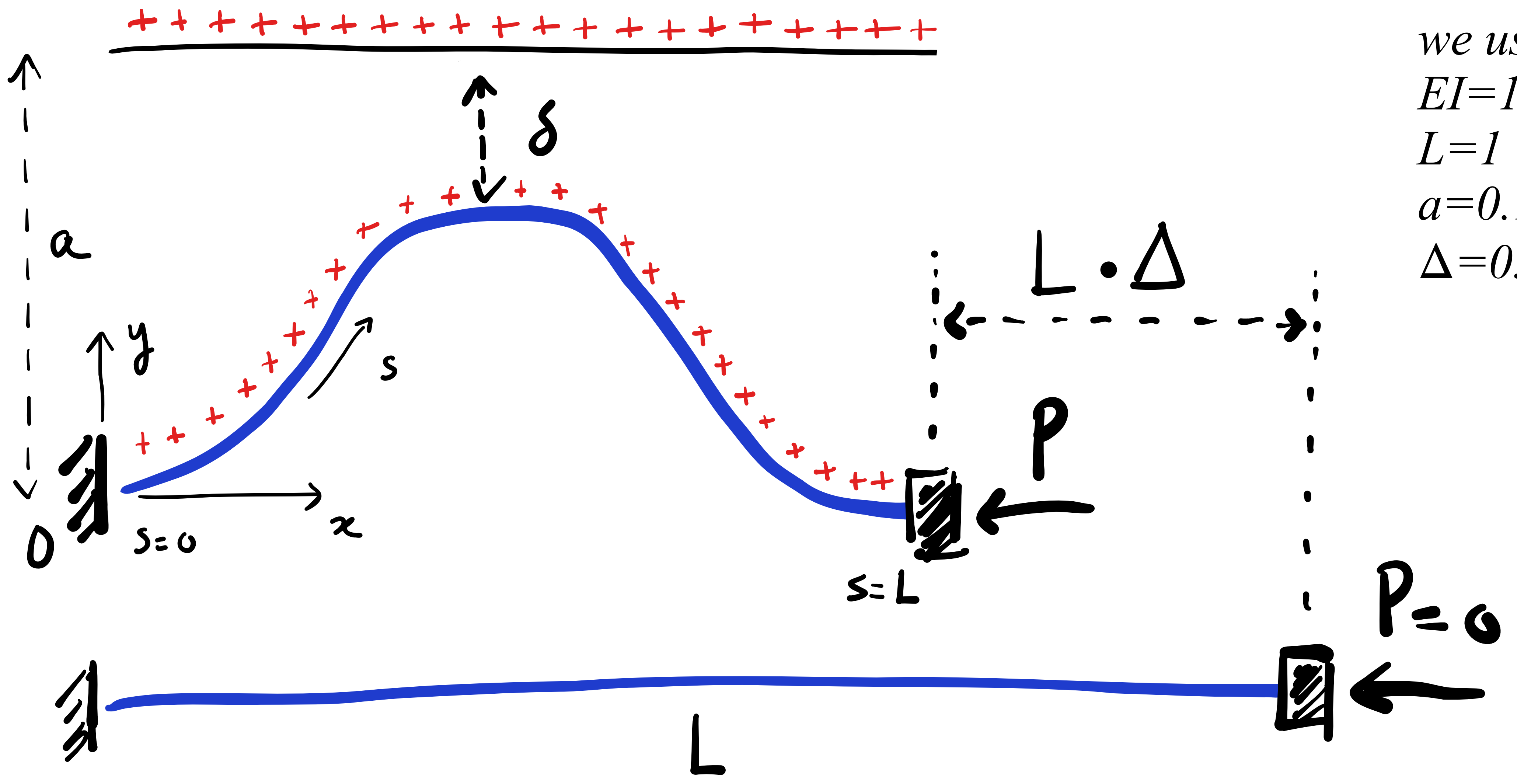
$$H = \underbrace{\frac{1}{2} EI\,\kappa^2(s) + tension}_{h(s)} - V[y(s)]$$

$$H'(s) = 0 \ \forall s$$

$$h'(s) \neq 0$$



CONSERVATIVE EXTERNAL POTENTIAL



we use
EI=1
L=1
a=0.1
Δ=0.026

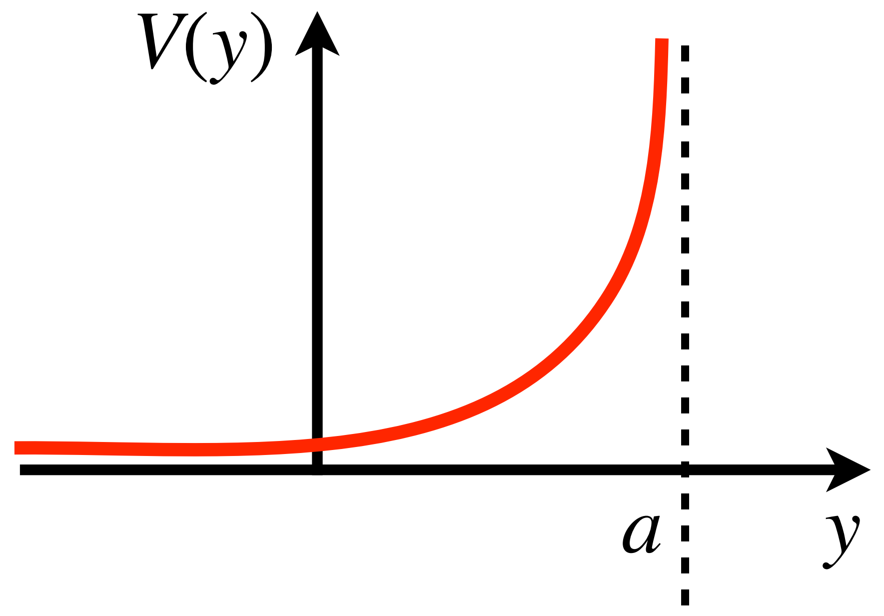
$$H = \underbrace{\frac{1}{2} EI\,\kappa^2(s) + tension}_{h(s)} - V[y(s)]$$

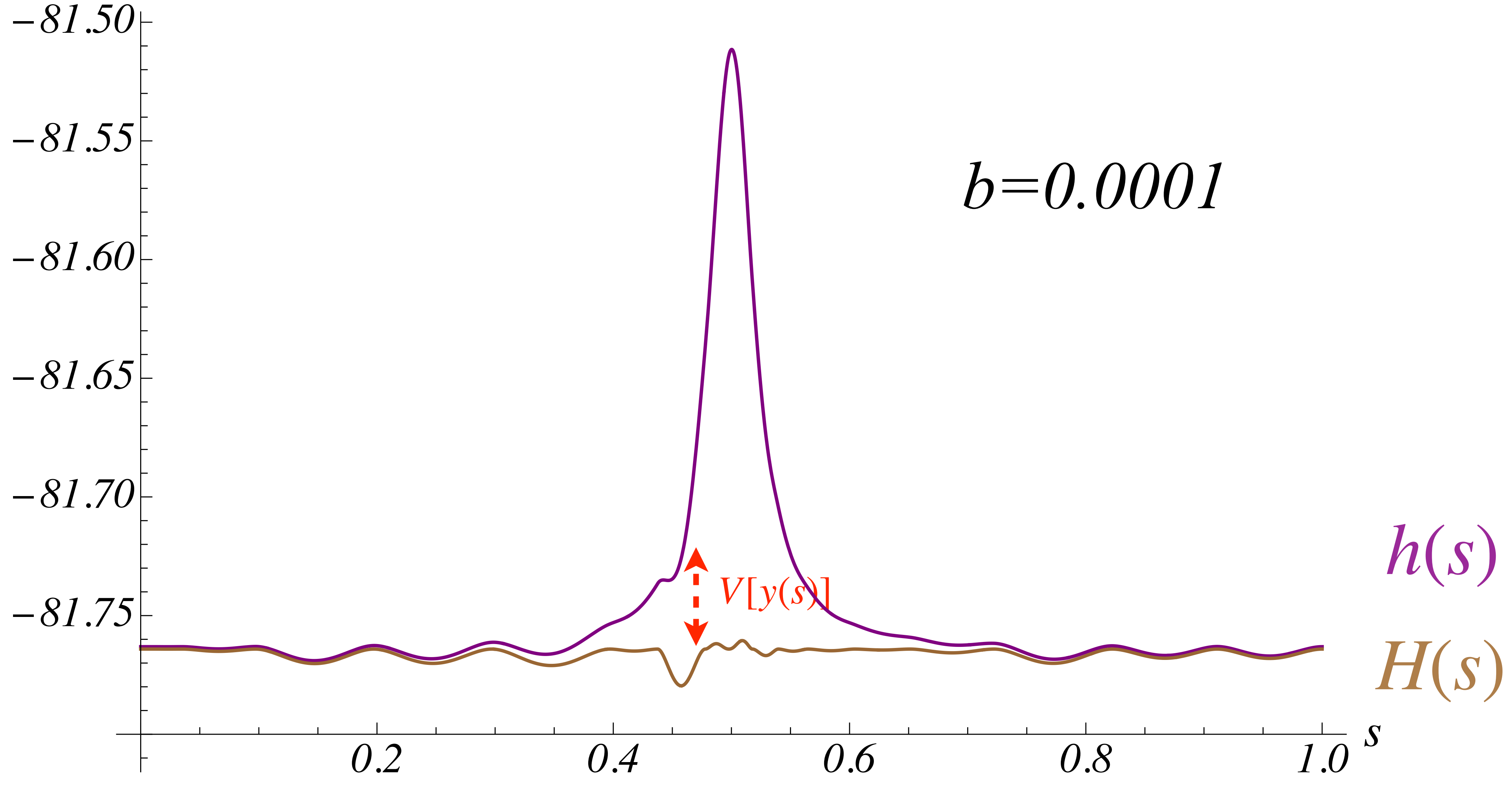
$$H'(s) = 0 \;\forall s$$

$$h'(s) \neq 0$$

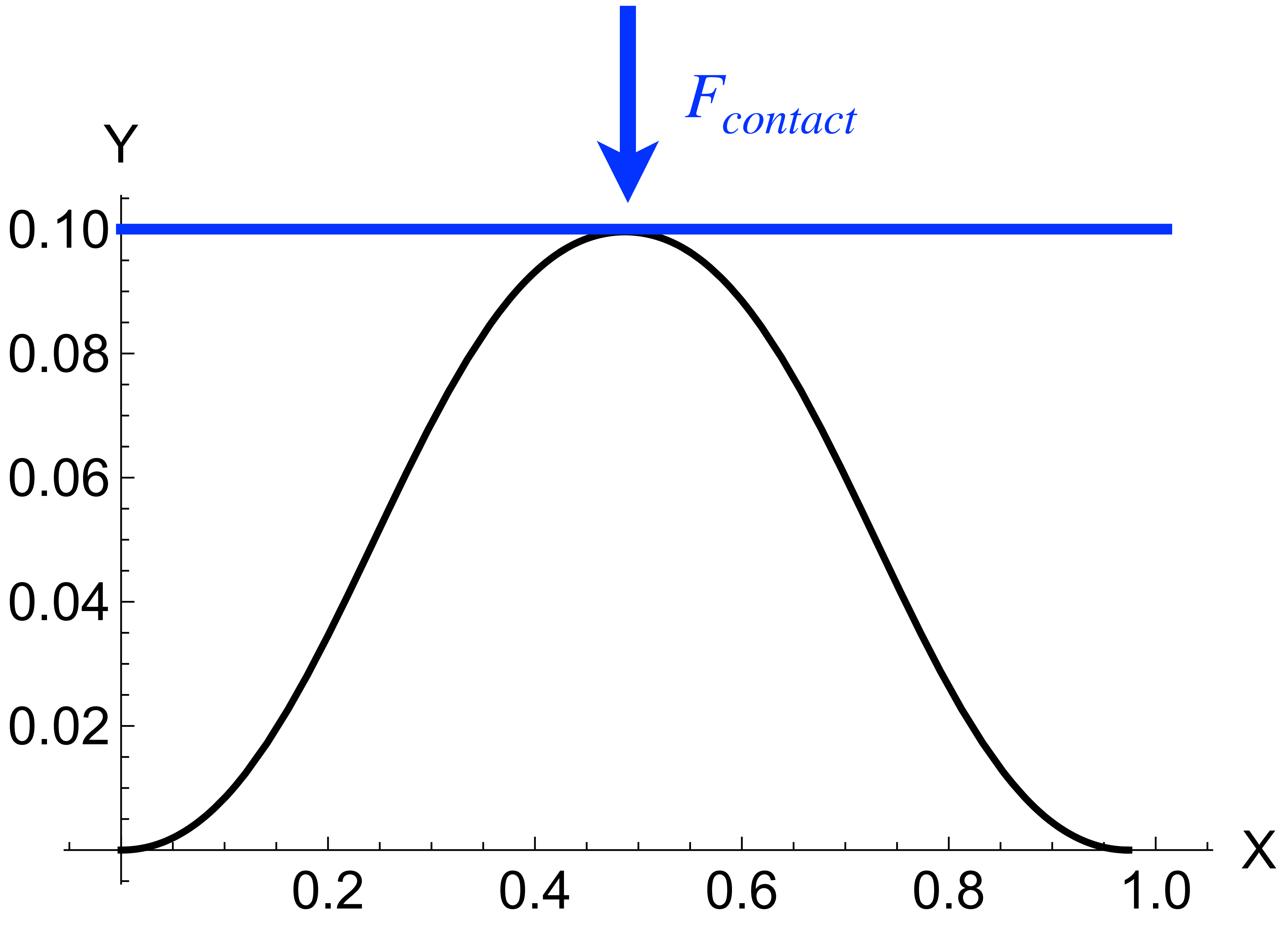
soft-wall (or barrier) potential

$$V(y) = \frac{b}{a - y(s)}$$





CONSERVATIVE EXTERNAL POTENTIAL

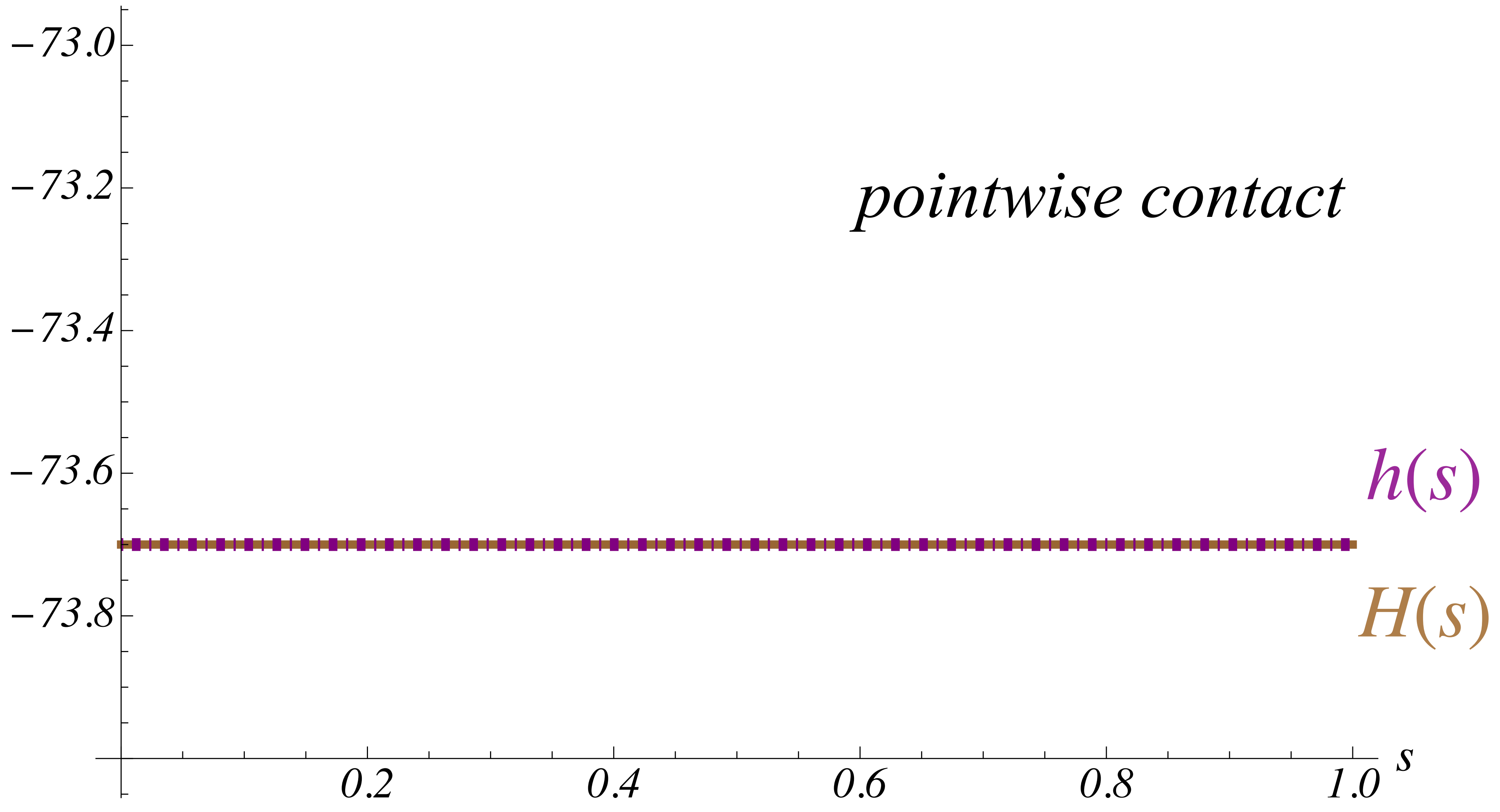


we use
EI=1
L=1
a=0.1
Δ=0.026

$$H = \underbrace{\frac{1}{2}EI\,\kappa^2(s) + tension}_{h(s)} - 0$$

$$H'(s) = 0\ \forall s$$

$$h'(s) = 0\ \forall s$$



potential $$\int_0^L V(y) = \frac{b}{a - y(s)}\,ds \xrightarrow[b\to 0]{} 0$$

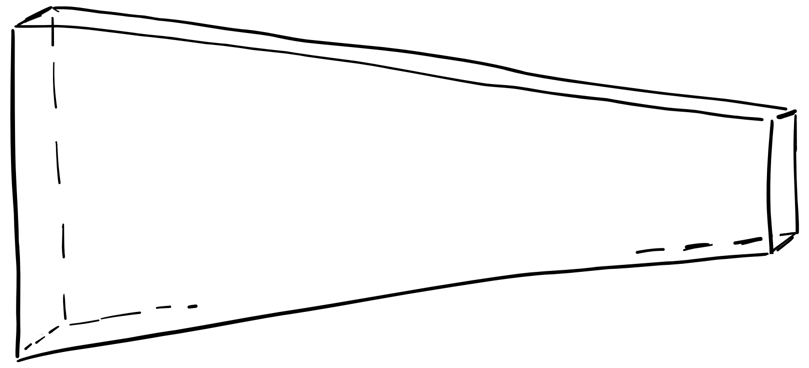
force $$\int_0^L \frac{\partial V}{\partial y} = \frac{b}{(a - y(s))^2}\,ds \xrightarrow[b\to 0]{} F_{contact}$$

THE INVARIANT: WHEN DOES IT FAIL?

$$\mathscr{L} = \frac{1}{2} EI(s)\,(\kappa(s)-\hat{\kappa}(s))^2 + \frac{1}{2} EA(s)\,(e(s)-1)^2 + V[x(s),y(s)] + \lambda_1(x'-e\,\cos\theta) + \lambda_2(y'-e\,\sin\theta) + \lambda_3(\theta'-\kappa)$$

explicit s dependence

Example: tapered beam



$$\mathscr{L} = \frac{1}{2} EI(s)\,\kappa^2(s) + \lambda_1(x'-\,\cos\theta) + \lambda_2(y'-\,\sin\theta) + \lambda_3(\theta'-\kappa)$$

Equilibrium: $\left[EI(s)\,\theta'(s)\right]' = -\,P\,\sin\theta(s)$

Invariant? $H(s) = \frac{1}{2} EI(s)\,\theta'(s)^2 - P\,\cos\theta(s)$

$$H'(s) = \ldots = -\frac{1}{2} EI'(s)\,\theta'(s)^2 \neq 0$$

reminder

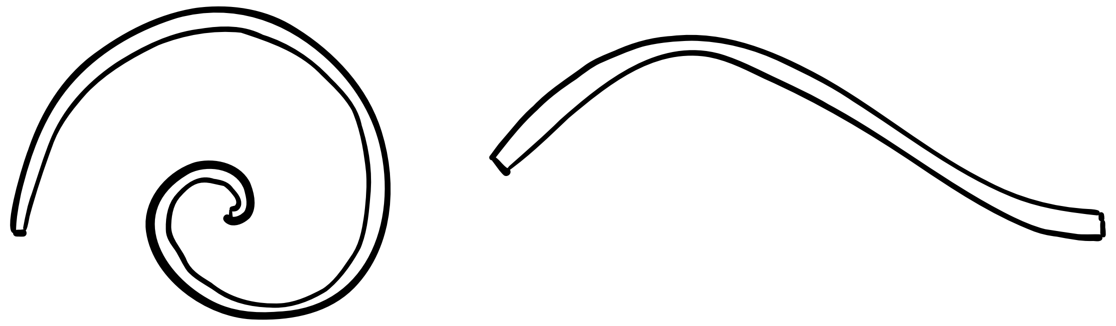
$\kappa(s) \equiv \theta'(s)$

THE INVARIANT: WHEN DOES IT FAIL?

$$\mathscr{L} = \frac{1}{2}\,EI(s)\,(\kappa(s) - \hat{\kappa}(s))^2 + \frac{1}{2}\,EA(s)\,(e(s) - 1)^2 + V[x(s), y(s)] + \lambda_1(x' - e\,\cos\theta) + \lambda_2(y' - e\,\sin\theta) + \lambda_3(\theta' - \kappa)$$

explicit s dependence

Example: curved beam



$$\mathscr{L} = \frac{1}{2}\,EI\,(\kappa(s) - \hat{\kappa}(s))^2 + \lambda_1(x' - \cos\theta) + \lambda_2(y' - \sin\theta) + \lambda_3(\theta' - \kappa)$$

Equilibrium: $EI\,\left[\theta'(s) - \hat{\kappa}(s)\right]' = -P\,\sin\theta(s)$

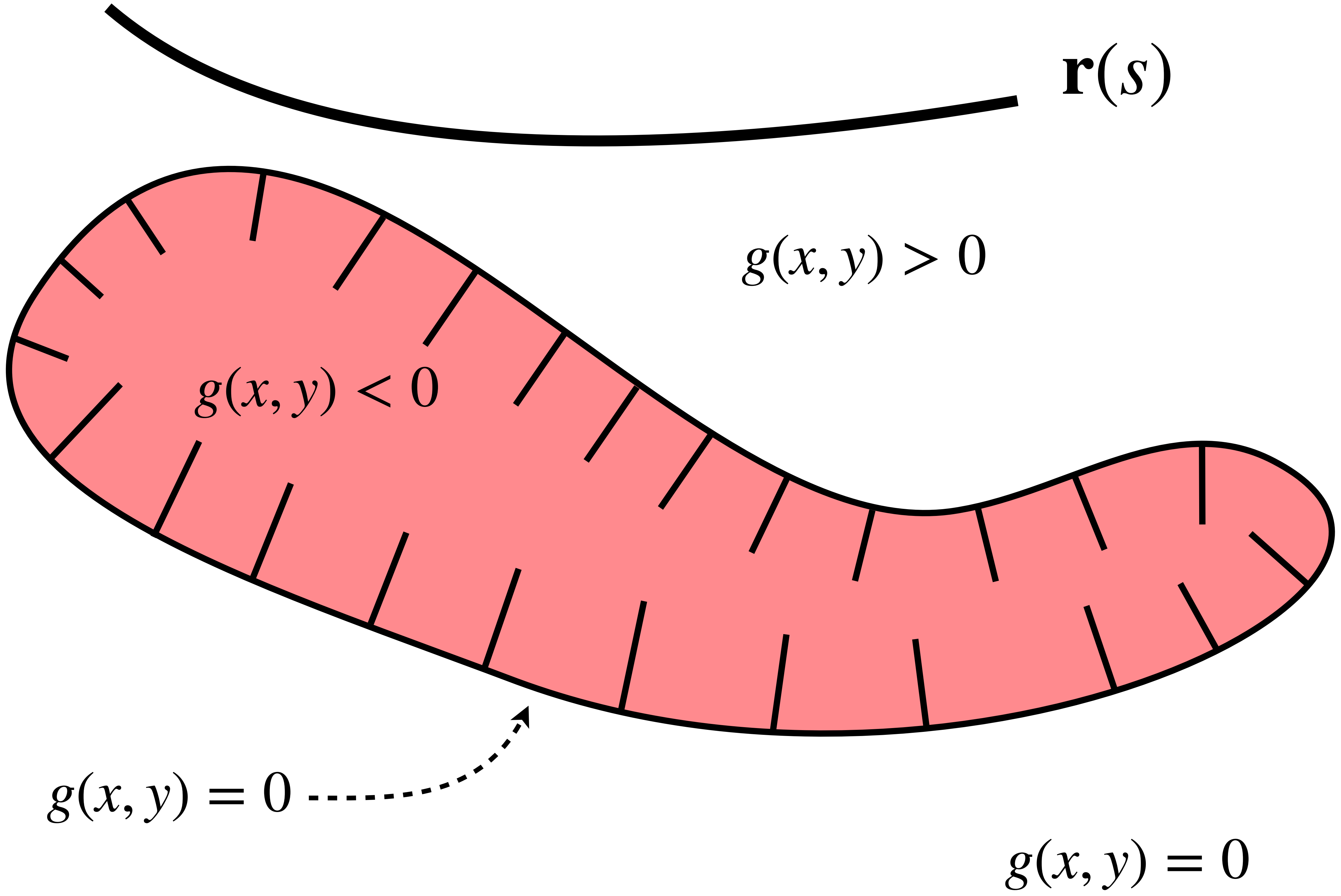
Invariant? $H(s) = \frac{1}{2}\,EI\,(\theta'(s)^2 - \hat{\kappa}^2(s)) - P\,\cos\theta(s)$

$$H'(s) = \ldots = -\frac{1}{2}\,EI\,\hat{\kappa}'(s)\,(\theta' - \hat{\kappa}) \neq 0$$

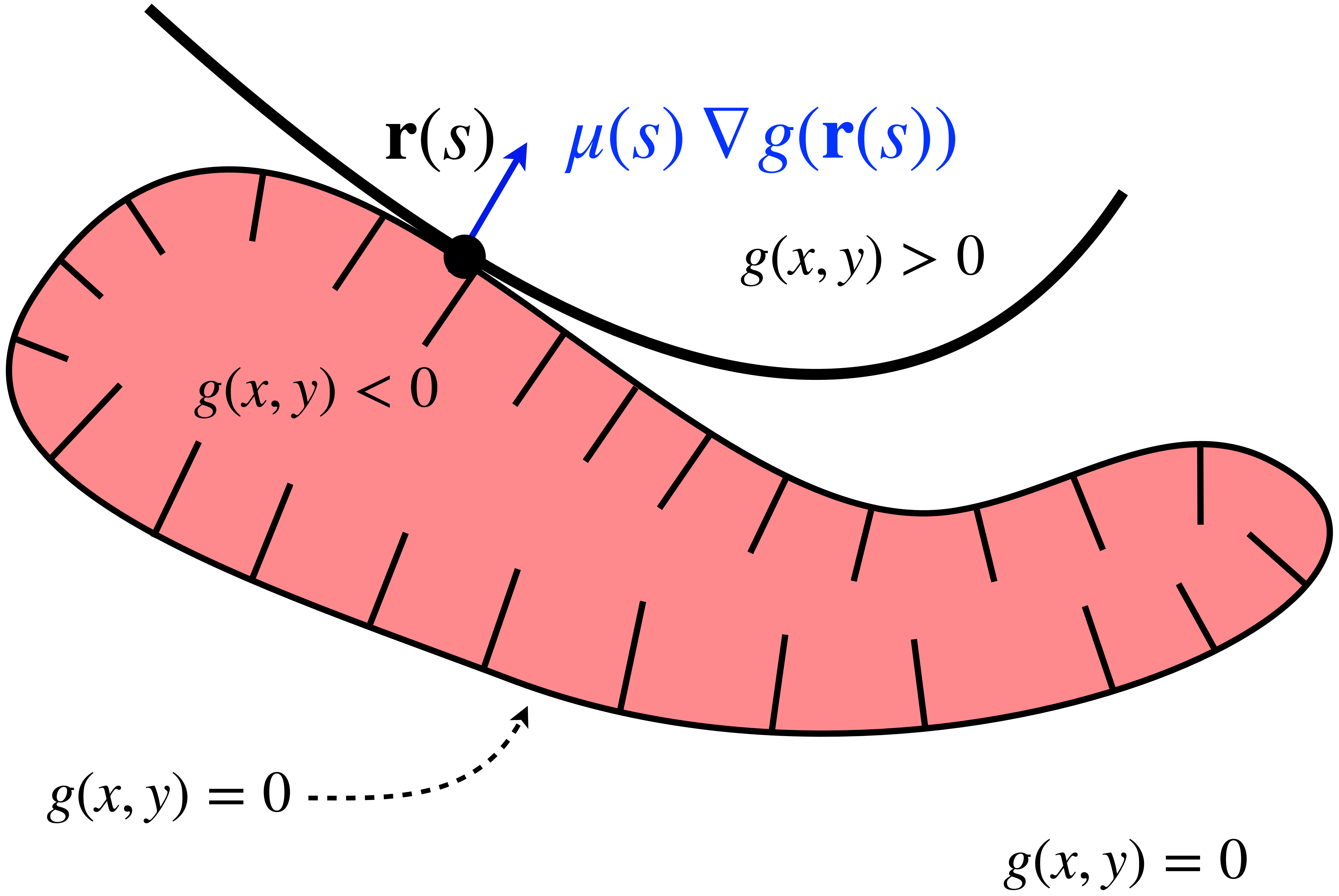
reminder

$\kappa(s) \equiv \theta'(s)$

BACK TO CONTACT

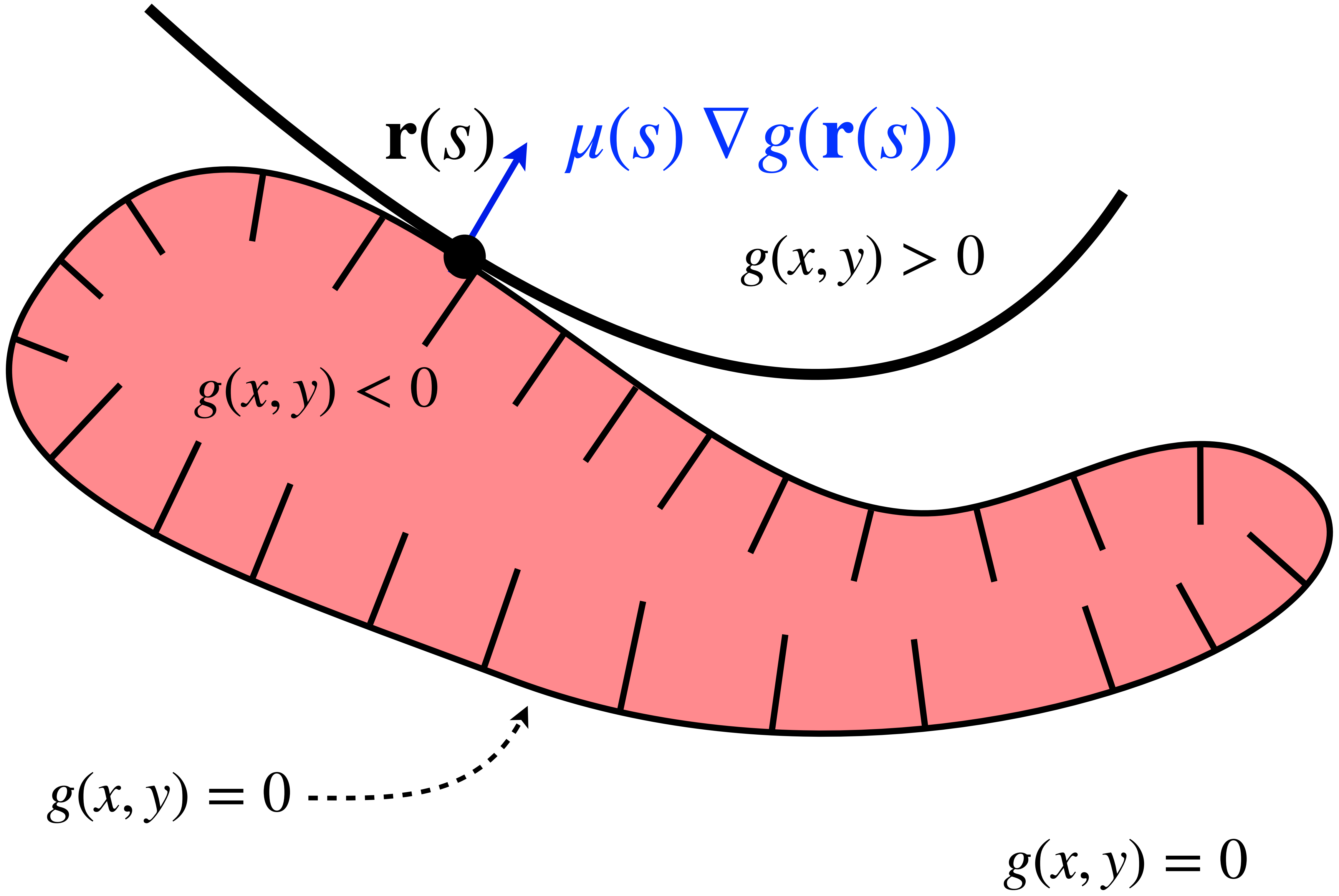


BACK TO CONTACT



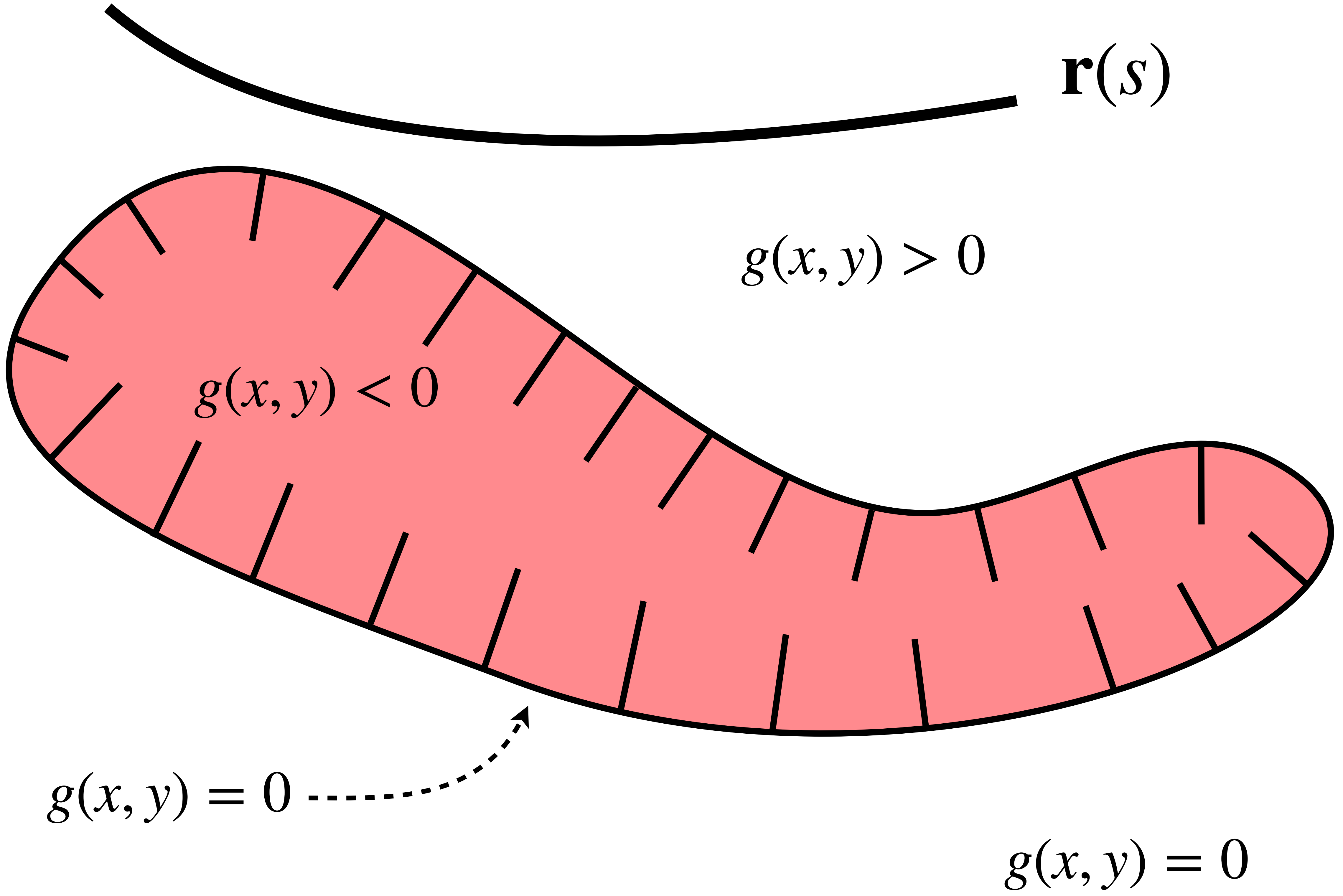
BACK TO CONTACT

New term in the Lagrangian: $\mathscr{L}_{\mathsf{C}} = -\mu(s)\, g(\mathbf{r}(s))$



BACK TO CONTACT

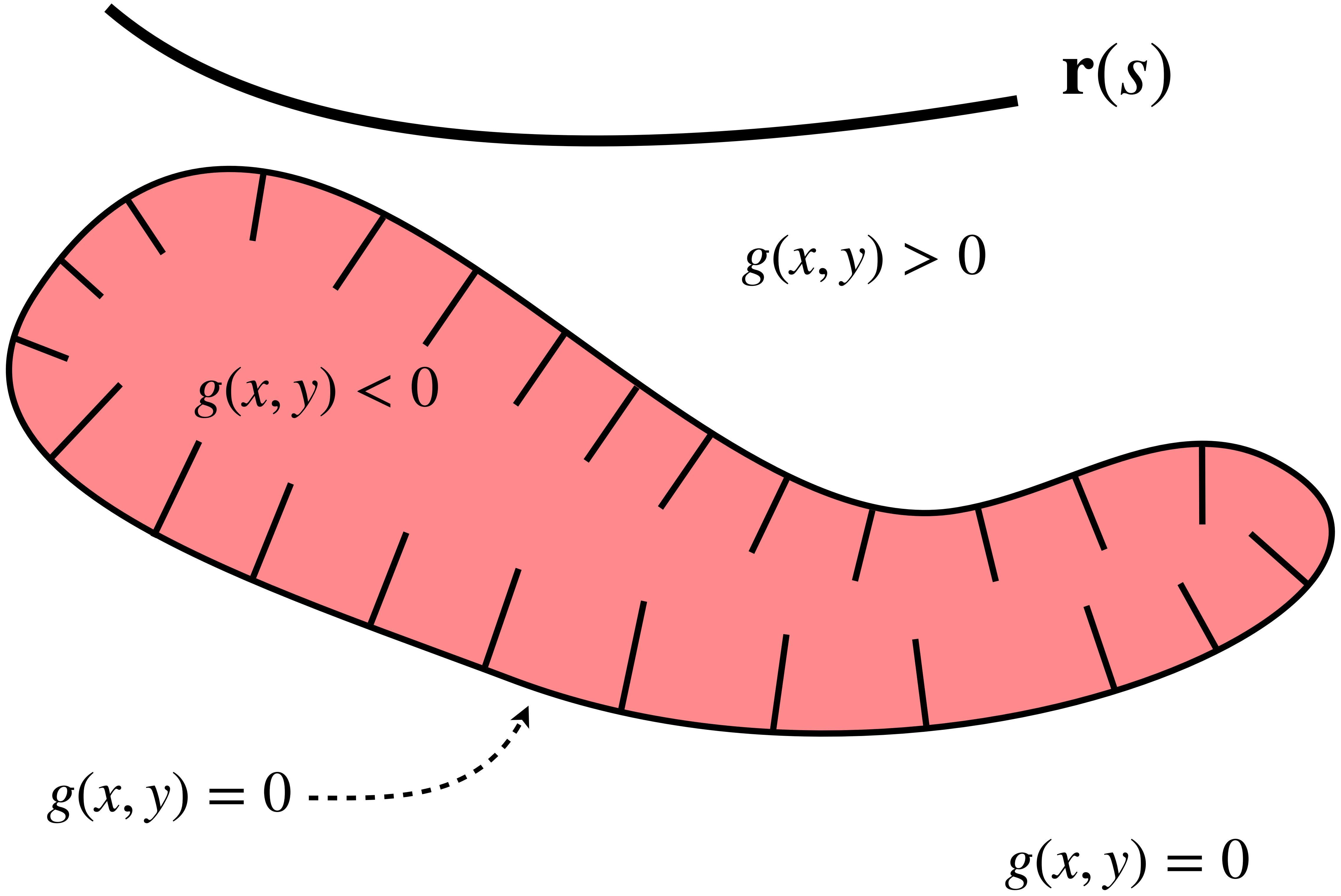
New term in the Lagrangian: $\mathscr{L}_{\mathsf{C}} = -\mu(s)\, g(\mathbf{r}(s))$



Does it break Noether's invariant ?

BACK TO CONTACT

New term in the Lagrangian: $\mathscr{L}_{\mathrm{C}} = -\mu(s)\, g(\mathbf{r}(s))$



Does it break Noether's invariant ?

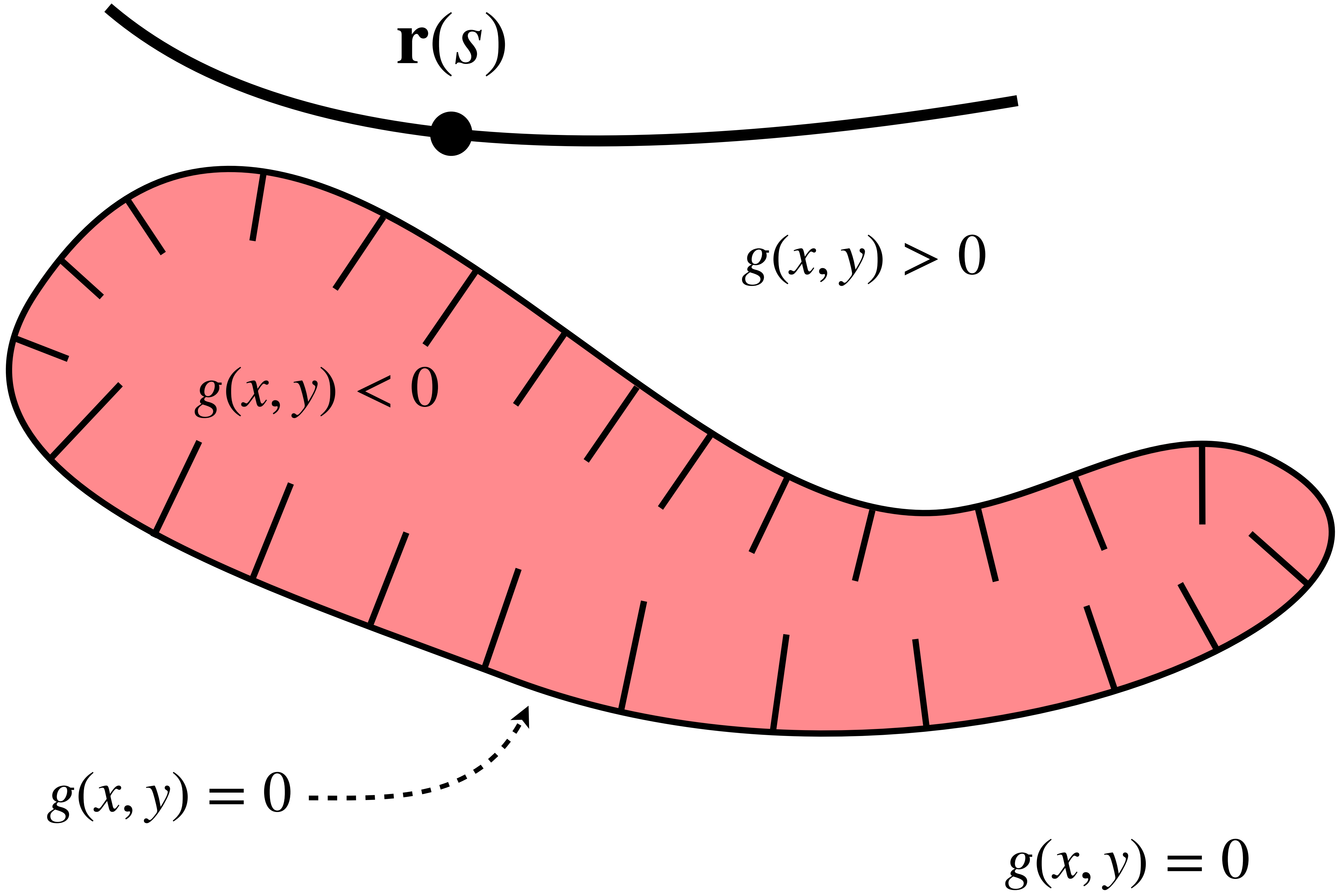
—> No!

Signorini law for frictionless contact

$$0 \leq g(\mathbf{r}(s)) \perp \mu(s) \geq 0$$

BACK TO CONTACT

New term in the Lagrangian: $\mathscr{L}_{\mathrm{C}} = -\mu(s)\, g(\mathbf{r}(s))$



Does it break Noether's invariant ?

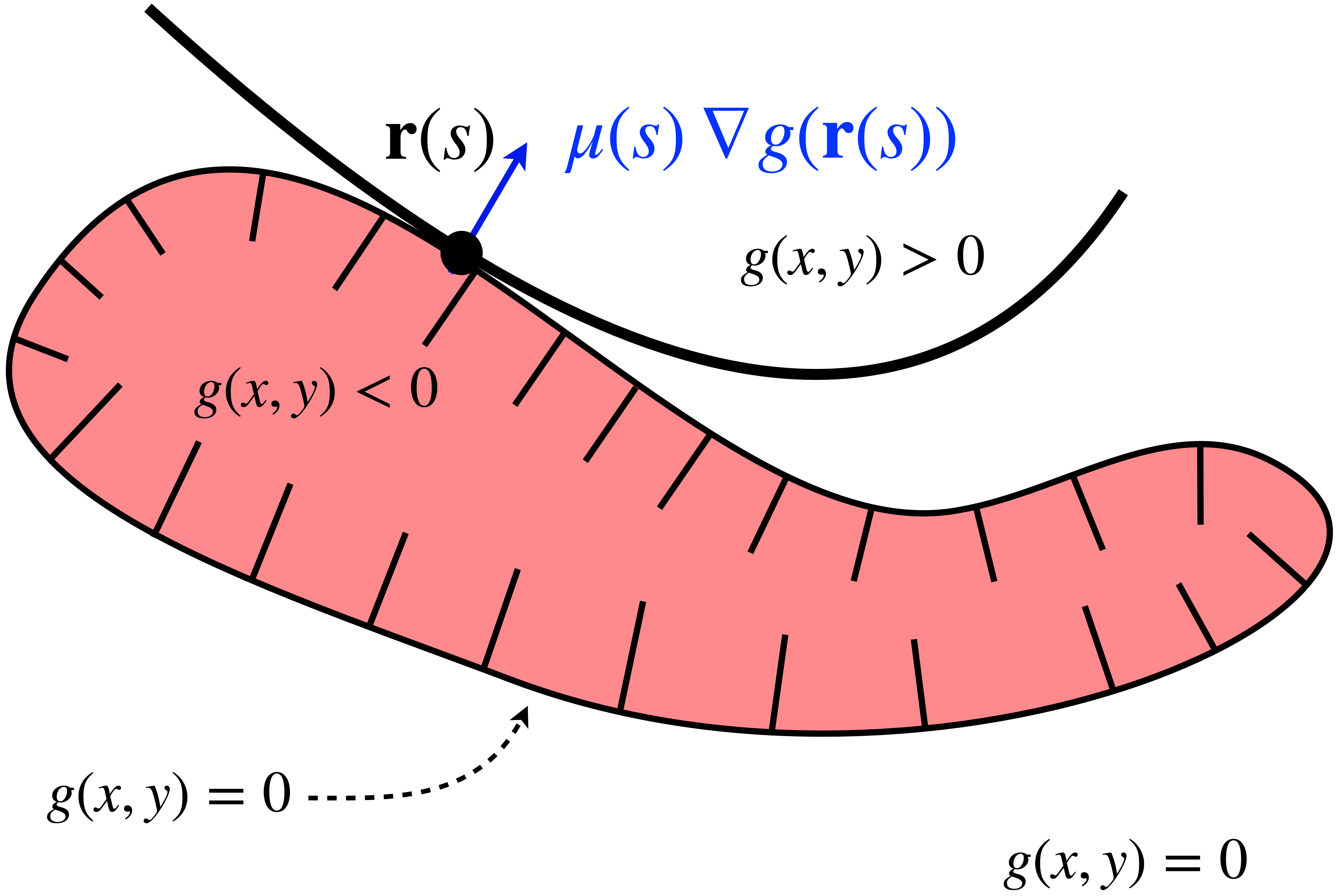
—> No!

Signorini law for frictionless contact

$$0 \leq g(\mathbf{r}(s)) \perp \mu(s) \geq 0$$

BACK TO CONTACT

New term in the Lagrangian: $\mathscr{L}_{\mathrm{C}} = -\mu(s)\, g(\mathbf{r}(s))$



Does it break Noether's invariant ?

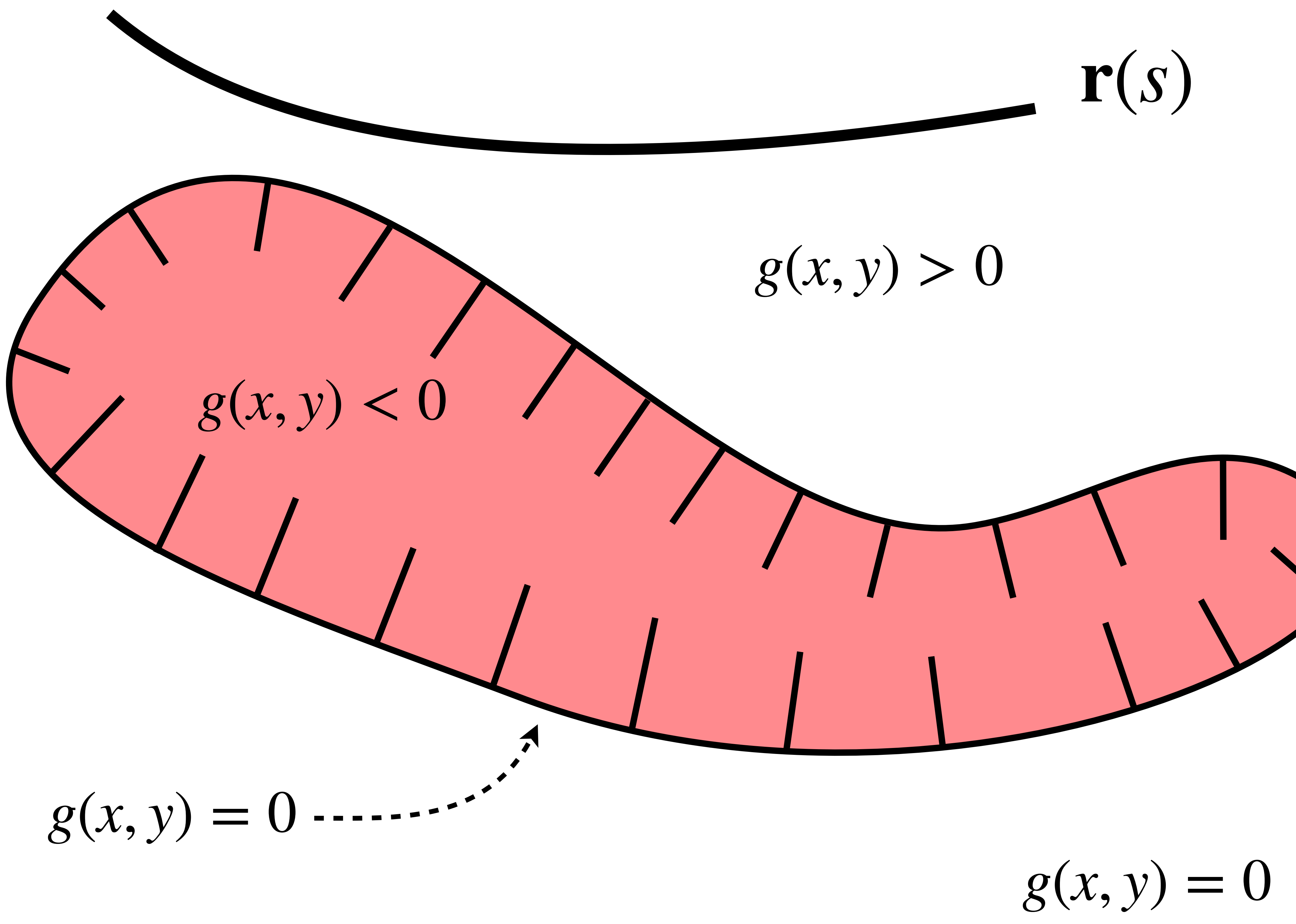
—> No!

Signorini law for frictionless contact

$$0 \leq g(\mathbf{r}(s)) \perp \mu(s) \geq 0$$

BACK TO CONTACT

New term in the Lagrangian: $\mathscr{L}_{\mathrm{C}} = -\mu(s)\, g(\mathbf{r}(s))$



Does it break Noether's invariant ?

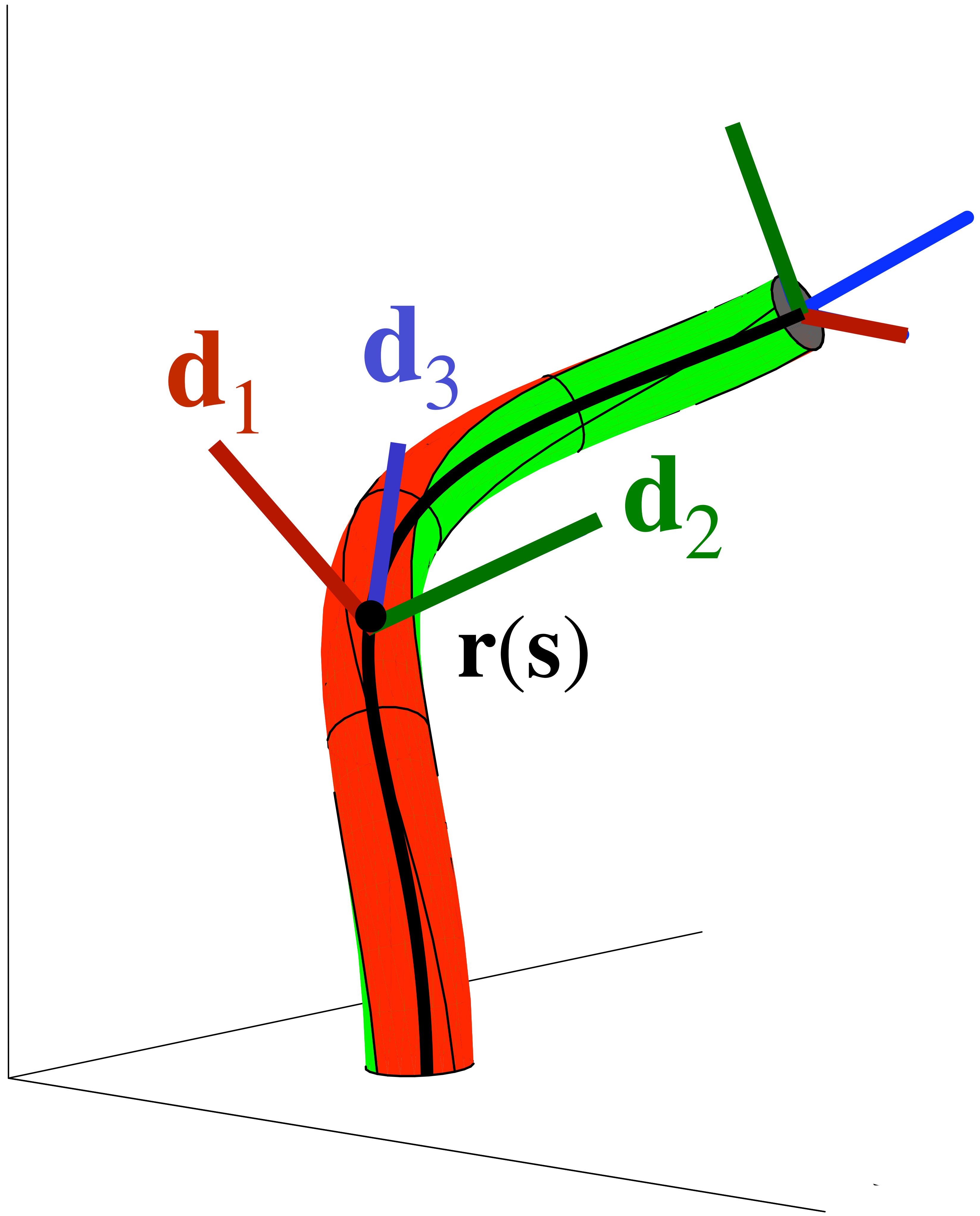
—> No!

Signorini law for frictionless contact

$$0 \leq g(\mathbf{r}(s)) \perp \mu(s) \geq 0$$

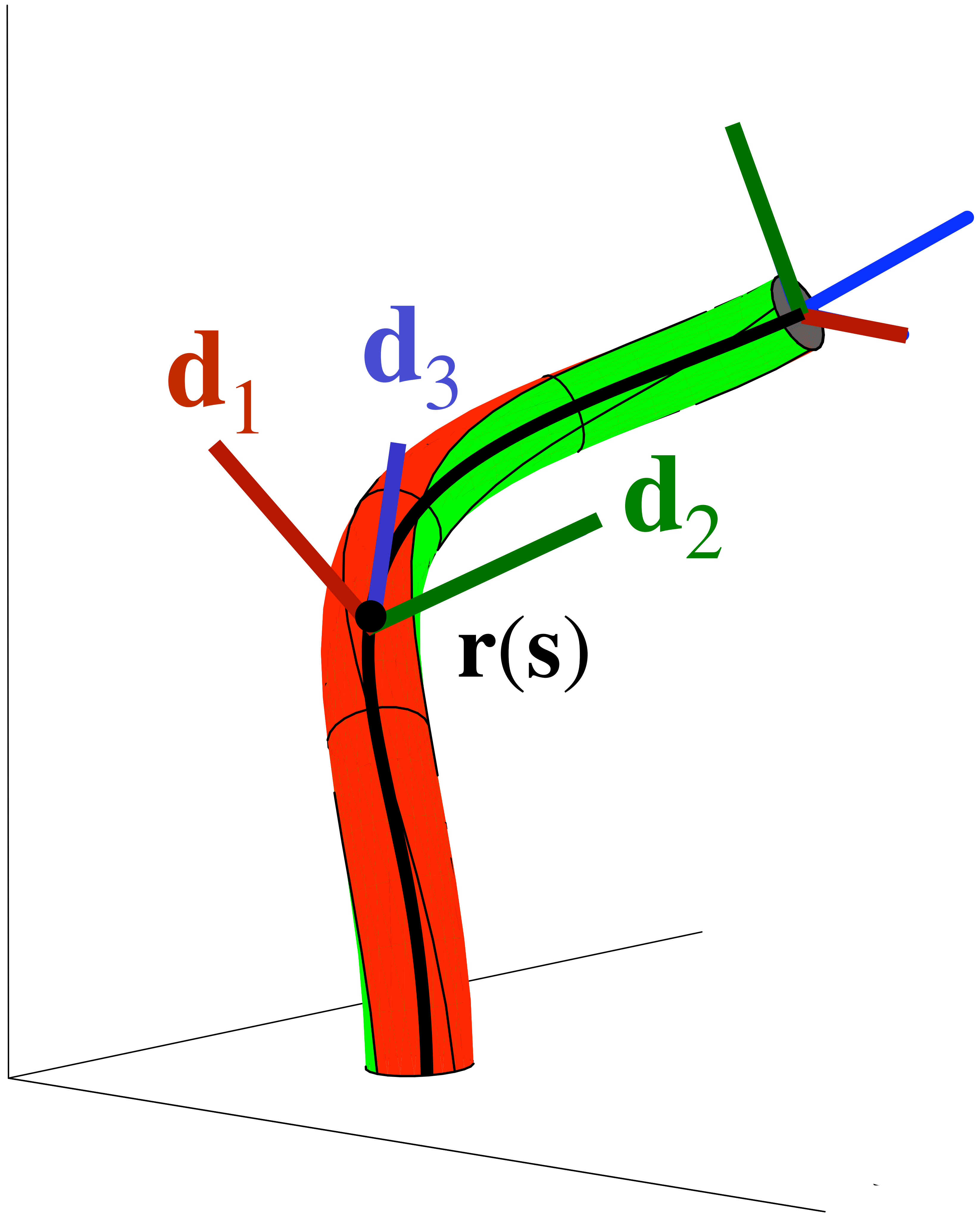
Moreover, H's formula is unchanged !

3D RODS: KINEMATICS



Centreline	$\mathbf{r}(\mathbf{s})$
Material frame	$\mathscr{R}(s) = \{\mathbf{d}_1(\mathbf{s}), \mathbf{d}_2(\mathbf{s}), \mathbf{d}_3(\mathbf{s})\}$

3D RODS: KINEMATICS



Centreline $\mathbf{r}(\mathbf{s})$

Material frame $\mathscr{R}(s) = \{\mathbf{d_1(s)}, \mathbf{d_2(s)}, \mathbf{d_3(s)}\}$

Two types of constraints:

1/ Coupling between $\mathbf{r}$ and $\mathscr{R}$

e.g.,

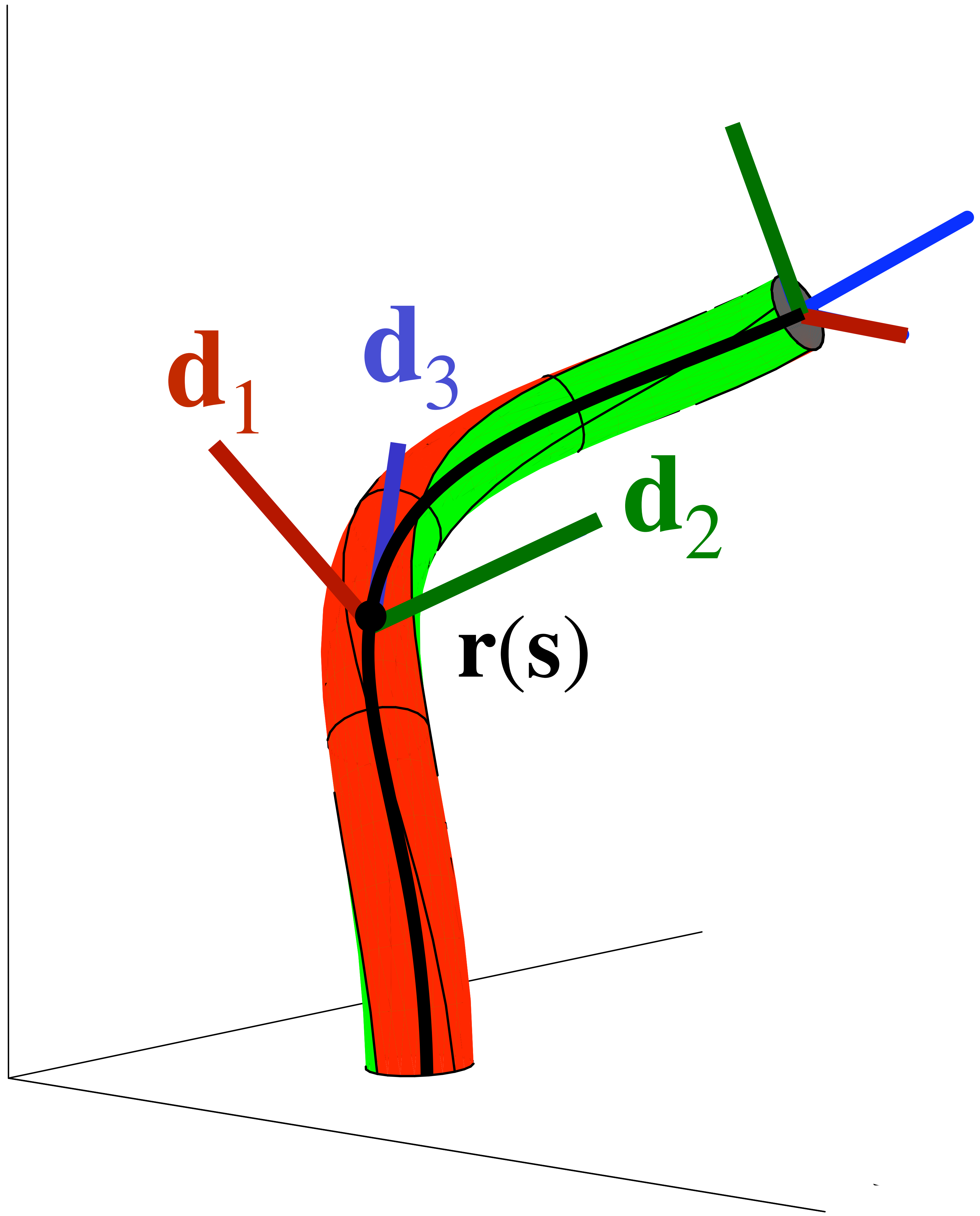
$$\mathbf{r}' = \mathbf{d}_3 \qquad \text{(Kirchhoff)}$$

or

$$\mathbf{r}' = \overset{\boxed{\text{shear}}}{\nu_1\,\mathbf{d}_1 + \nu_2\,\mathbf{d}_2} + \overset{\boxed{\text{1+extension}}}{\nu_3}\,\mathbf{d}_3 \qquad \text{(Reissner)}$$

→ General case handled here

3D RODS: KINEMATICS



Centreline $\mathbf{r(s)}$

Material frame $\mathscr{R}(s) = \{\mathbf{d}_1(s), \mathbf{d}_2(s), \mathbf{d}_3(s)\}$

Two types of constraints:

2/ $\mathscr{R} \in SO(3)$ Rotation group

→ Existence of a vector $\mathbf{u} = \{u_1, u_2, u_3\}_{d_j}$ such that

curvatures

twist

$$\begin{cases} \mathbf{d}_1' = \mathbf{u} \times \mathbf{d}_1 \\ \mathbf{d}_2' = \mathbf{u} \times \mathbf{d}_2 \\ \mathbf{d}_3' = \mathbf{u} \times \mathbf{d}_3 \end{cases}$$

(Darboux equations)

THE INVARIANT IN THE 3D CASE: TWO METHODS

Method $\mathbb{R}^n$ ('flattened') :

$\mathscr{L} = \mathscr{L}[\mathbf{q}(s), \mathbf{q}'(s)]$ with $\mathbf{q} = (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, u_1, u_2, u_3, v_1, v_2, v_3)$ and $\mathbf{q}' = (\mathbf{r}', \mathbf{d}_1', \mathbf{d}_2', \mathbf{d}_3', \ldots)$

→ Add constraints 1 and 2 in $\mathscr{L}$ with multipliers

Classical Noether invariant: $H = \dfrac{\partial \mathscr{L}}{\partial \mathbf{q}'} \cdot \mathbf{q}' - \mathscr{L}$

THE INVARIANT IN THE 3D CASE: TWO METHODS

Method $\mathbb{R}^n$ ('flattened') :

$\mathscr{L} = \mathscr{L}[\mathbf{q}(s), \mathbf{q}'(s)]$ with $\mathbf{q} = (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, u_1, u_2, u_3, v_1, v_2, v_3)$ and $\mathbf{q}' = (\mathbf{r}', \mathbf{d}_1', \mathbf{d}_2', \mathbf{d}_3', \dots)$

→ Add constraints 1 and 2 in $\mathscr{L}$ with multipliers

Classical Noether invariant: $H = \dfrac{\partial \mathscr{L}}{\partial \mathbf{q}'} \cdot \mathbf{q}' - \mathscr{L}$

Method $\mathbb{R}^n \times SO(3)$ ('compact') :

$\mathscr{L} = \mathscr{L}[\mathbf{q}(s), \tilde{\mathbf{q}}(s)]$ with $\mathbf{q} = (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, v_1, v_2, v_3)$ and $\tilde{\mathbf{q}} = (\mathbf{r}', u_1, u_2, u_3, \dots) \neq \mathbf{q}'$

→ Add only constraint 1 in $\mathscr{L}$ with multipliers

New Noether invariant, valid on $\mathbb{R}^n \times SO(3)$: $H = \dfrac{\partial \mathscr{L}}{\partial \mathbf{r}'} \cdot \mathbf{r}' + \dfrac{\partial \mathscr{L}}{\partial \bar{\mathbf{u}}} \cdot \bar{\mathbf{u}} - \mathscr{L}$

THE INVARIANT IN THE 3D CASE: TWO METHODS

Method $\mathbb{R}^n$ ('flattened') :

$\mathscr{L} = \mathscr{L}[\mathbf{q}(s), \mathbf{q}'(s)]$ with $\mathbf{q} = (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, u_1, u_2, u_3, v_1, v_2, v_3)$ and $\mathbf{q}' = (\mathbf{r}', \mathbf{d}_1', \mathbf{d}_2', \mathbf{d}_3', \ldots)$

→ Add constraints 1 and 2 in $\mathscr{L}$ with multipliers

Classical Noether invariant: $H = \dfrac{\partial \mathscr{L}}{\partial \mathbf{q}'} \cdot \mathbf{q}' - \mathscr{L}$

Method $\mathbb{R}^n \times SO(3)$ ('compact') :

$\mathscr{L} = \mathscr{L}[\mathbf{q}(s), \tilde{\mathbf{q}}(s)]$ with $\mathbf{q} = (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, v_1, v_2, v_3)$ and $\tilde{\mathbf{q}} = (\mathbf{r}', u_1, u_2, u_3, \ldots) \neq \mathbf{q}'$

→ Add only constraint 1 in $\mathscr{L}$ with multipliers

New Noether invariant, valid on $\mathbb{R}^n \times SO(3)$: $H = \dfrac{\partial \mathscr{L}}{\partial \mathbf{r}'} \cdot \mathbf{r}' + \dfrac{\partial \mathscr{L}}{\partial \bar{\mathbf{u}}} \cdot \bar{\mathbf{u}} - \mathscr{L}$ $\quad \begin{cases} \bar{\mathbf{u}} = \{u_1, u_2, u_3\} \\ \text{s.t.} \quad \mathbf{u} = \mathscr{R}\,\bar{\mathbf{u}} \end{cases}$

THE INVARIANT IN THE 3D CASE: TWO METHODS

Method $\mathbb{R}^n$ ('flattened') :

$\mathscr{L} = \mathscr{L}[\mathbf{q}(s), \mathbf{q}'(s)]$ with $\mathbf{q} = (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, u_1, u_2, u_3, v_1, v_2, v_3)$ and $\mathbf{q}' = (\mathbf{r}', \mathbf{d}_1', \mathbf{d}_2', \mathbf{d}_3', \ldots)$

→ Add constraints 1 and 2 in $\mathscr{L}$ with multipliers

Classical Noether invariant: $H = \dfrac{\partial \mathscr{L}}{\partial \mathbf{q}'} \cdot \mathbf{q}' - \mathscr{L}$

Method $\mathbb{R}^n \times SO(3)$ ('compact') :

$\mathscr{L} = \mathscr{L}[\mathbf{q}(s), \tilde{\mathbf{q}}(s)]$ with $\mathbf{q} = (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, v_1, v_2, v_3)$ and $\tilde{\mathbf{q}} = (\mathbf{r}', u_1, u_2, u_3, \ldots) \neq \mathbf{q}'$

→ Add only constraint 1 in $\mathscr{L}$ with multipliers

New Noether invariant, valid on $\mathbb{R}^n \times SO(3)$: $H = \underbrace{\dfrac{\partial \mathscr{L}}{\partial \mathbf{r}'} \cdot \mathbf{r}'}_{\text{translation}} + \underbrace{\dfrac{\partial \mathscr{L}}{\partial \bar{\mathbf{u}}} \cdot \bar{\mathbf{u}}}_{\text{rotation}} - \mathscr{L} \quad \begin{cases} \bar{\mathbf{u}} = \{u_1, u_2, u_3\} \\ \text{s.t.} \quad \mathbf{u} = \mathscr{R}\,\bar{\mathbf{u}} \end{cases}$

THE INVARIANT IN THE 3D CASE

$$\left.\begin{aligned} W_{bend} &= \frac{1}{2}B_1\left(u_1(s)-\hat{u}_1\right)^2+\frac{1}{2}B_2\left(u_2(s)-\hat{u}_2\right)^2+\frac{1}{2}B_3\left(u_3(s)-\hat{u}_3\right)^2 \\ W_{shear} &= \frac{1}{2}A_1\left(v_1(s)-\hat{v}_1\right)^2+\frac{1}{2}A_2\left(v_2(s)-\hat{v}_2\right)^2+\frac{1}{2}A_3\left(v_3(s)-\hat{v}_3\right)^2 \end{aligned}\right\}\ \text{Reissner model}$$

THE INVARIANT IN THE 3D CASE

$$\left.\begin{aligned} W_{bend} &= \frac{1}{2}B_1\left(u_1(s)-\hat{u}_1\right)^2+\frac{1}{2}B_2\left(u_2(s)-\hat{u}_2\right)^2+\frac{1}{2}B_3\left(u_3(s)-\hat{u}_3\right)^2 \\ W_{shear} &= \frac{1}{2}A_1\left(v_1(s)-\hat{v}_1\right)^2+\frac{1}{2}A_2\left(v_2(s)-\hat{v}_2\right)^2+\frac{1}{2}A_3\left(v_3(s)-\hat{v}_3\right)^2 \end{aligned}\right\}\text{ Reissner model}$$

Gravity
Electrostatics
Soft contact

$$\mathscr{L} = W_{bend} + W_{shear} + V(\mathbf{r}(s)) +$$

THE INVARIANT IN THE 3D CASE

$$W_{bend} = \frac{1}{2}B_1\left(u_1(s) - \hat{u}_1\right)^2 + \frac{1}{2}B_2\left(u_2(s) - \hat{u}_2\right)^2 + \frac{1}{2}B_3\left(u_3(s) - \hat{u}_3\right)^2$$

$$W_{shear} = \frac{1}{2}A_1\left(v_1(s) - \hat{v}_1\right)^2 + \frac{1}{2}A_2\left(v_2(s) - \hat{v}_2\right)^2 + \frac{1}{2}A_3\left(v_3(s) - \hat{v}_3\right)^2$$

} Reissner model

Gravity
Electrostatics
Soft contact

$$\mathscr{L} = W_{bend} + W_{shear} + V(\mathbf{r}(s)) +$$

constraints

$$\lambda_{\mathbf{r}} \cdot \left(\mathbf{r}'(s) - v_1(s)\,\mathbf{d}_1 - v_2(s)\,\mathbf{d}_2 - v_3(s)\,\mathbf{d}_3\right) + \lambda_{u_1}\left(\mathbf{d}_2' \cdot \mathbf{d}_3 - u_1\right) + \lambda_{u_2}\left(\mathbf{d}_3' \cdot \mathbf{d}_1 - u_2\right) + \lambda_{u_3}\left(\mathbf{d}_1' \cdot \mathbf{d}_2 - u_3\right) +$$

$$\lambda_{11}\frac{1}{2}\left(1 - \mathbf{d}_1 \cdot \mathbf{d}_1\right) + \lambda_{22}\frac{1}{2}\left(1 - \mathbf{d}_2 \cdot \mathbf{d}_2\right) + \lambda_{33}\frac{1}{2}\left(1 - \mathbf{d}_3 \cdot \mathbf{d}_3\right) + \lambda_{12}\,\mathbf{d}_1 \cdot \mathbf{d}_2 + \lambda_{23}\,\mathbf{d}_2 \cdot \mathbf{d}_3 + \lambda_{31}\,\mathbf{d}_3 \cdot \mathbf{d}_1$$

where $\mathscr{L} = \mathscr{L}[\mathbf{q}(s), \mathbf{q}'(s)]$ with $\mathbf{q} = (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, u_1, u_2, u_3, v_1, v_2, v_3)$ and $\mathbf{q}' = (\mathbf{r}', \mathbf{d}_1', \mathbf{d}_2', \mathbf{d}_3', \ldots)$

Method $\mathbb{R}^n$

$$H = \frac{\partial \mathscr{L}}{\partial \mathbf{q}'} \cdot \mathbf{q}' - \mathscr{L}$$

THE INVARIANT IN THE 3D CASE

$$\left.\begin{aligned} W_{bend} &= \frac{1}{2}B_1\left(u_1(s)-\hat{u}_1\right)^2+\frac{1}{2}B_2\left(u_2(s)-\hat{u}_2\right)^2+\frac{1}{2}B_3\left(u_3(s)-\hat{u}_3\right)^2 \\ W_{shear} &= \frac{1}{2}A_1\left(v_1(s)-\hat{v}_1\right)^2+\frac{1}{2}A_2\left(v_2(s)-\hat{v}_2\right)^2+\frac{1}{2}A_3\left(v_3(s)-\hat{v}_3\right)^2 \end{aligned}\right\}$$

Reissner model

Gravity
Electrostatics
Soft contact

$$\mathscr{L} = W_{bend} + W_{shear} + V(\mathbf{r}(s)) +$$

$$\lambda_{\mathbf{r}} \cdot \left(\mathbf{r}'(s) - v_1(s)\,\mathbf{d}_1 - v_2(s)\,\mathbf{d}_2 - v_3(s)\,\mathbf{d}_3\right)$$

constraints

where $\mathscr{L} = \mathscr{L}[\mathbf{q}(s), \tilde{\mathbf{q}}(s)]$ with $\mathbf{q} = (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, v_1, v_2, v_3)$ and $\tilde{\mathbf{q}} = (\mathbf{r}', u_1, u_2, u_3, \ldots) \neq \mathbf{q}'$

Method $\mathbb{R}^n \times SO(3)$

$$H = \frac{\partial \mathscr{L}}{\partial \mathbf{r}'} \cdot \mathbf{r}' + \frac{\partial \mathscr{L}}{\partial \bar{\mathbf{u}}} \cdot \bar{\mathbf{u}} - \mathscr{L}$$

THE INVARIANT IN THE 3D CASE

$$\left.\begin{aligned} W_{bend} &= \frac{1}{2}B_1\left(u_1(s)-\hat{u}_1\right)^2+\frac{1}{2}B_2\left(u_2(s)-\hat{u}_2\right)^2+\frac{1}{2}B_3\left(u_3(s)-\hat{u}_3\right)^2 \\ W_{shear} &= \frac{1}{2}A_1\left(v_1(s)-\hat{v}_1\right)^2+\frac{1}{2}A_2\left(v_2(s)-\hat{v}_2\right)^2+\frac{1}{2}A_3\left(v_3(s)-\hat{v}_3\right)^2 \end{aligned}\right\} \text{Reissner model}$$

Gravity

Electrostatics

Soft contact

$$\mathscr{L} = W_{bend} + W_{shear} + V(\mathbf{r}(s)) +$$

constraints

$$\lambda_{\mathbf{r}} \cdot \left(\mathbf{r}'(s) - v_1(s)\,\mathbf{d}_1 - v_2(s)\,\mathbf{d}_2 - v_3(s)\,\mathbf{d}_3\right)$$

where $\mathscr{L} = \mathscr{L}[\mathbf{q}(s), \tilde{\mathbf{q}}(s)]$ with $\mathbf{q} = (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, v_1, v_2, v_3)$ and $\tilde{\mathbf{q}} = (\mathbf{r}', u_1, u_2, u_3, \ldots) \neq \mathbf{q}'$

Method $\mathbb{R}^n \times SO(3)$

$$H = \boxed{\frac{\partial \mathscr{L}}{\partial \mathbf{r}'}} \cdot \mathbf{r}' + \boxed{\frac{\partial \mathscr{L}}{\partial \bar{\mathbf{u}}}} \cdot \bar{\mathbf{u}} - \mathscr{L}$$

Internal force

Internal moment

HOW DOES THE TENSION ENTER THE INVARIANT FORMULA (3D)

Noether invariant:

$$
\begin{aligned}
H &= \mathbf{n}\cdot\mathbf{v} + \mathbf{m}\cdot\mathbf{u} - W_{bend} - W_{shear} - V(\mathbf{r}) \\
&= \frac{1}{2}B_i\,\left[u_i^2(s) - \hat{u}_i^2\right] + \frac{1}{2}A_i\left[v_i^2(s) - \hat{v}_i^2\right] - V(\mathbf{r}) \\
&= \frac{1}{2}B_i\,\left[u_i(s) - \hat{u}_i\right]^2 + m_i\,\hat{u}_i + \frac{1}{2}A_i\left[v_i(s) - \hat{v}_i\right]^2 + n_i\,\hat{v}_i - V(\mathbf{r})
\end{aligned}
$$

HOW DOES THE TENSION ENTER THE INVARIANT FORMULA (3D)

Noether invariant:

$$\left.\begin{aligned} H &= \mathbf{n}\cdot\mathbf{v}+\mathbf{m}\cdot\mathbf{u}-W_{bend}-W_{shear}-V(\mathbf{r}) \\ &= \frac{1}{2}B_i\left[u_i^2(s)-\hat{u}_i^2\right]+\frac{1}{2}A_i\left[v_i^2(s)-\hat{v}_i^2\right]-V(\mathbf{r}) \\ &= \frac{1}{2}B_i\left[u_i(s)-\hat{u}_i\right]^2+m_i\,\hat{u}_i+\frac{1}{2}A_i\left[v_i(s)-\hat{v}_i\right]^2+n_i\,\hat{v}_i-V(\mathbf{r}) \end{aligned}\right\}$$

general case

not the energies !

How does the Tension enter the Invariant Formula (3D)

Noether invariant:

$$
\begin{aligned}
H &= \mathbf{n}\cdot\mathbf{v} + \mathbf{m}\cdot\mathbf{u} - W_{bend} - W_{shear} - V(\mathbf{r}) \\
&= \frac{1}{2}B_i\left[u_i^2(s) - \hat{u}_i^2\right] + \frac{1}{2}A_i\left[v_i^2(s) - \hat{v}_i^2\right] - V(\mathbf{r}) \\
&= \frac{1}{2}B_i\left[u_i(s) - \hat{u}_i\right]^2 + m_i\hat{u}_i + \frac{1}{2}A_i\left[v_i(s) - \hat{v}_i\right]^2 + n_i\hat{v}_i - V(\mathbf{r})
\end{aligned}
$$

general case

not the energies !

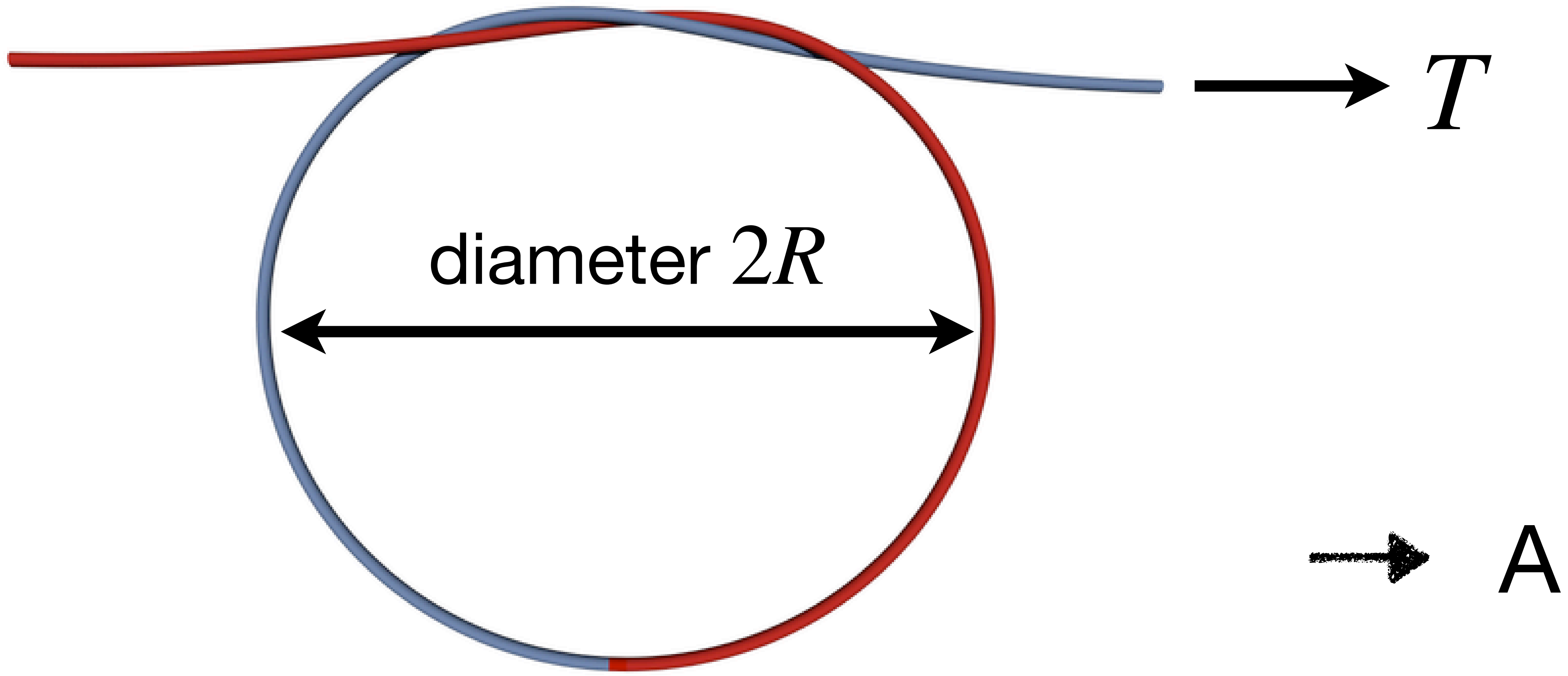
$$
\begin{aligned}
&= \frac{1}{2}B_i u_i^2(s) + \frac{1}{2}A_i v_i^2(s) + n_3 - V(\mathbf{r}) \\
&= W_{bend} + W_{shear} + tension - V(\mathbf{r})
\end{aligned}
$$

if $\hat{u}_i = 0$ and $\hat{v}_{1,2} = 0$ and $\hat{v}_3 = 1$

(naturally flat rod, with naturally straight cross-section)

Maddocks 1994
(Intuited case $V = 0$)

MAKING USE OF THE INVARIANT IN 3D: THE TREFOIL KNOT

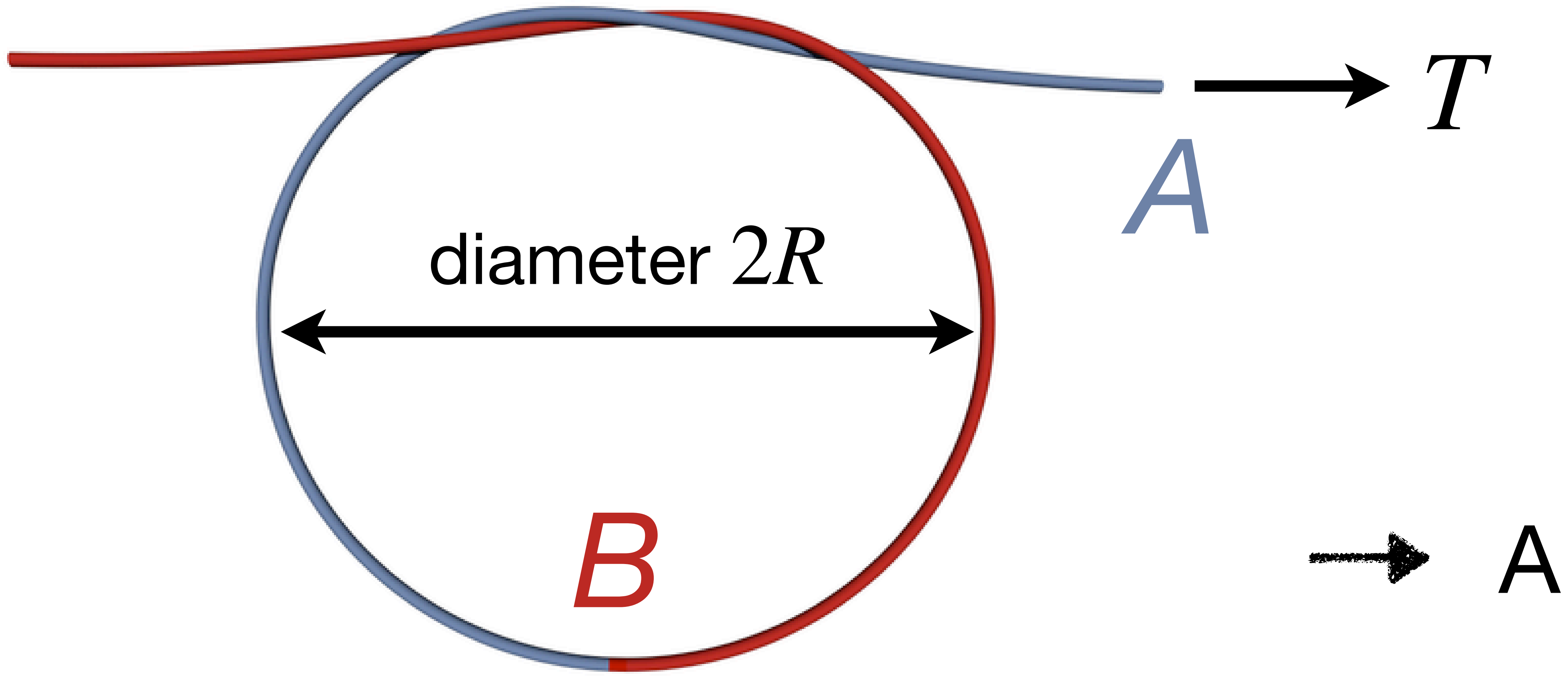


Limit case of vanishing thickness

$$H_{3D} = curvature + twist + tension$$

→ A simple way to retrieve quickly a theory

MAKING USE OF THE INVARIANT IN 3D: THE TREFOIL KNOT



Limit case of vanishing thickness

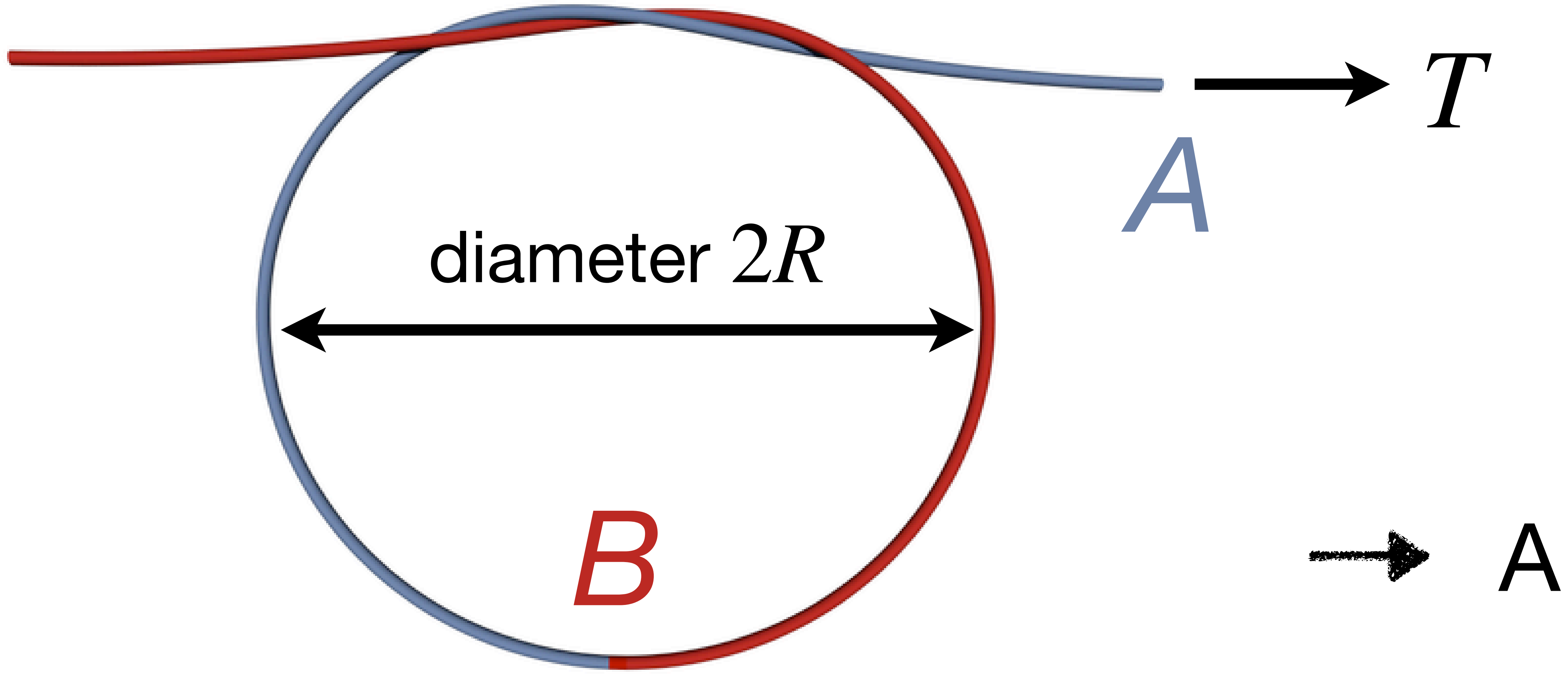
$$H_{3D} = curvature + twist + tension$$

→ A simple way to retrieve quickly a theory

$$H_A = 0 + 0 + T$$

$$H_B = \frac{1}{2}\frac{EI}{R^2} + 0 + 0$$

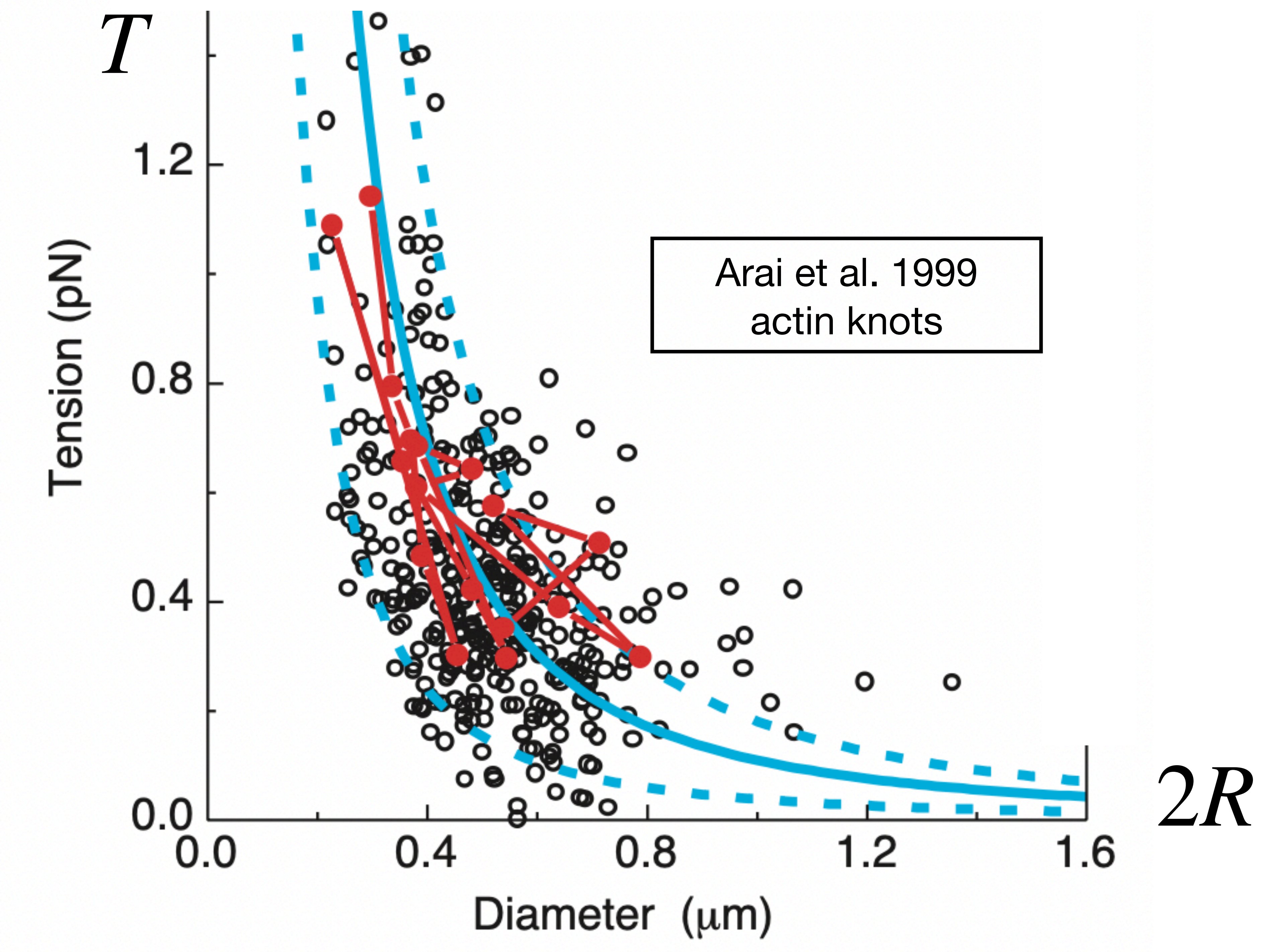
MAKING USE OF THE INVARIANT IN 3D: THE TREFOIL KNOT



Limit case of vanishing thickness

$$H_{3D} = curvature + twist + tension$$

→ A simple way to retrieve quickly a theory

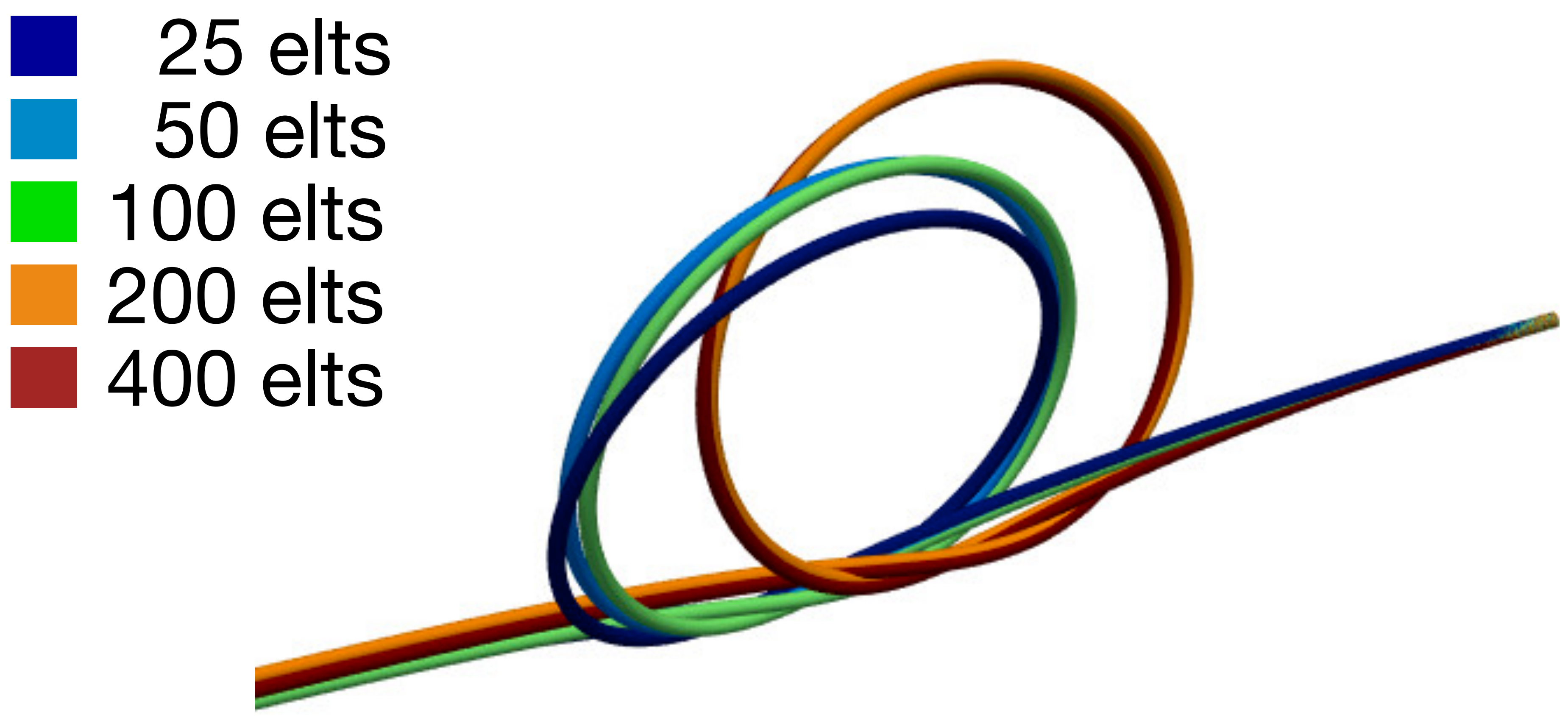


$$H_A = 0 + 0 + T$$

$$H_B = \frac{1}{2}\frac{EI}{R^2} + 0 + 0$$

$$H_A = H_B \Rightarrow T = \frac{1}{2}\frac{EI}{R^2}$$

MAKING USE OF THE INVARIANT IN 3D: THE TREFOIL KNOT



Super-helix model
+ non-smooth contact solver

Joint work with T. Metivet and V. Romero (Inria)

Case of finite thickness (simulation)

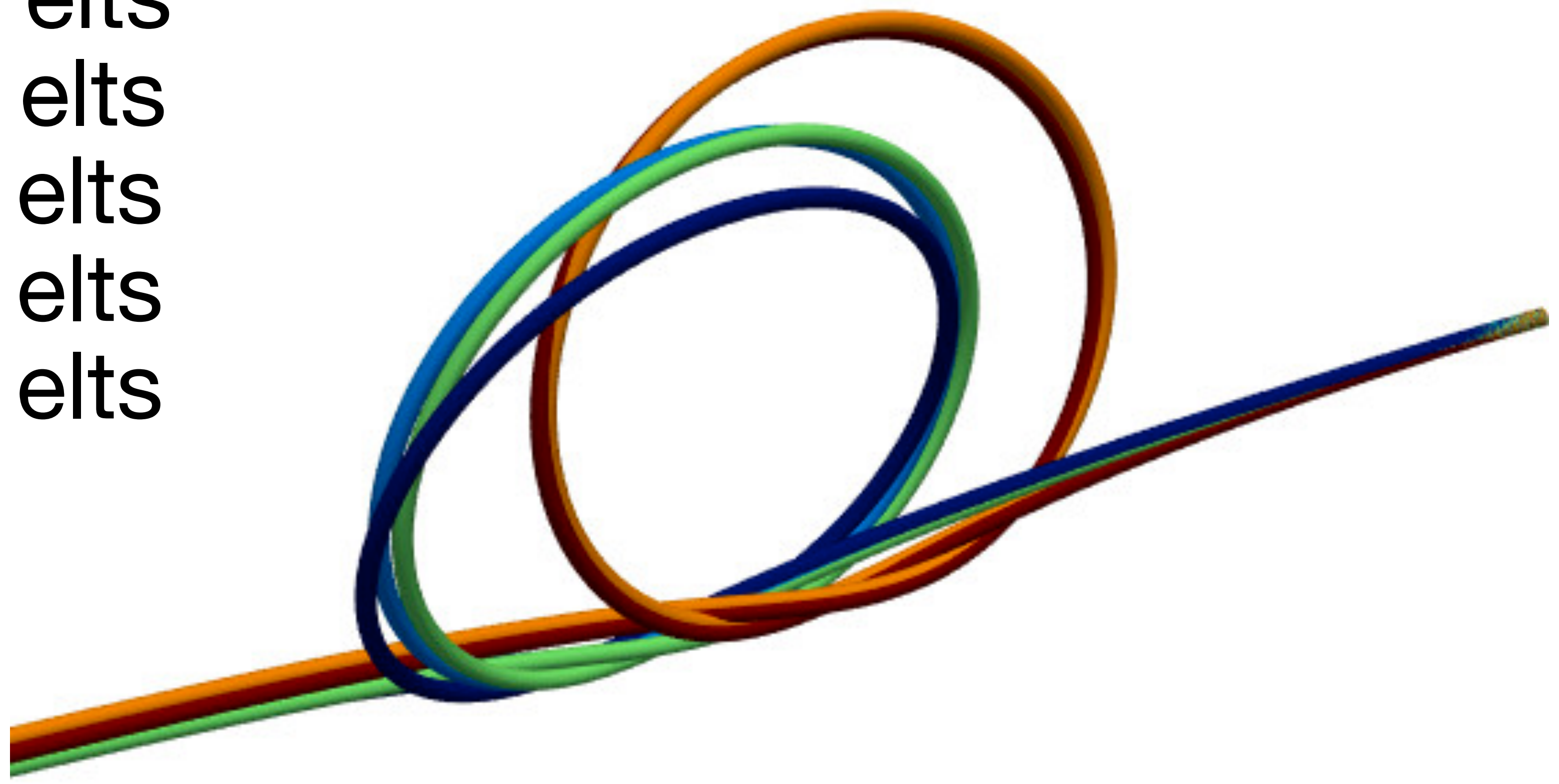
$$H_{3D} = curvature + twist + tension$$

→ A simple way to **validate** a simulator

MAKING USE OF THE INVARIANT IN 3D: THE TREFOIL KNOT

Joint work with T. Metivet and V. Romero (Inria)

🟦 25 elts
🟦 50 elts
🟩 100 elts
🟧 200 elts
🟥 400 elts

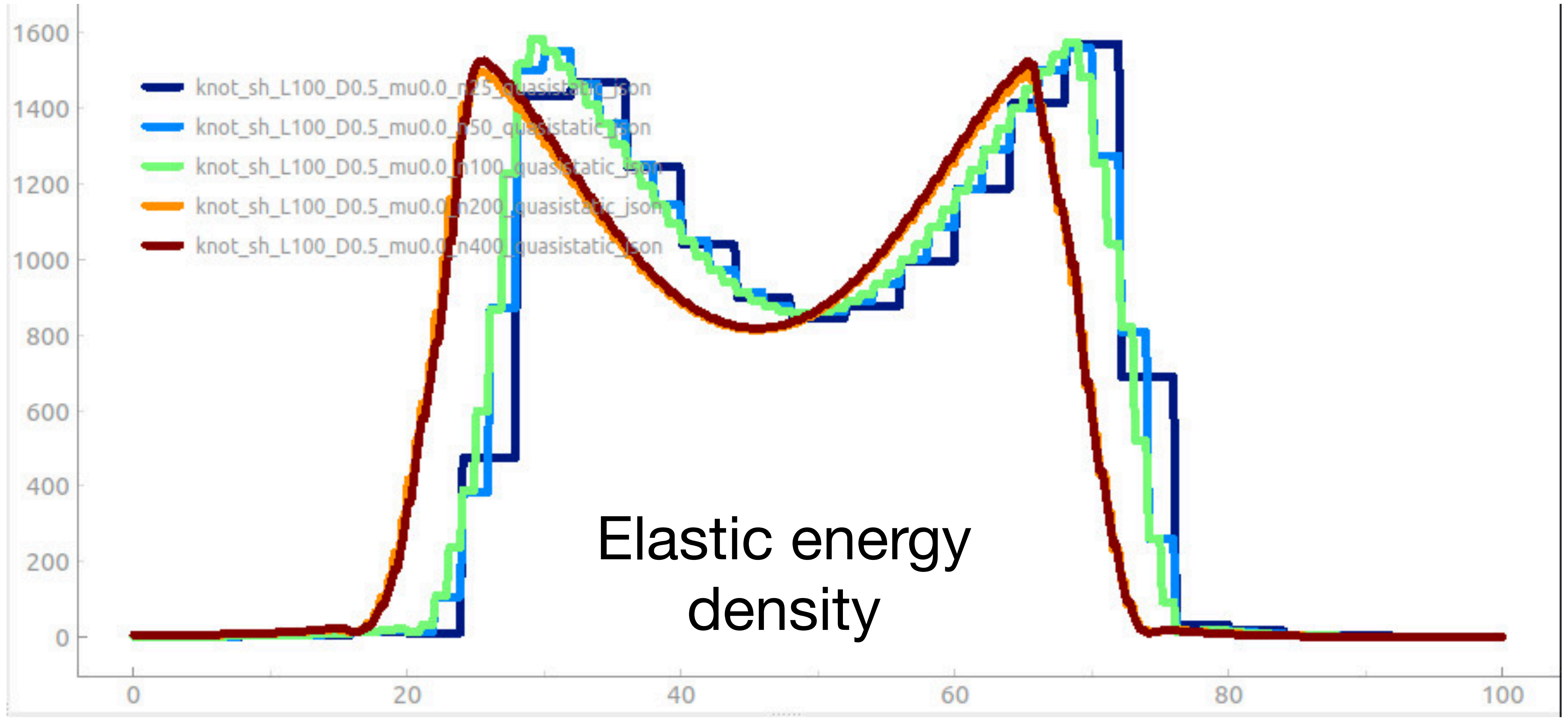


Super-helix model
+ non-smooth contact solver

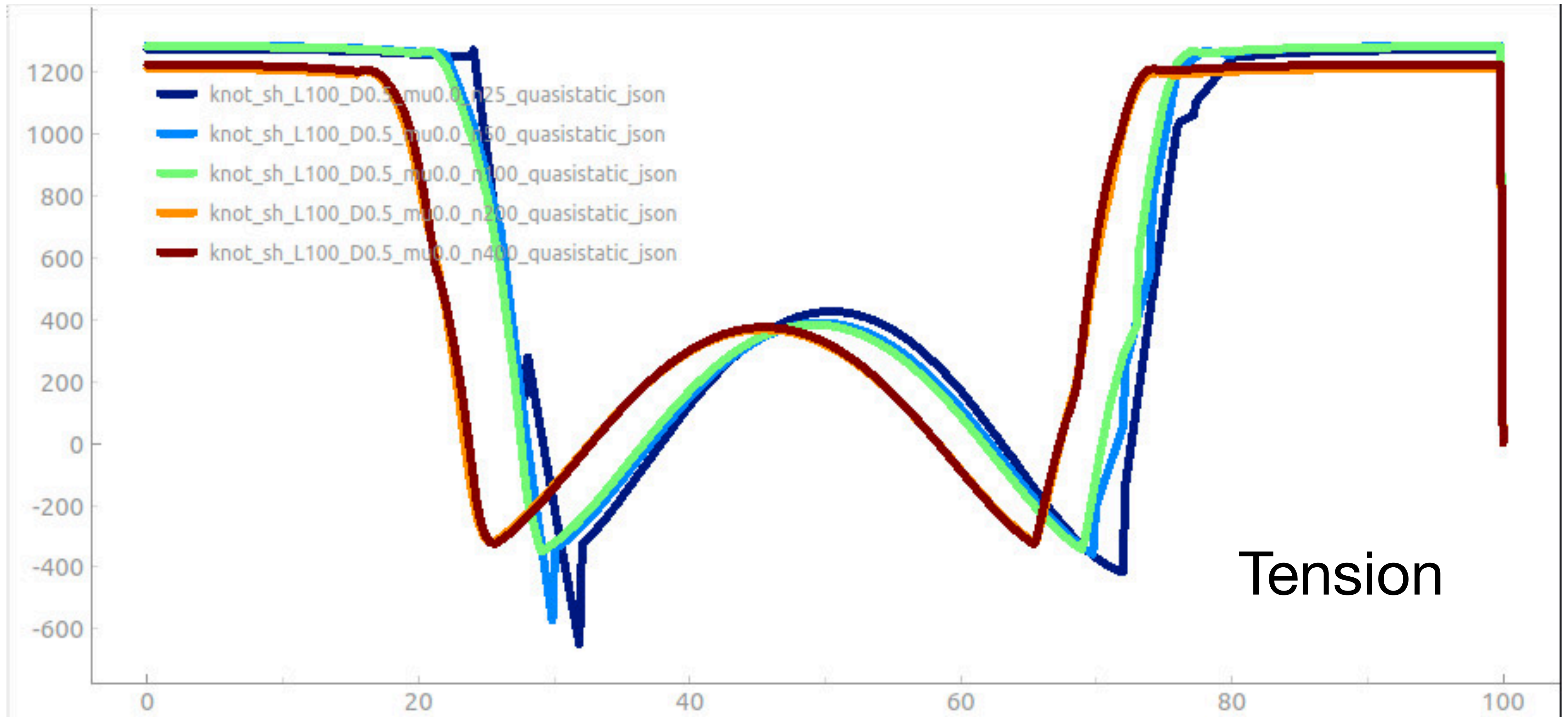
Case of finite thickness (simulation)

$$H_{3D} = \mathit{curvature} + \mathit{twist} + \mathit{tension}$$

→ A simple way to **validate** a simulator



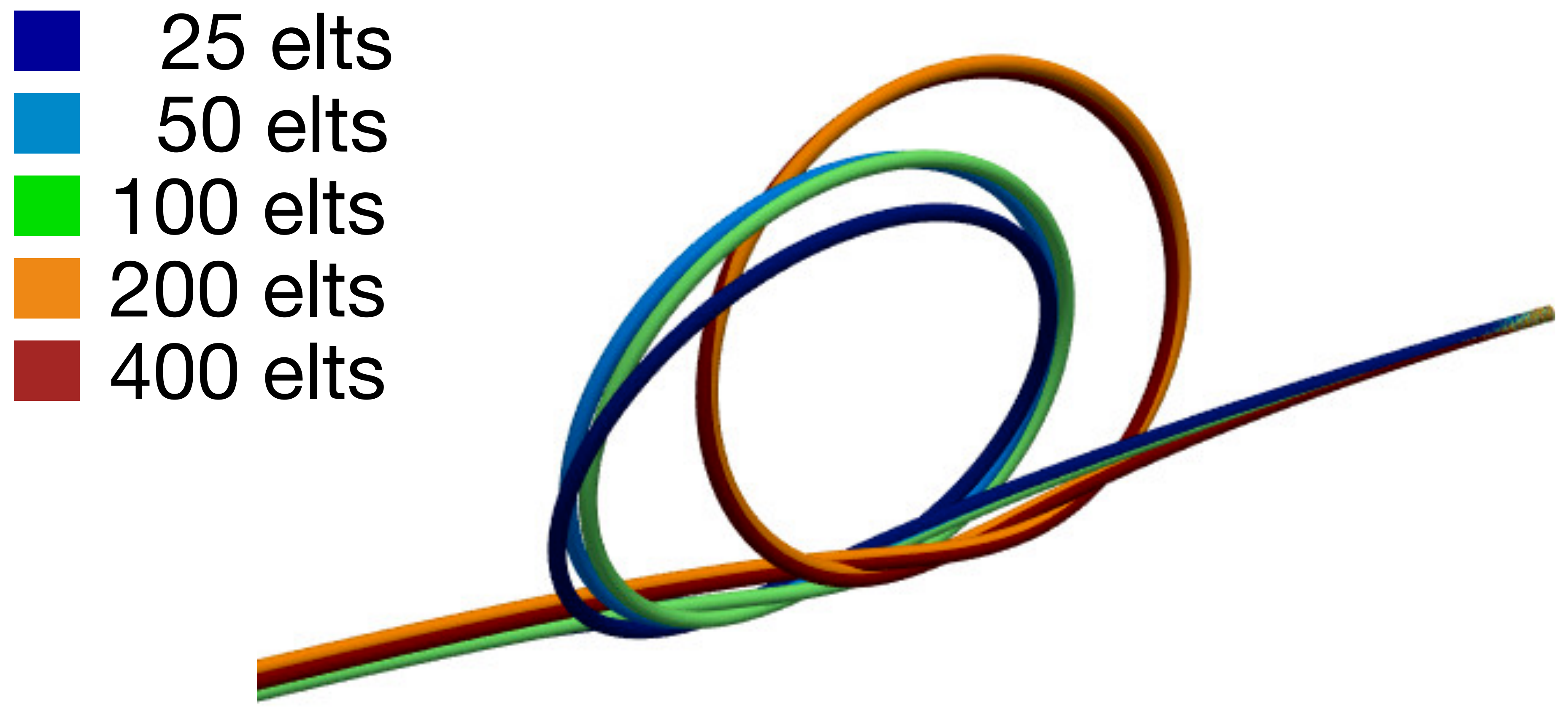
Elastic energy density



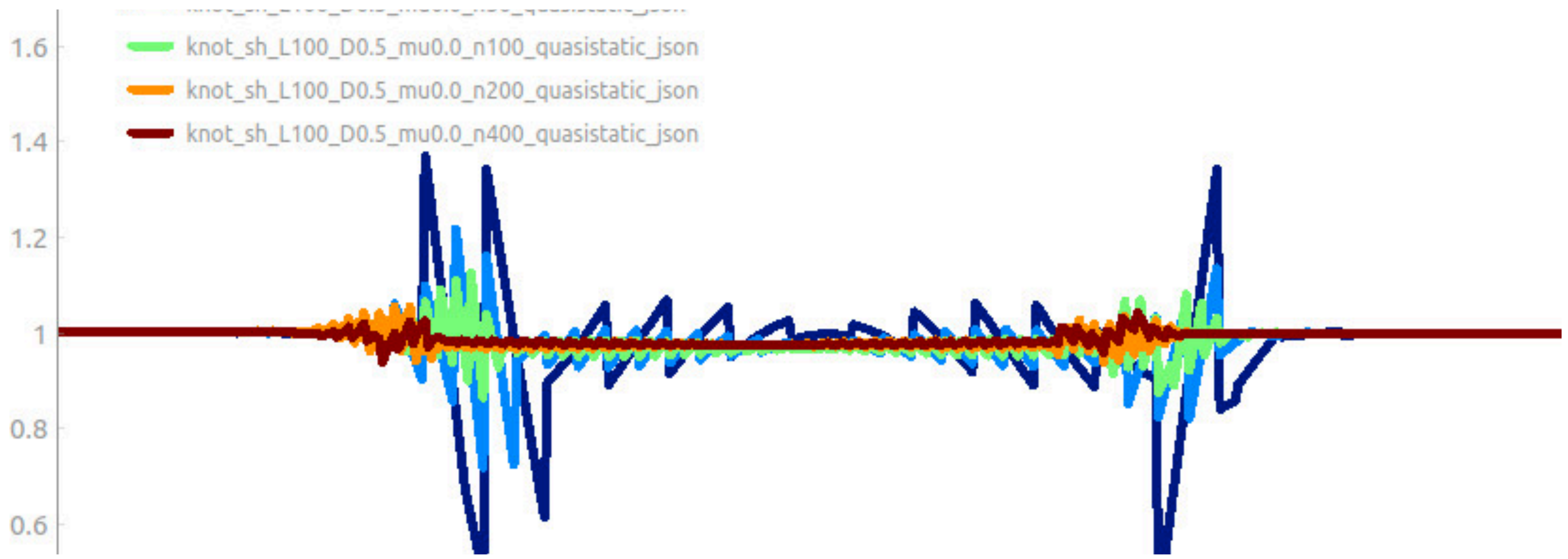
Tension

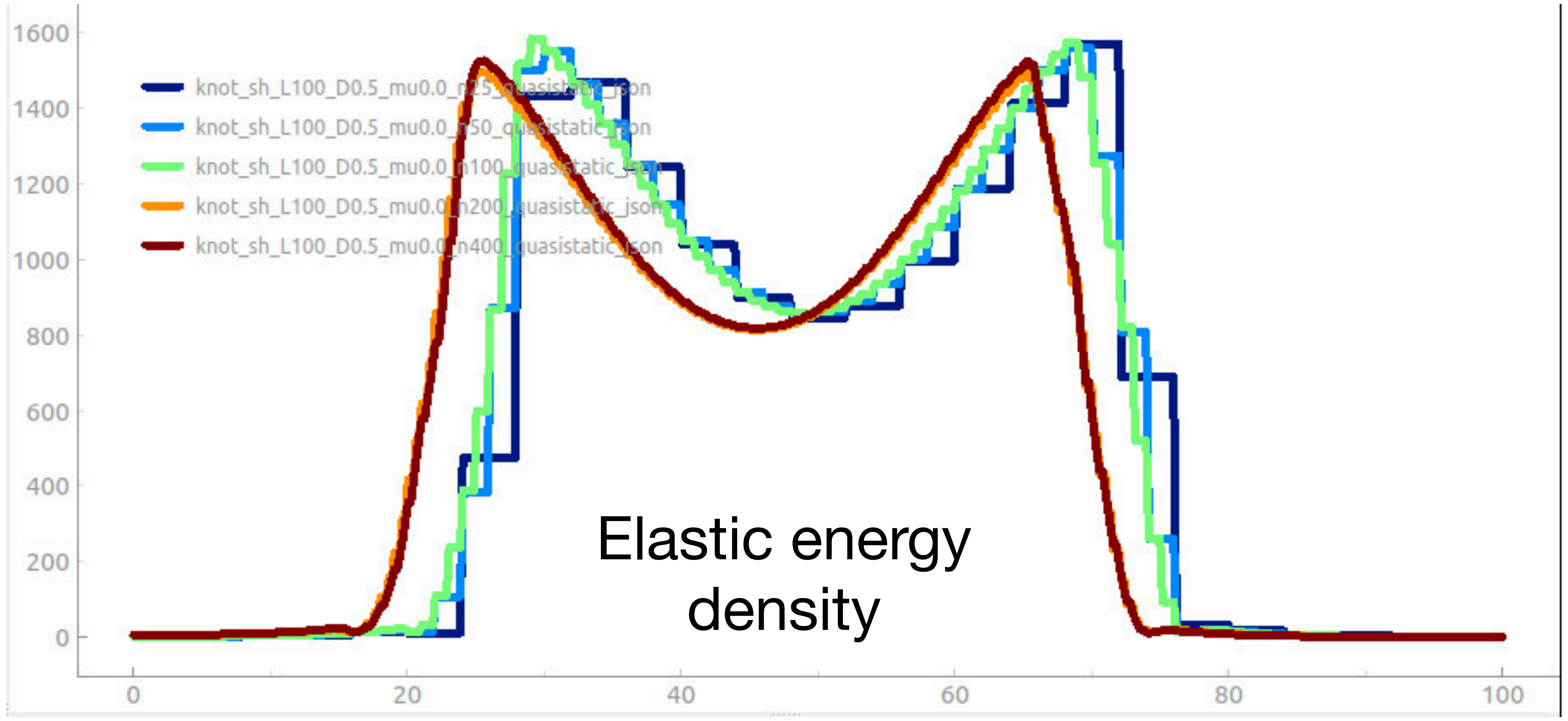
MAKING USE OF THE INVARIANT IN 3D: THE TREFOIL KNOT

Joint work with T. Metivet and V. Romero (Inria)

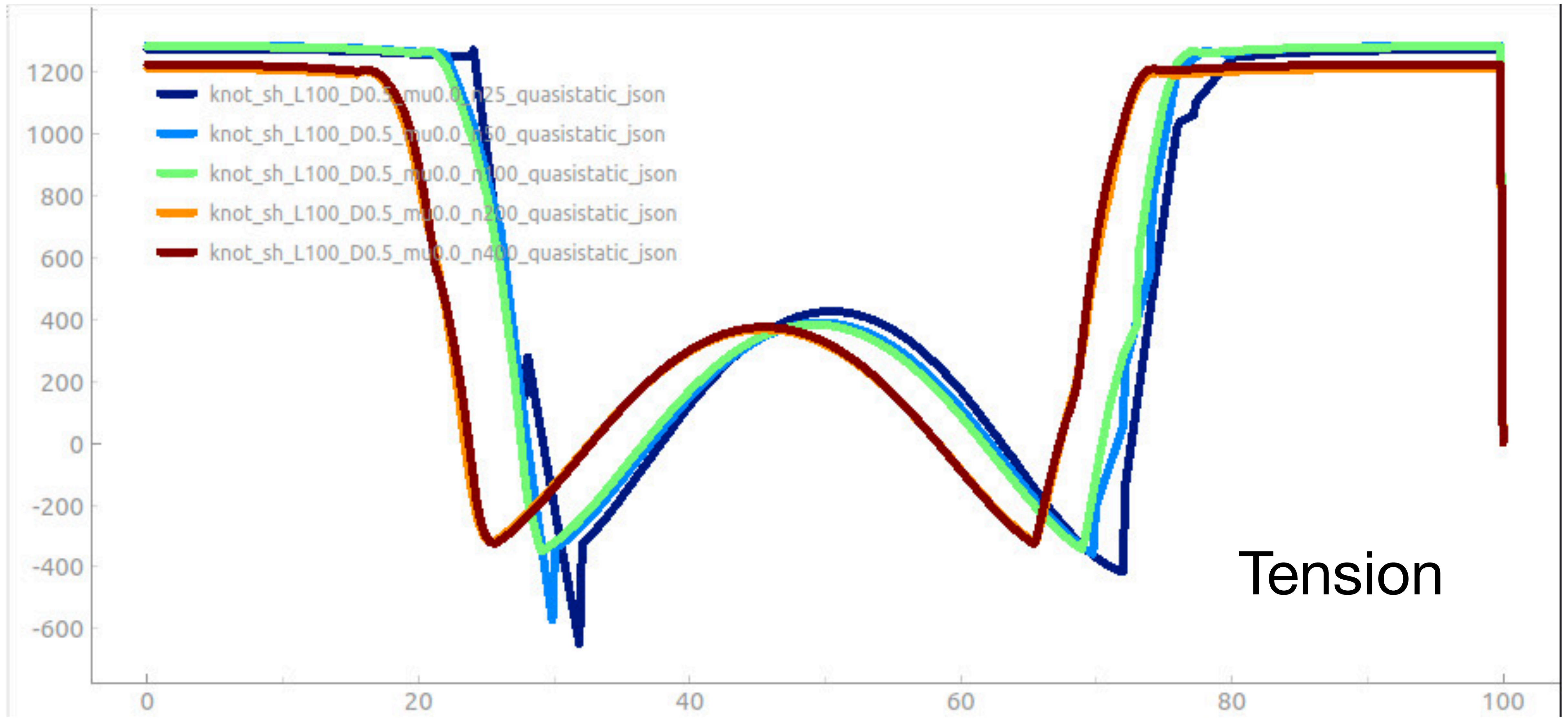


Super-helix model
+ non-smooth contact solver



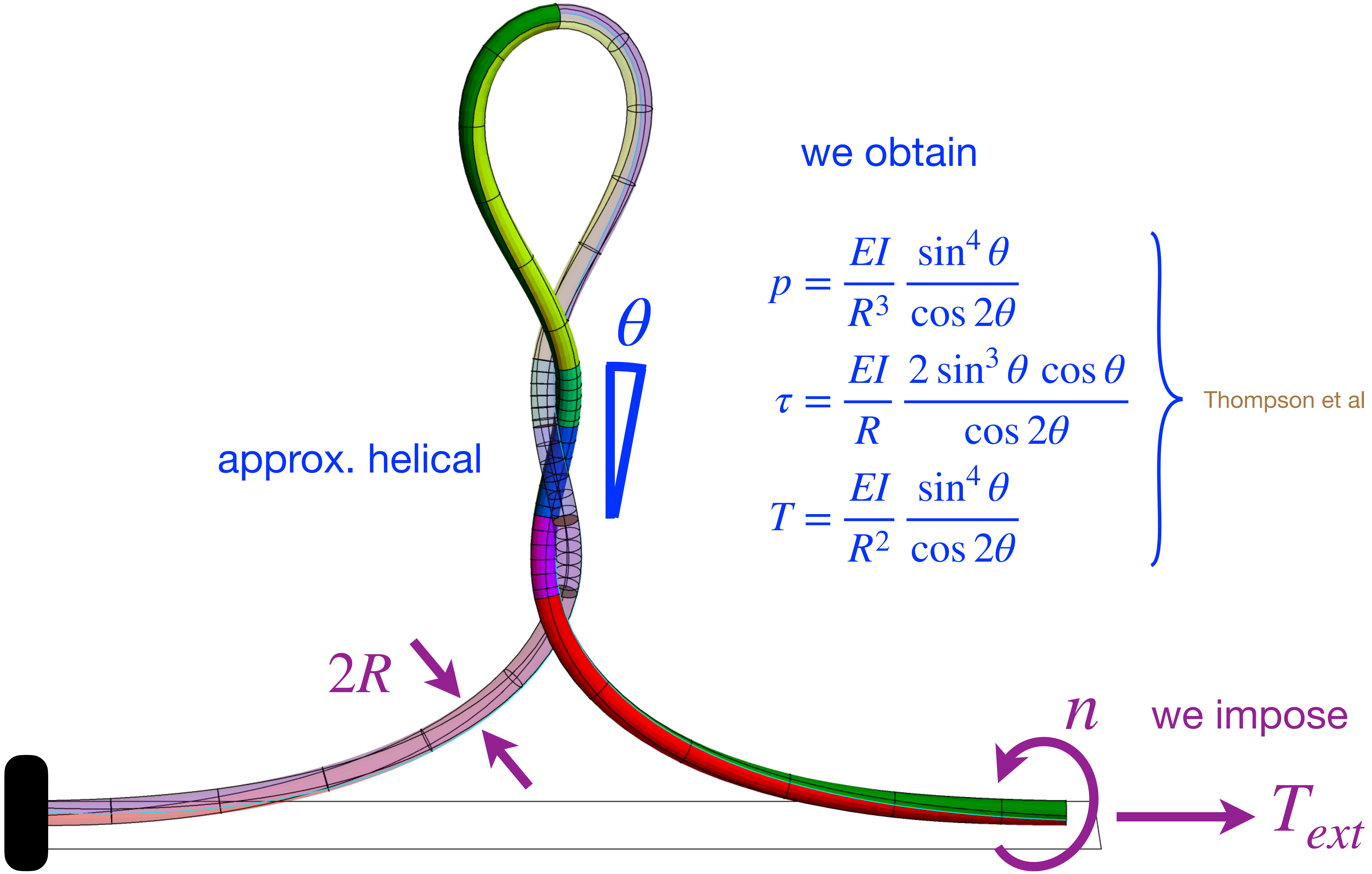


Elastic energy density



Tension

PLECTONEMES



we obtain

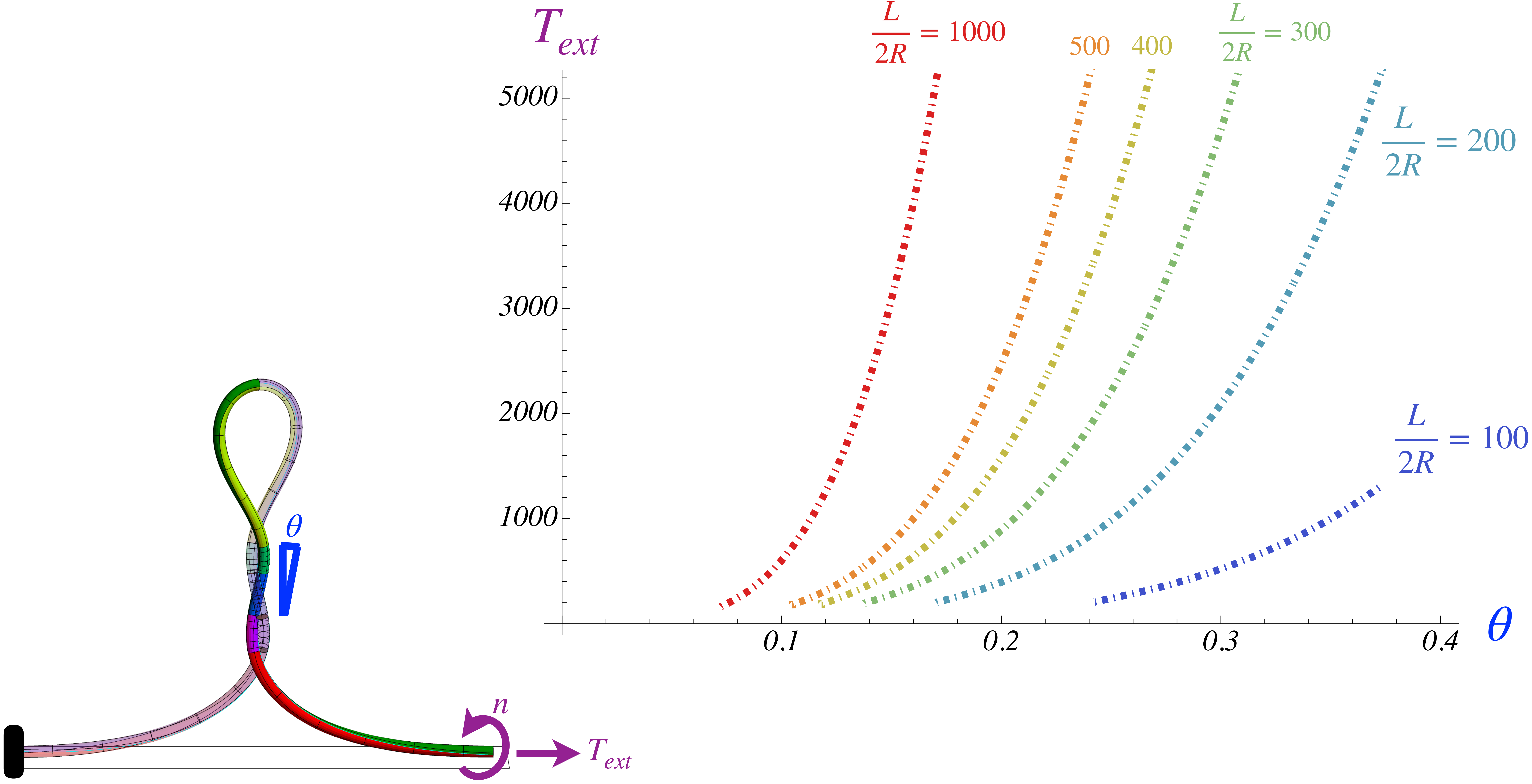
$$p = \frac{EI}{R^3}\,\frac{\sin^4\theta}{\cos 2\theta}$$

$$\tau = \frac{EI}{R}\,\frac{2\sin^3\theta\cos\theta}{\cos 2\theta}$$

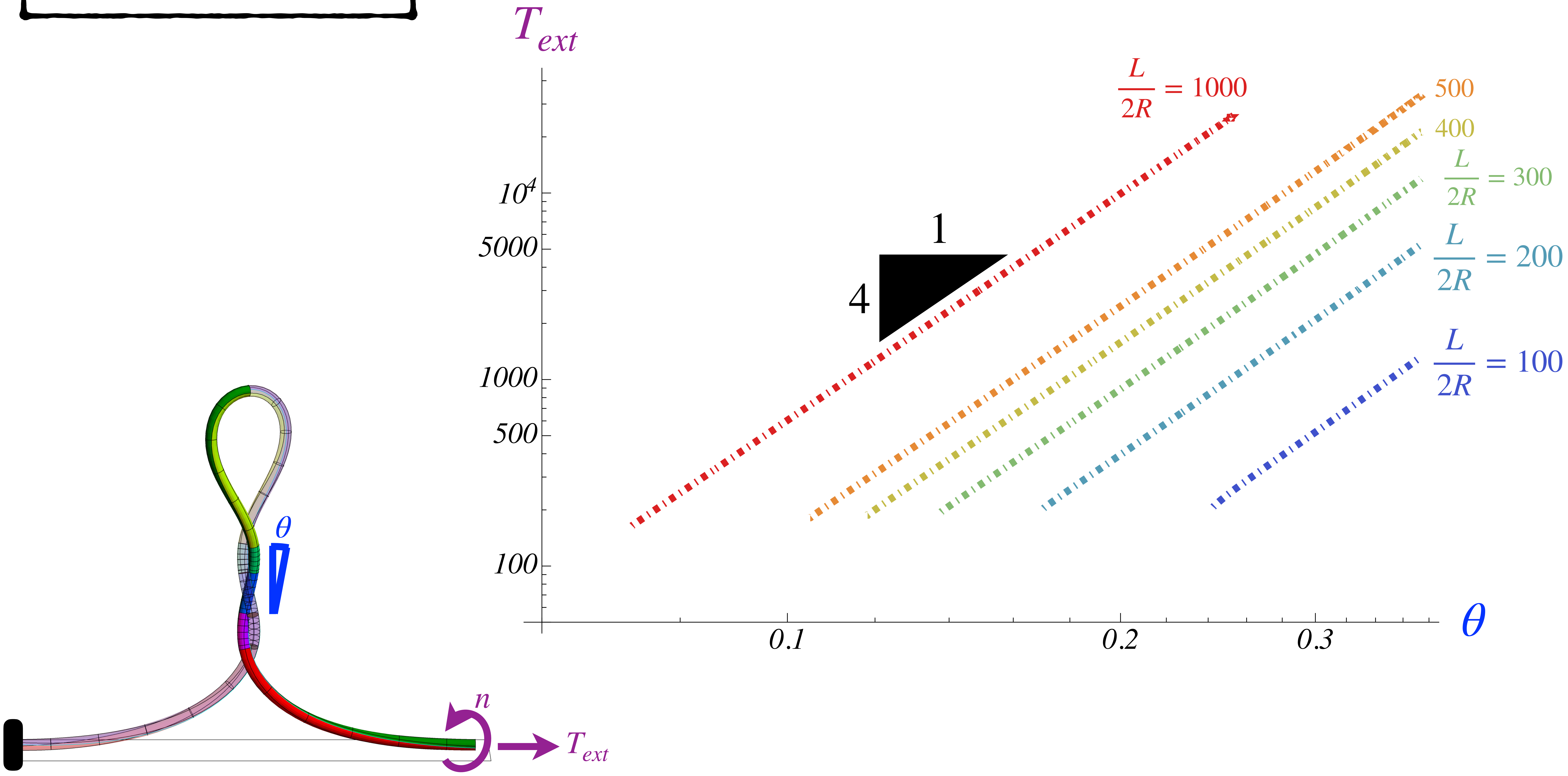
$$T = \frac{EI}{R^2}\,\frac{\sin^4\theta}{\cos 2\theta}$$

Thompson et al ProcSocA 2002

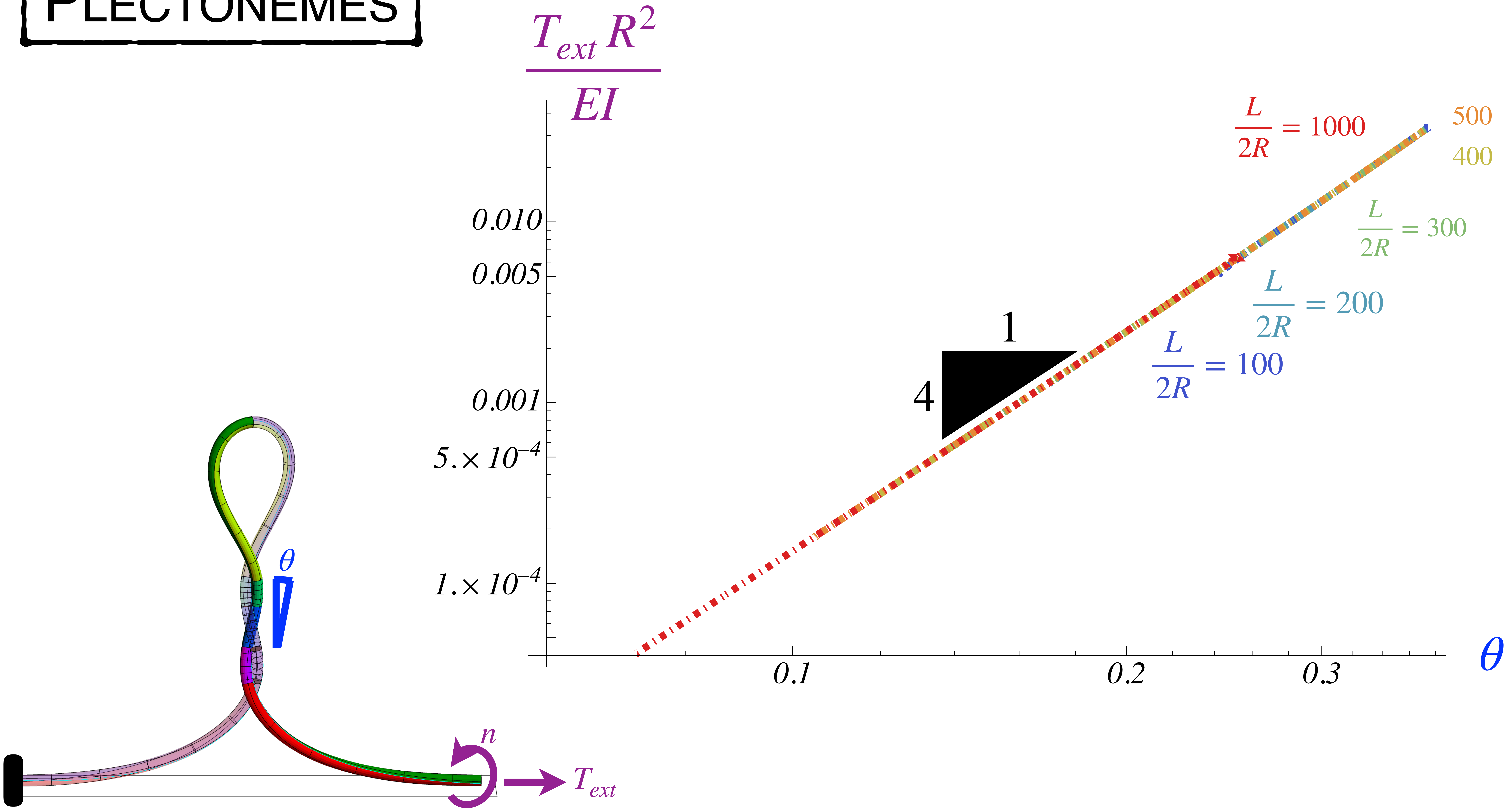
PLECTONEMES



PLECTONEMES

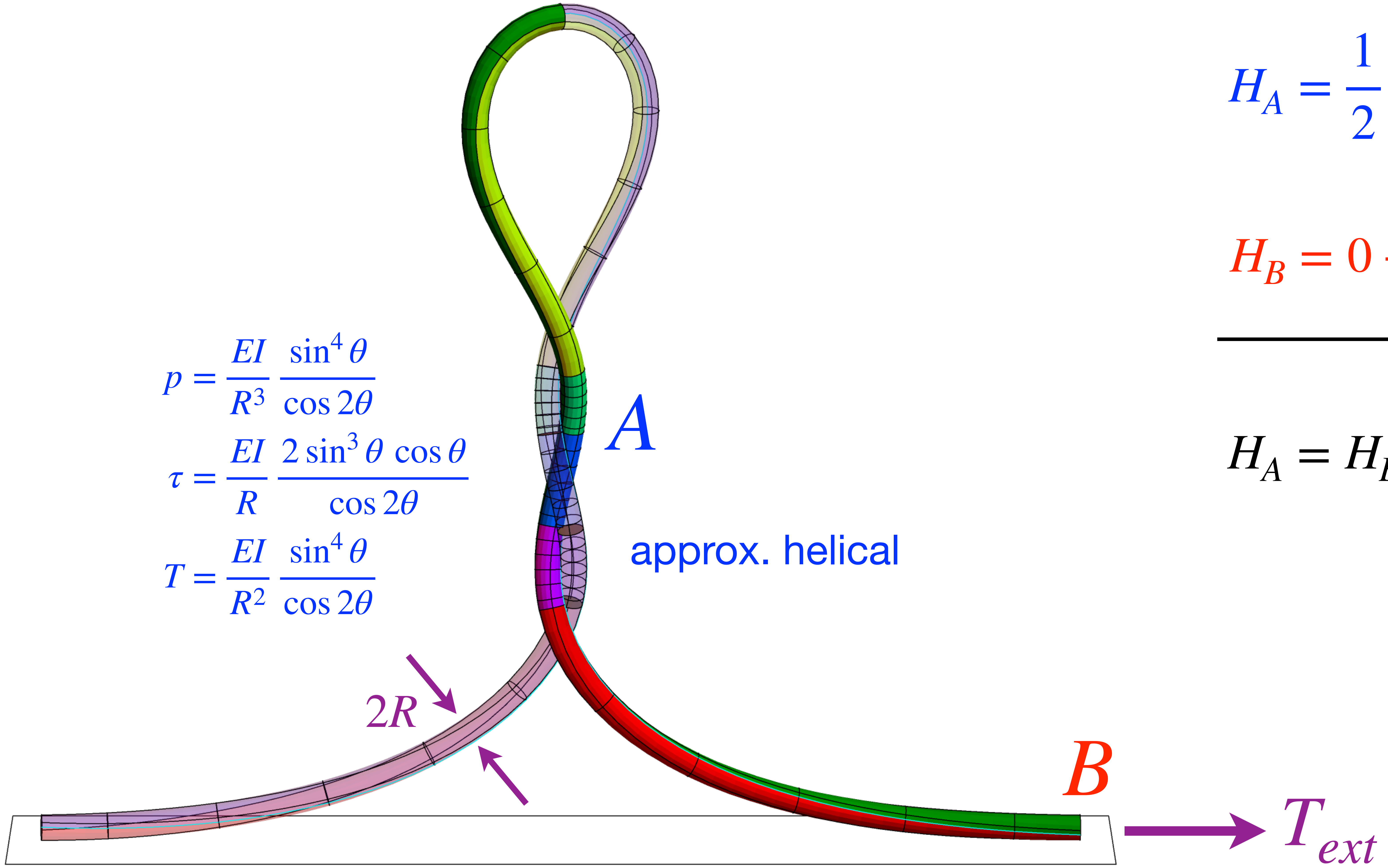


PLECTONEMES



PLECTONEMES

$$H = curvature + twist + tension$$



$$p = \frac{EI}{R^3}\frac{\sin^4\theta}{\cos 2\theta}$$

$$\tau = \frac{EI}{R}\frac{2\sin^3\theta\cos\theta}{\cos 2\theta}$$

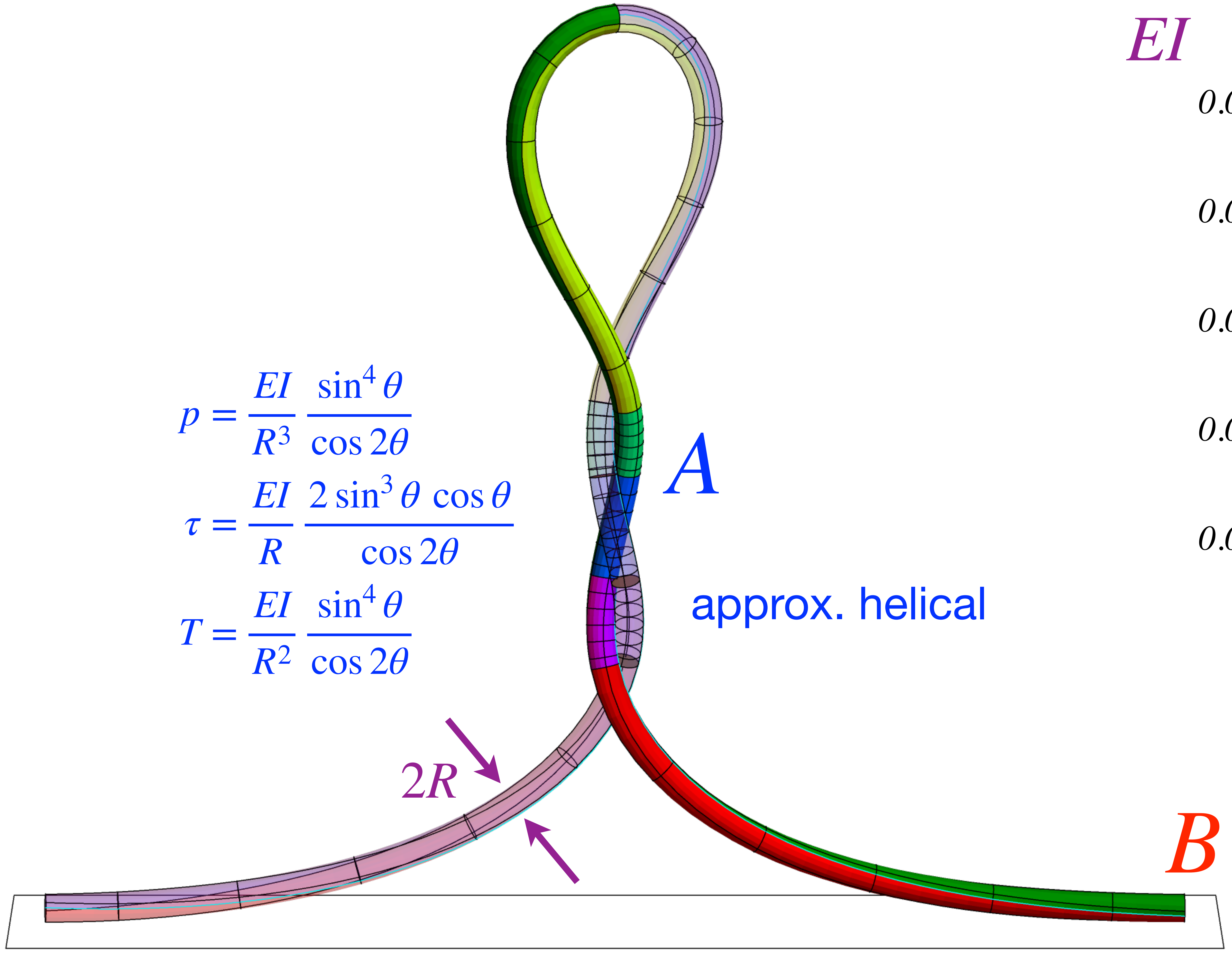
$$T = \frac{EI}{R^2}\frac{\sin^4\theta}{\cos 2\theta}$$

$$H_A = \frac{1}{2}\frac{EI}{R^2}\sin^4\theta + \frac{1}{2}GJ\tau^2 + \frac{EI}{R^2}\frac{\sin^4\theta}{\cos 2\theta}$$

$$H_B = 0 + \frac{1}{2}GJ\tau^2 + T_{ext}$$

$$H_A = H_B \Rightarrow T_{ext} = \frac{1}{2}\frac{EI}{R^2}\sin^4\theta\left(1 + \frac{2}{\cos 2\theta}\right)$$

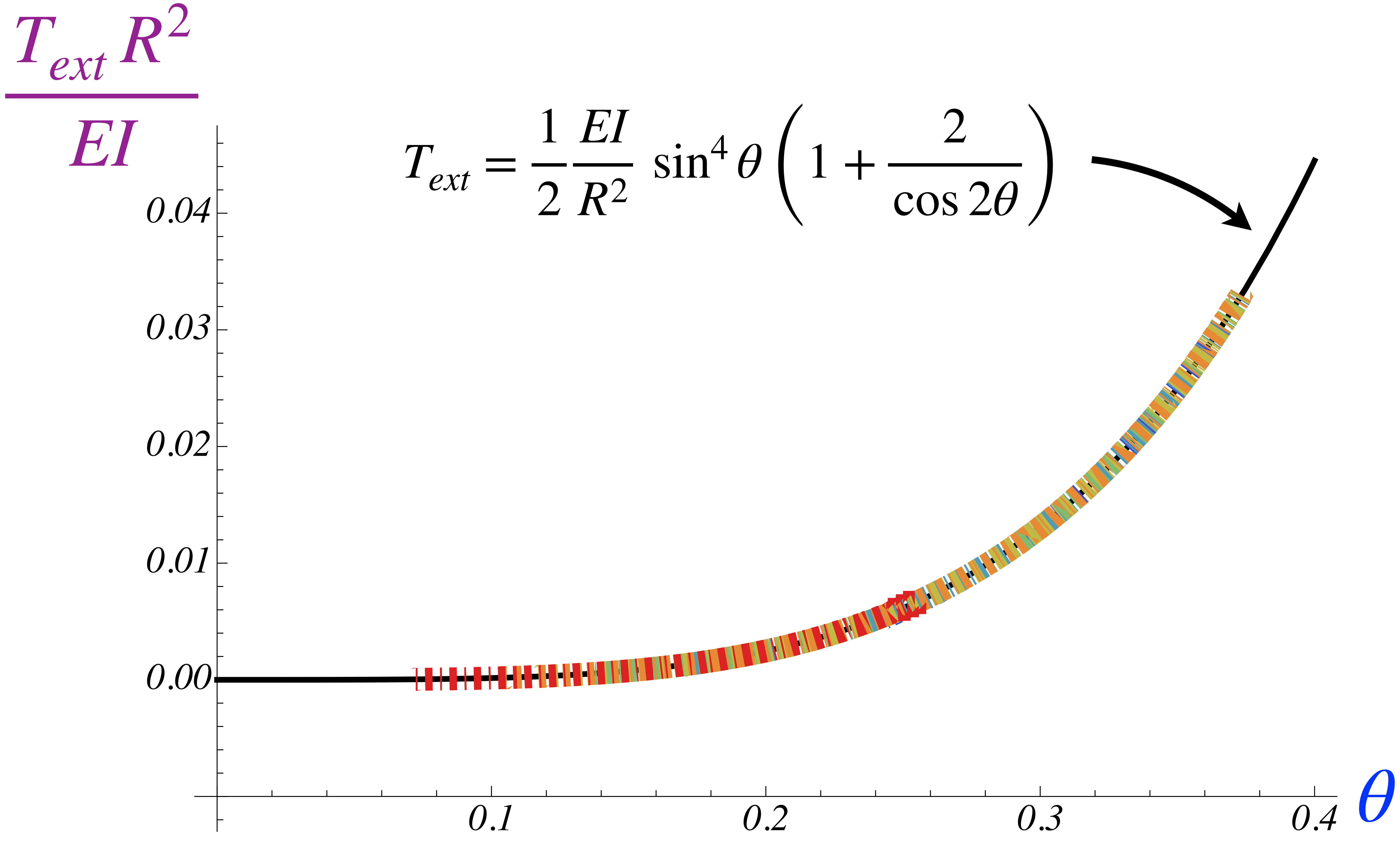
PLECTONEMES



$$p = \frac{EI}{R^3}\,\frac{\sin^4\theta}{\cos 2\theta}$$

$$\tau = \frac{EI}{R}\,\frac{2\sin^3\theta\,\cos\theta}{\cos 2\theta}$$

$$T = \frac{EI}{R^2}\,\frac{\sin^4\theta}{\cos 2\theta}$$



$$T_{ext} = \frac{1}{2}\,\frac{EI}{R^2}\,\sin^4\theta\left(1+\frac{2}{\cos 2\theta}\right)$$

CONCLUSION

preprint

- invariant for 2D and 3D Kirchhoff / Reissner elastic rods
- deduced from variational approach
- valid in case of (frictionless) contact / sleeves
- useful in numerical code validation
- dynamics: other invariants (statics/dynamics/vibrations)

CONCLUSION

preprint

- invariant for 2D and 3D Kirchhoff / Reissner elastic rods
- deduced from variational approach
- valid in case of (frictionless) contact / sleeves
- useful in numerical code validation
- dynamics: other invariants (statics/dynamics/vibrations)

EMMY NOETHER
(1882 - 1935)

- University of Erlangen, Göttingen then USA
- Exceptional mathematical contributions (algebra)
- Strongly discriminated against because was a woman
- Things have (fortunately) changed but… stay careful!