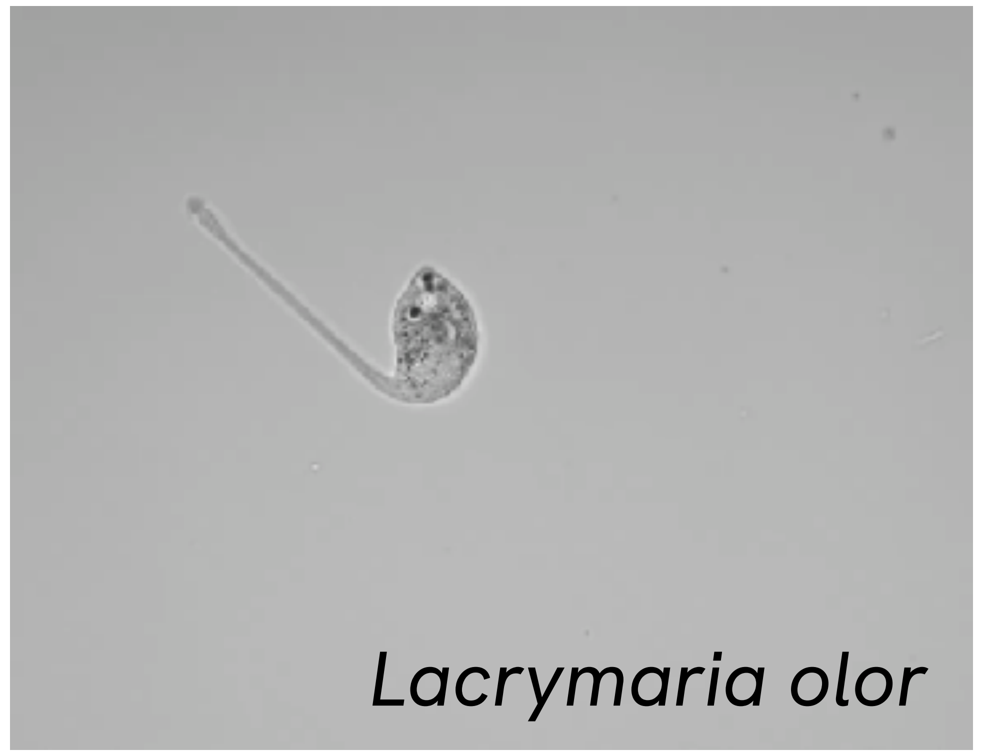
From slender microscopic locomotion to controllability of affine systems

Clément Moreau

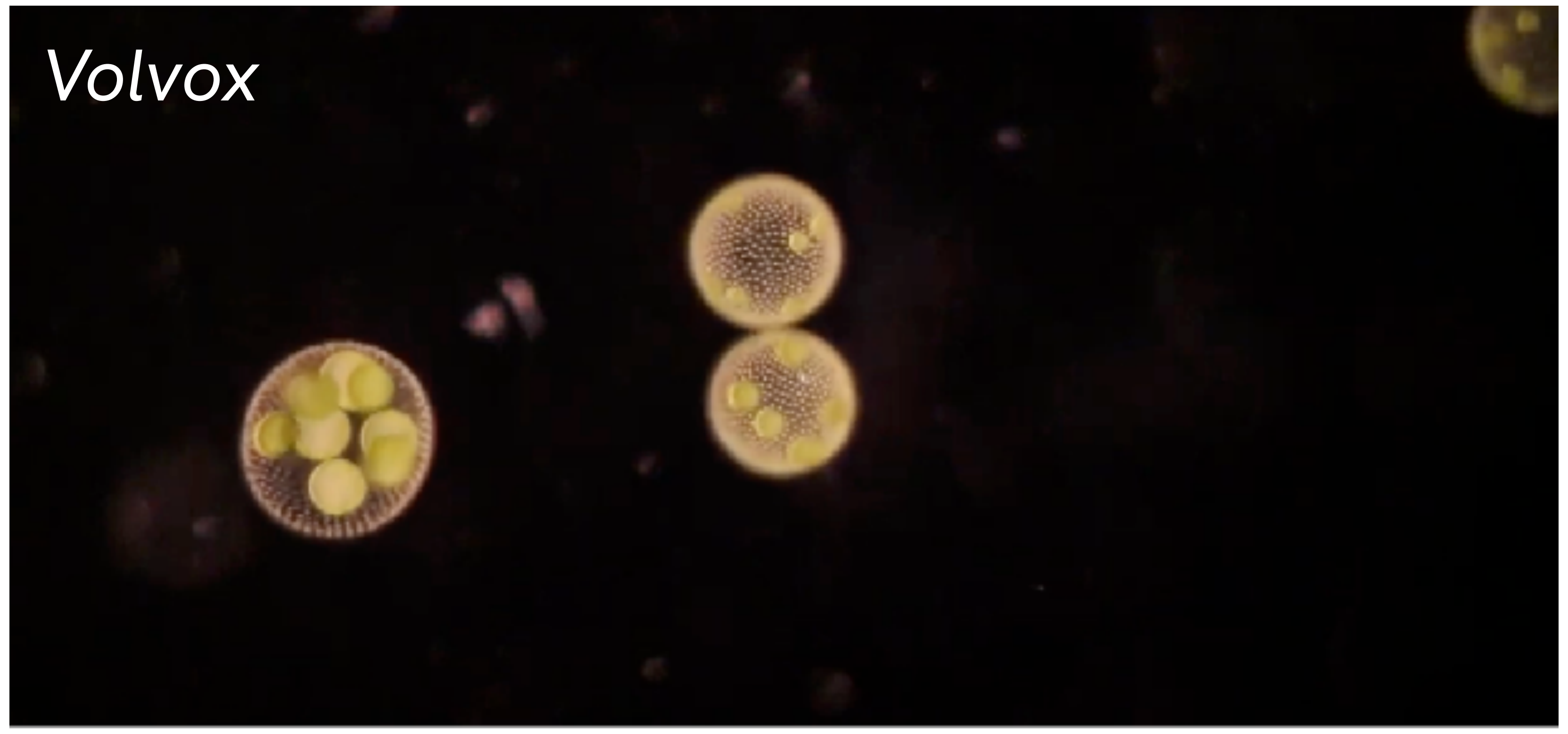
Laboratoire des Sciences
du Numérique de Nantes

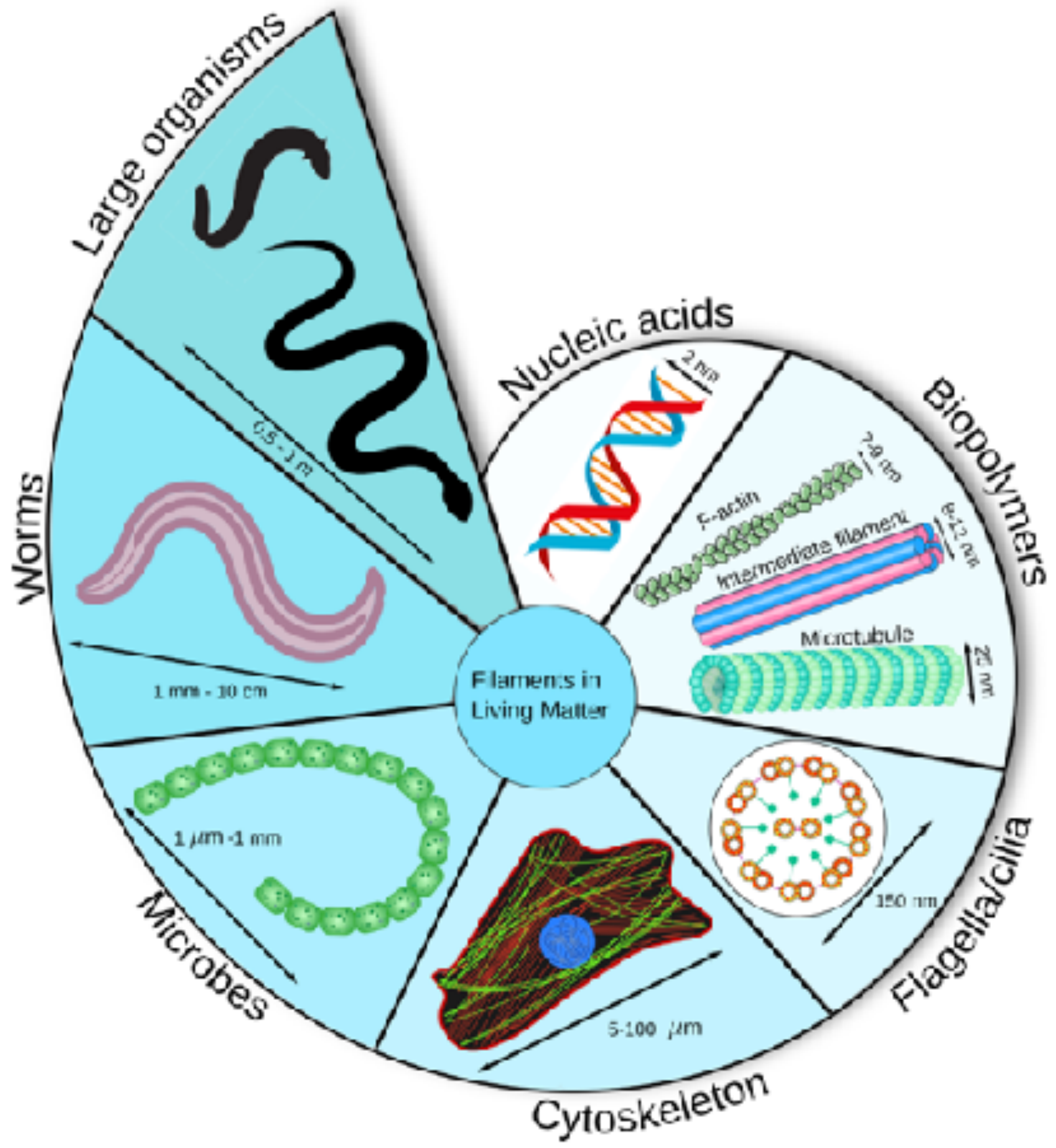
Atelier thématique du GDR-GDM — 29 avril 2026

Microscopic locomotors

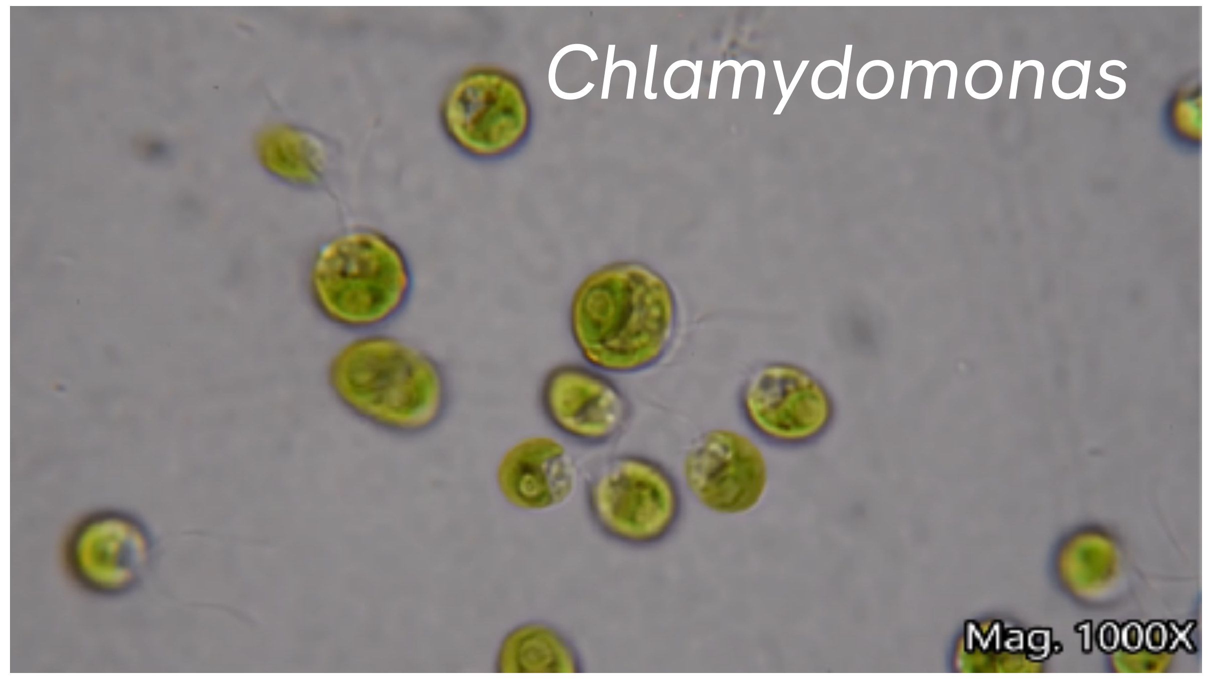

Lacrymaria olor


Euglena gracilis

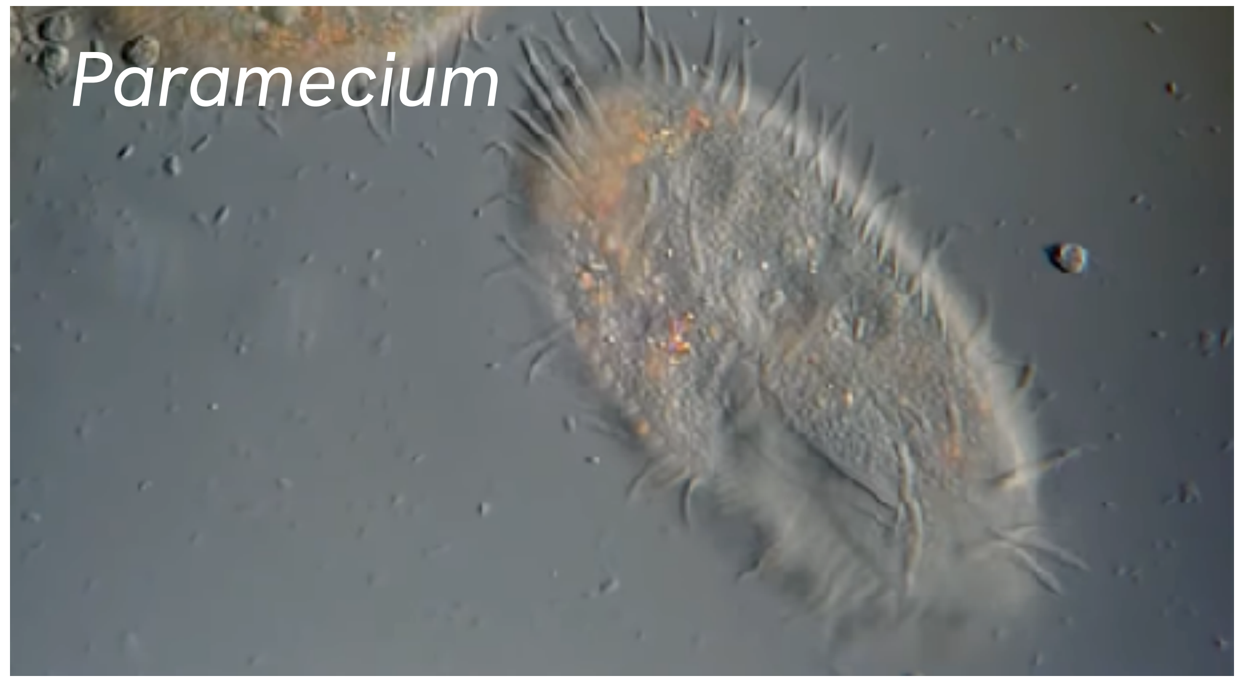

Volvox

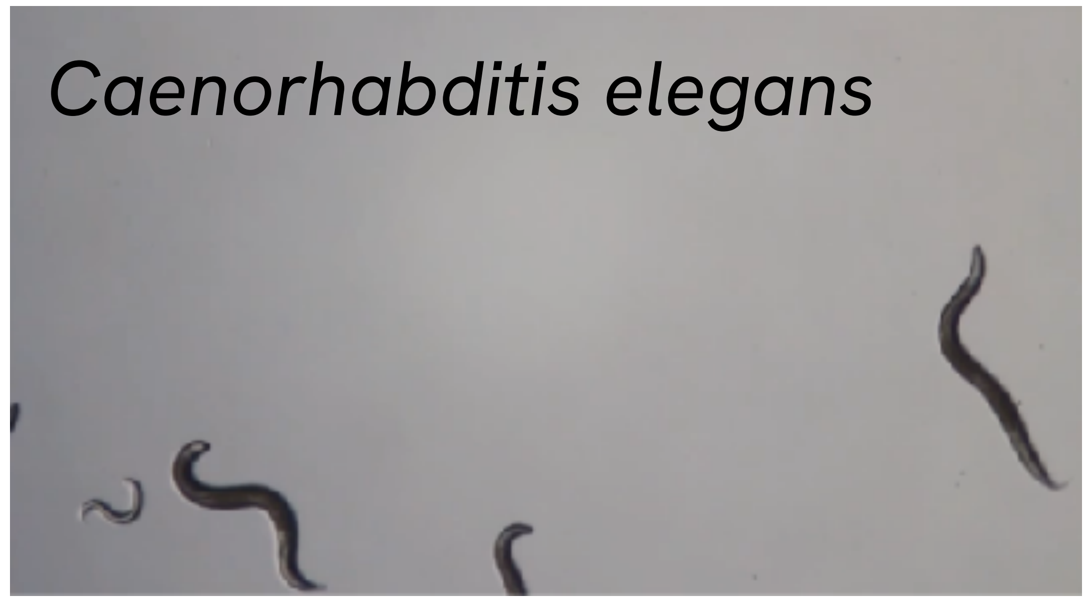


Cammann et al. arXiv:2502.12731


Chlamydomonas


Spermatozoon

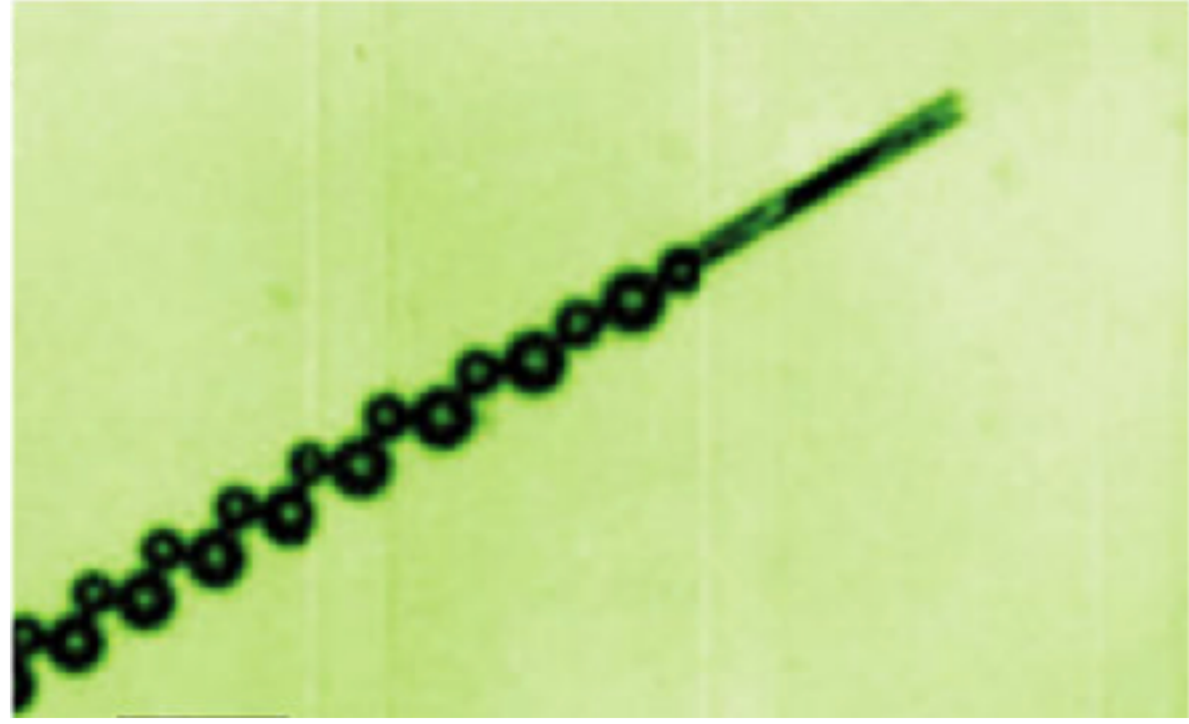

Paramecium


Caenorhabditis elegans

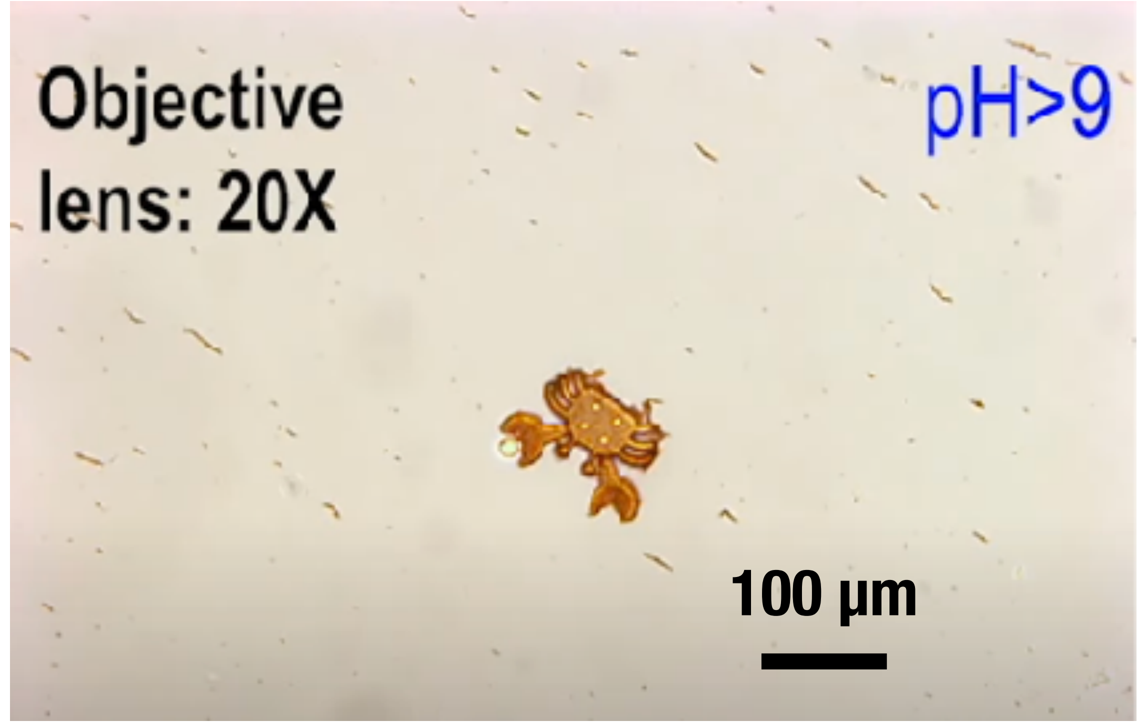
Slender structures are ubiquitous at microscopic scale

Micro-robots

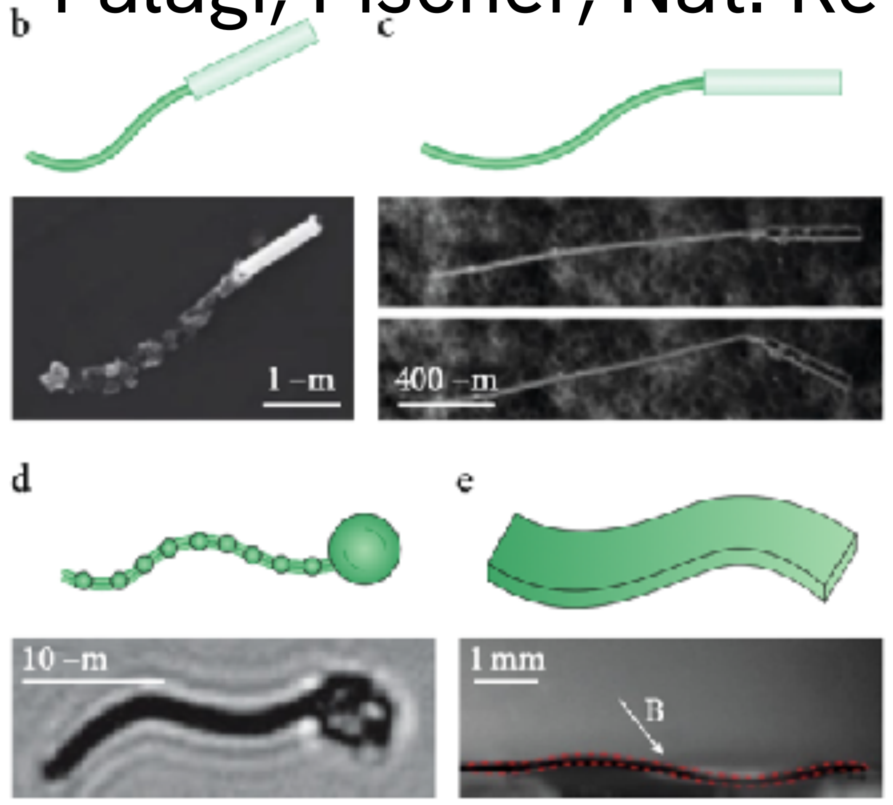
Sovolev et al., Small 5(14), 2009

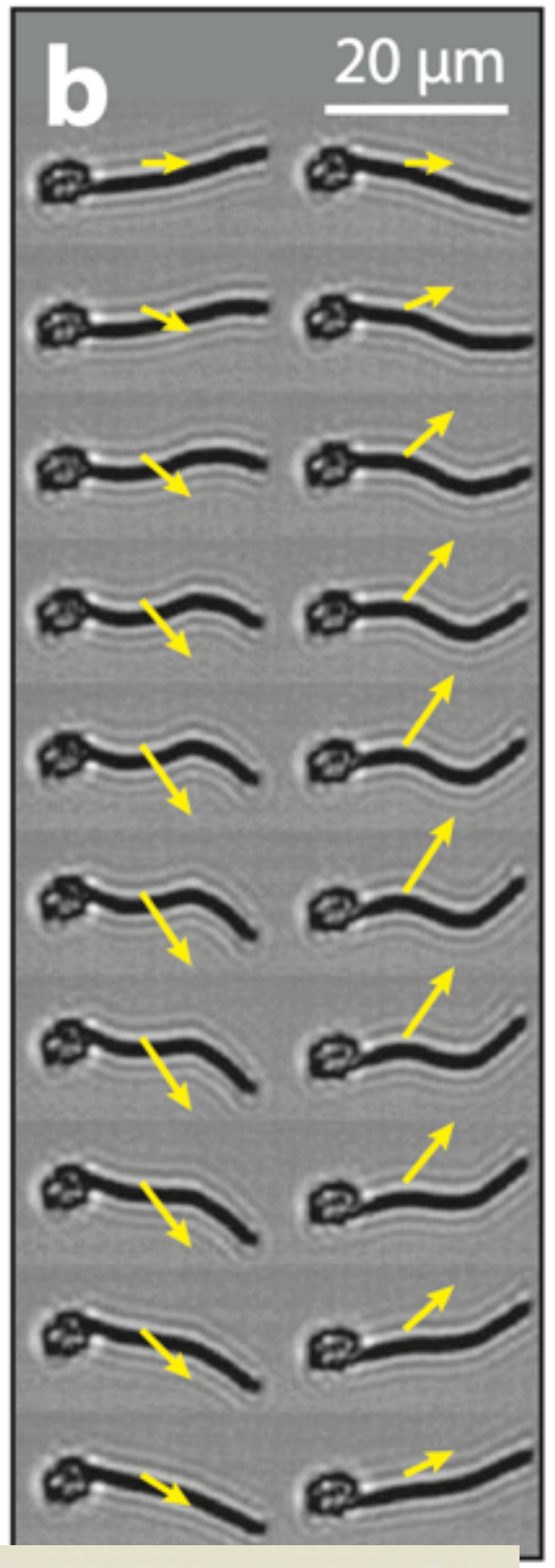


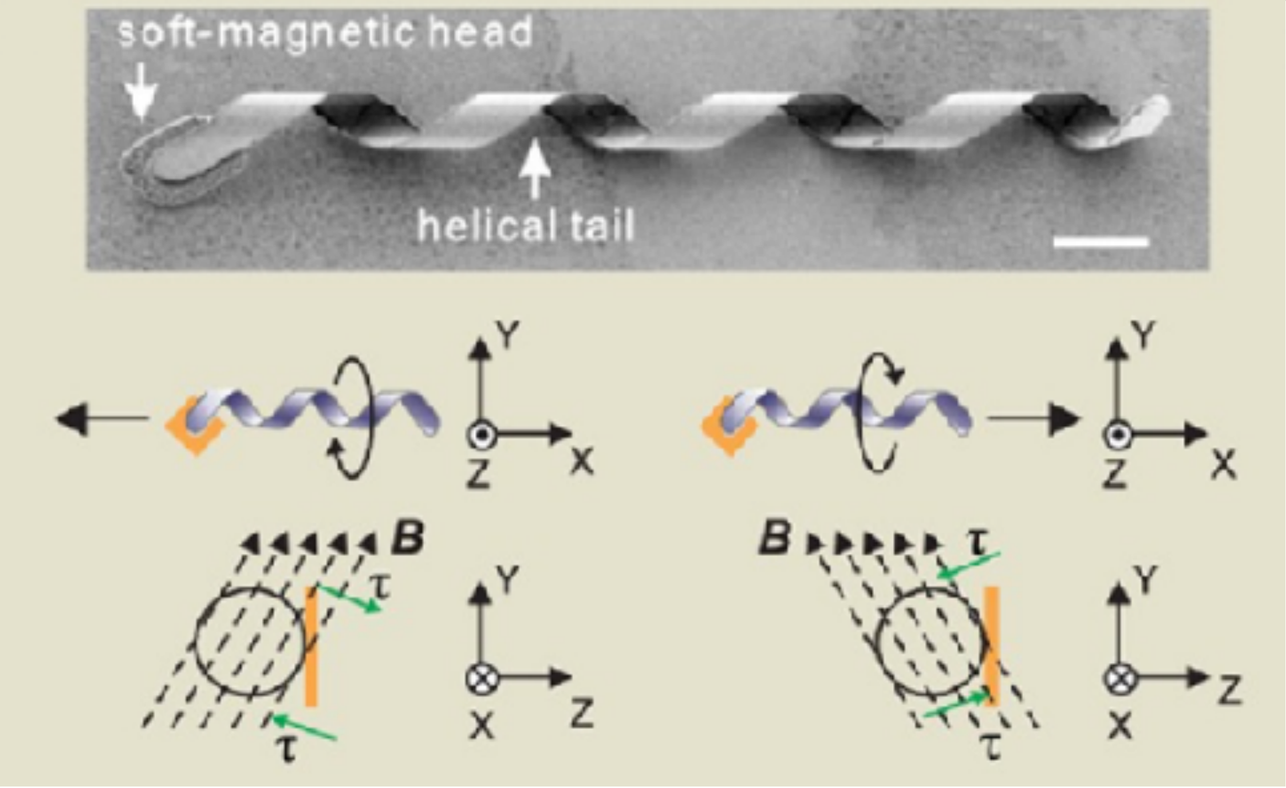
Xin et al., ACS Nano 15(11), 2021

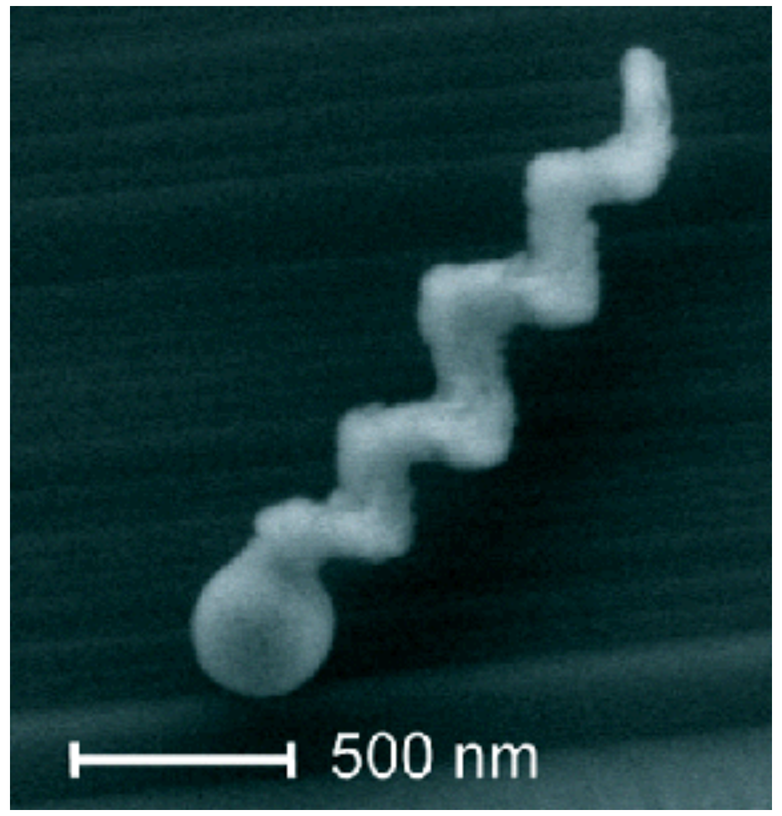


Palagi, Fischer, Nat. Rev. Materials 3 (2018)

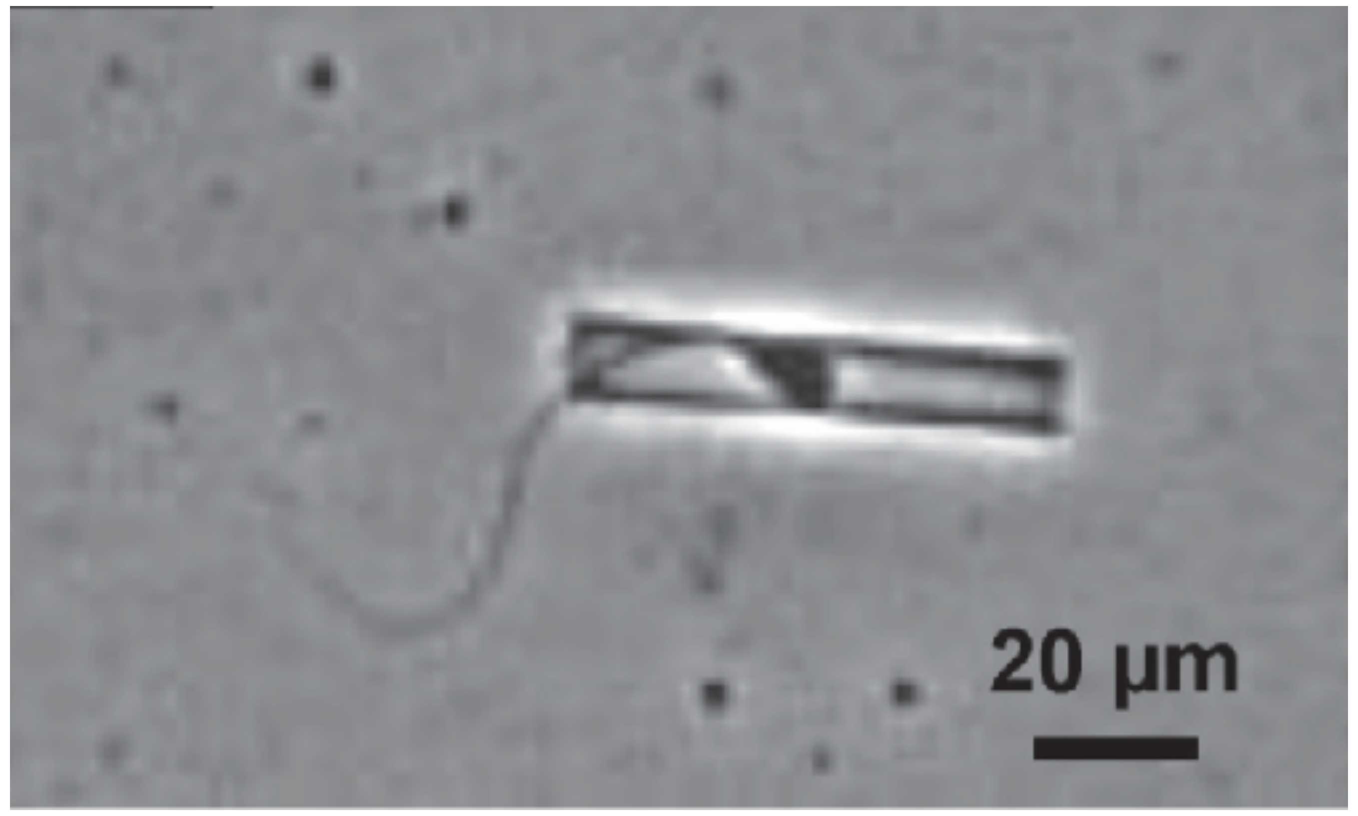






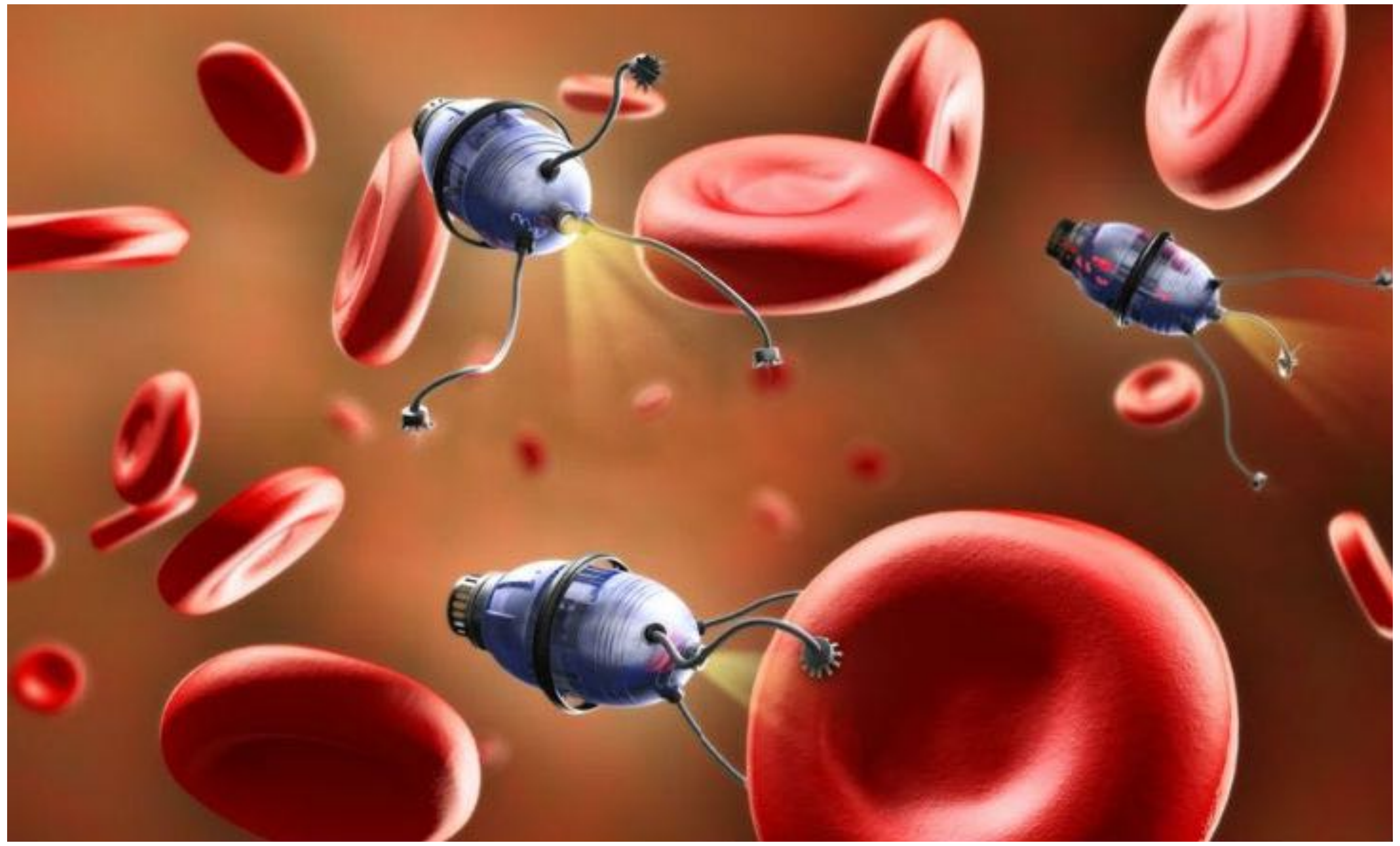


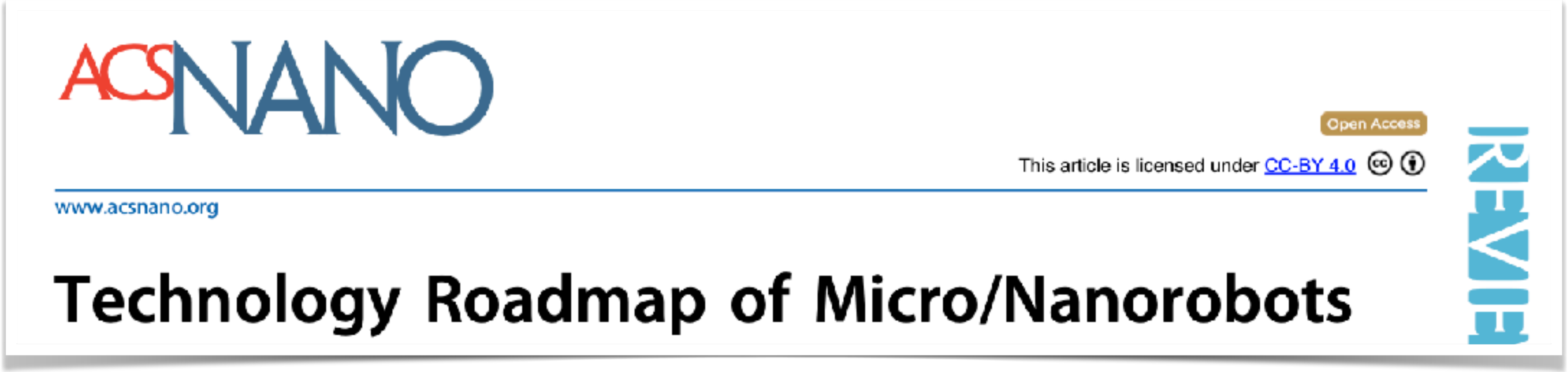
Ghosh & Fischer, Nano Letters 9(6), 2009



Magdanz et al., Adv. Funct. Mater. 25(18), 2015

Bio-inspired microscopic locomotion





ACSNANO

Open Access

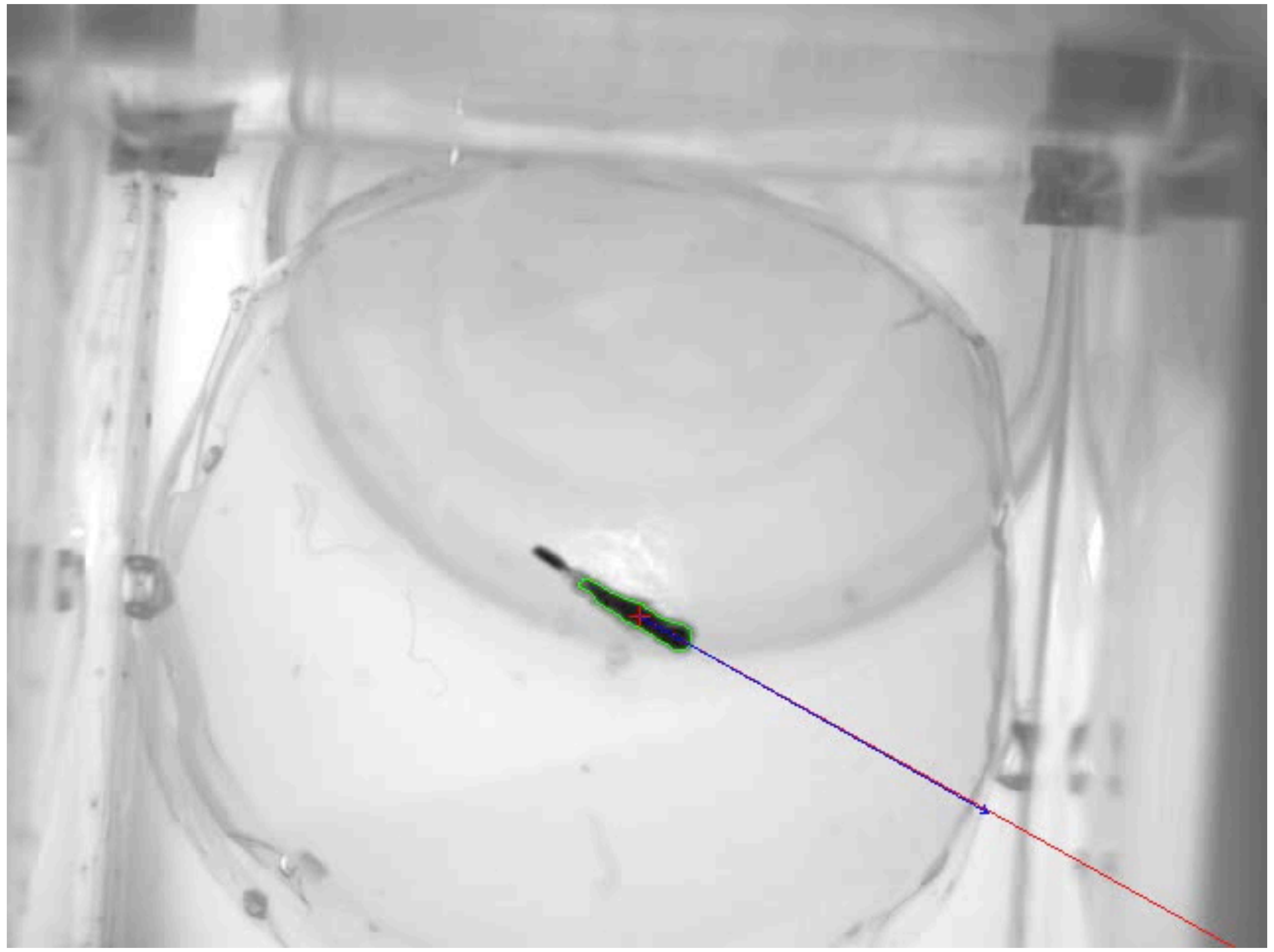
This article is licensed under CC-BY 4.0

www.acsnano.org

Technology Roadmap of Micro/Nanorobots

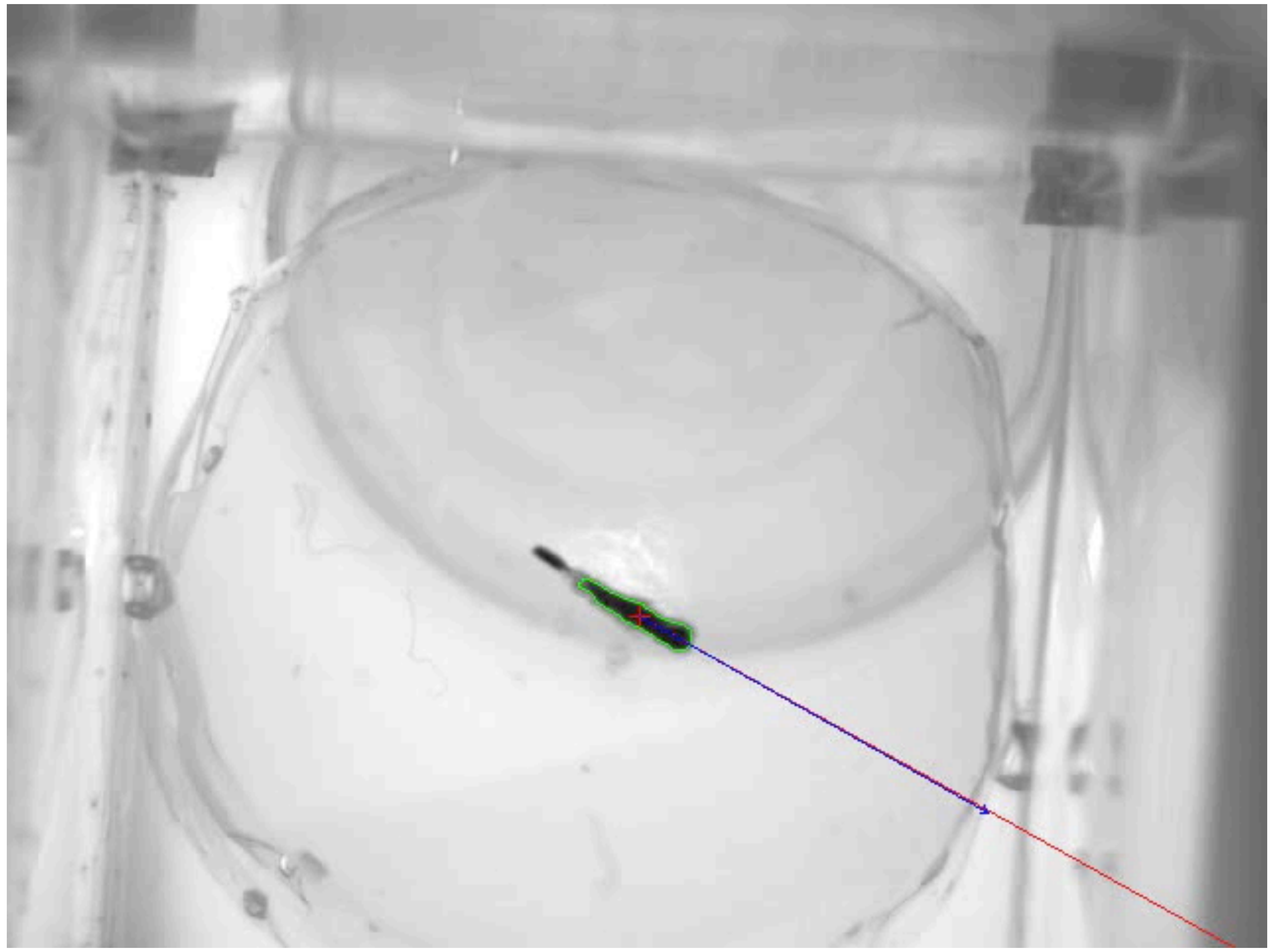
REVIE

Motivation: flexible magnetic robot



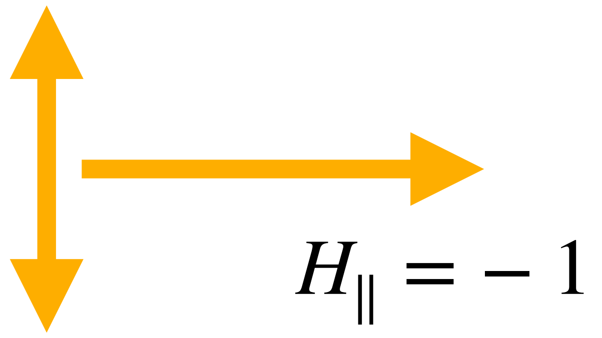
El Alaoui-Faris et al., Phys. Rev. E **101** (2020)

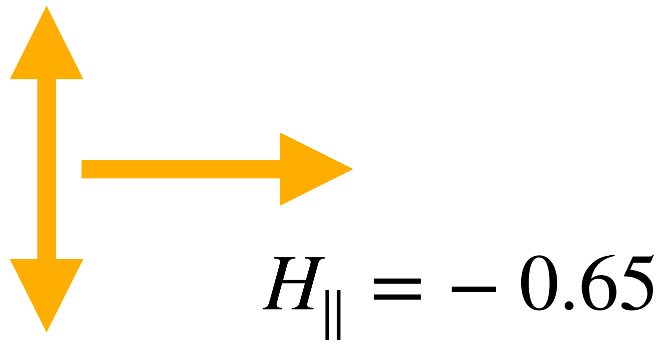
Motivation: flexible magnetic robot

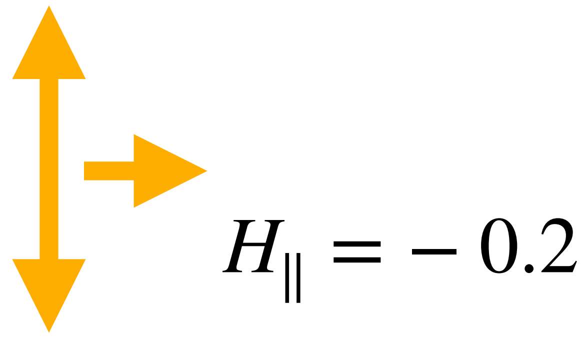


El Alaoui-Faris et al., Phys. Rev. E **101** (2020)

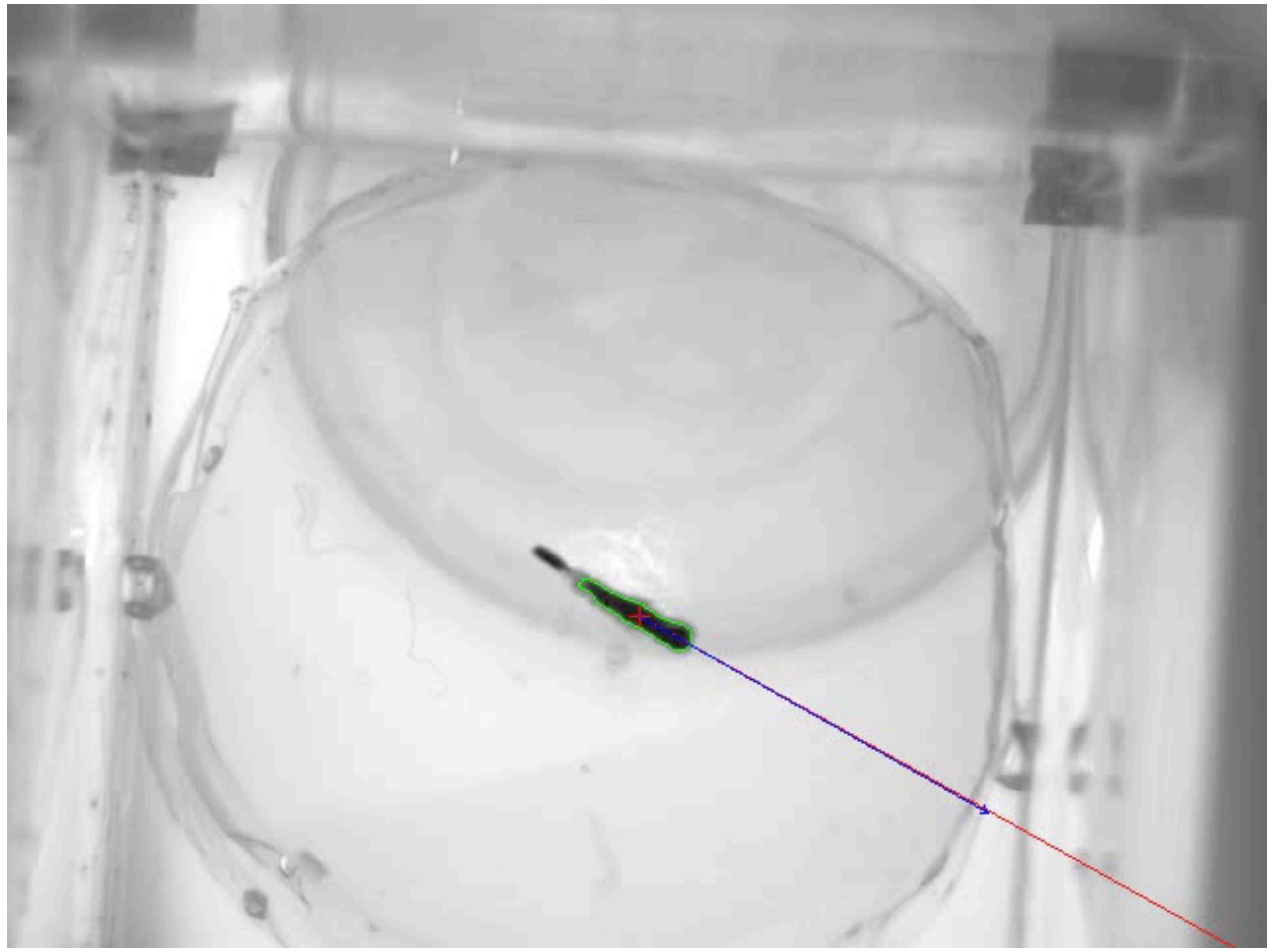
Magnetic field: vertical oscillating
+ constant horizontal







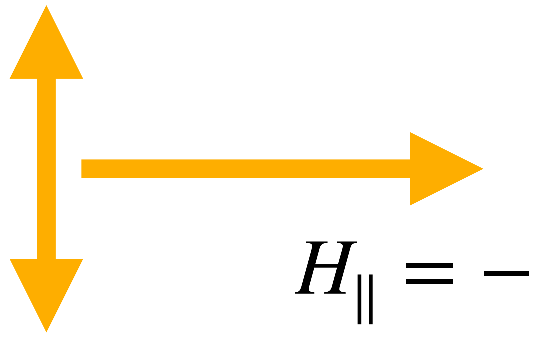
Motivation: flexible magnetic robot



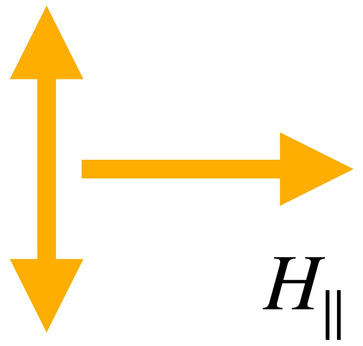
El Alaoui-Faris et al., Phys. Rev. E **101** (2020)

how and why?

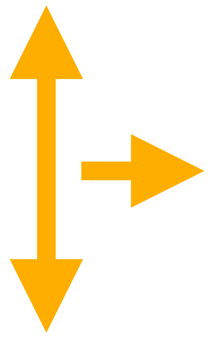
Magnetic field: vertical oscillating
+ constant horizontal



$H_{\|} = -1$

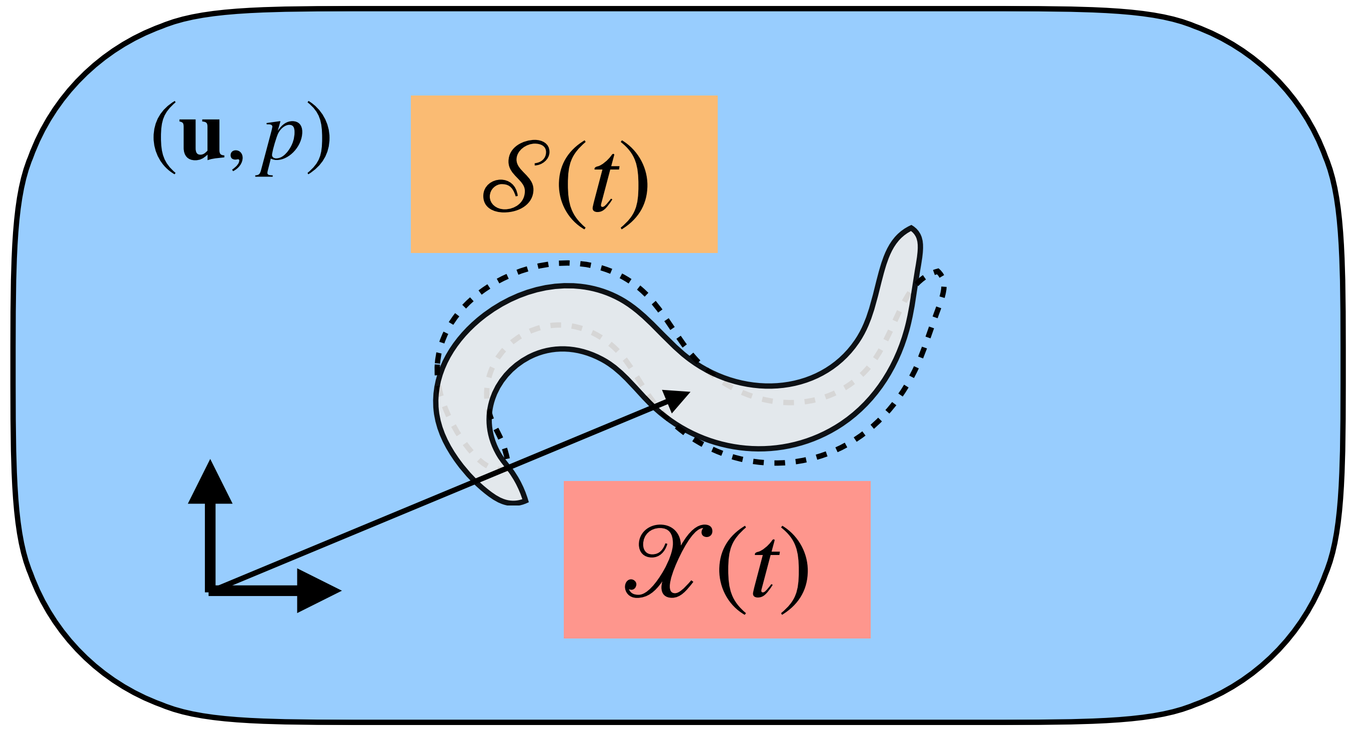


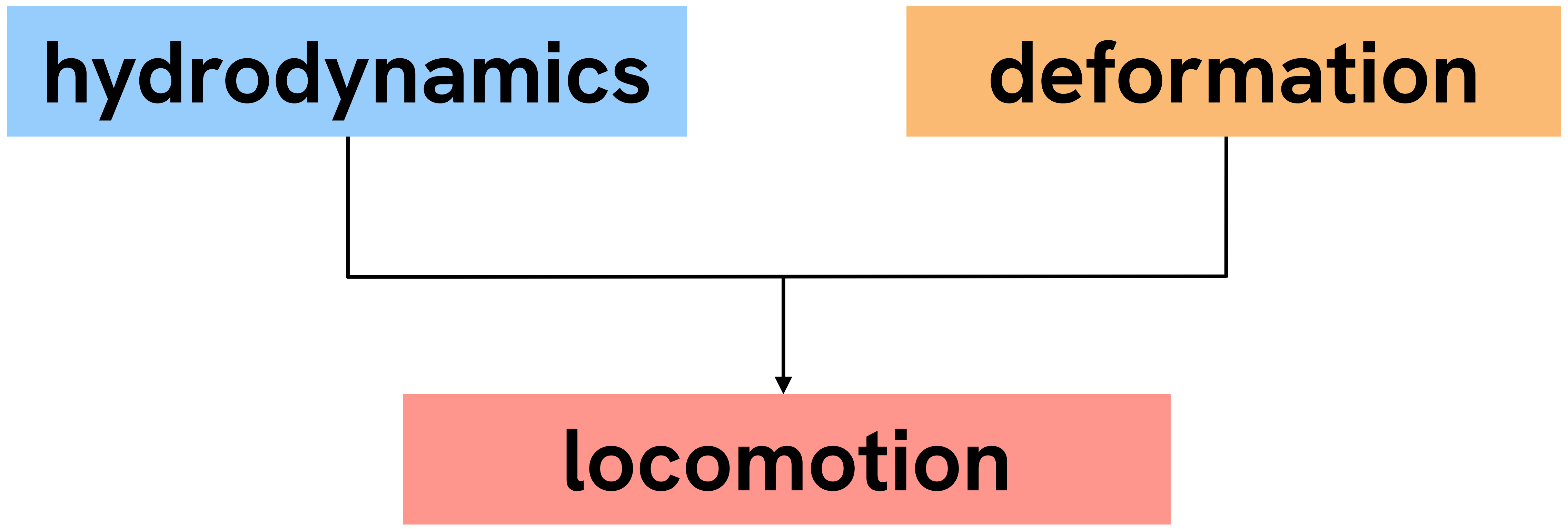
$H_{\|} = -0.65$



$H_{\|} = -0.2$

Principles of micro-swimming



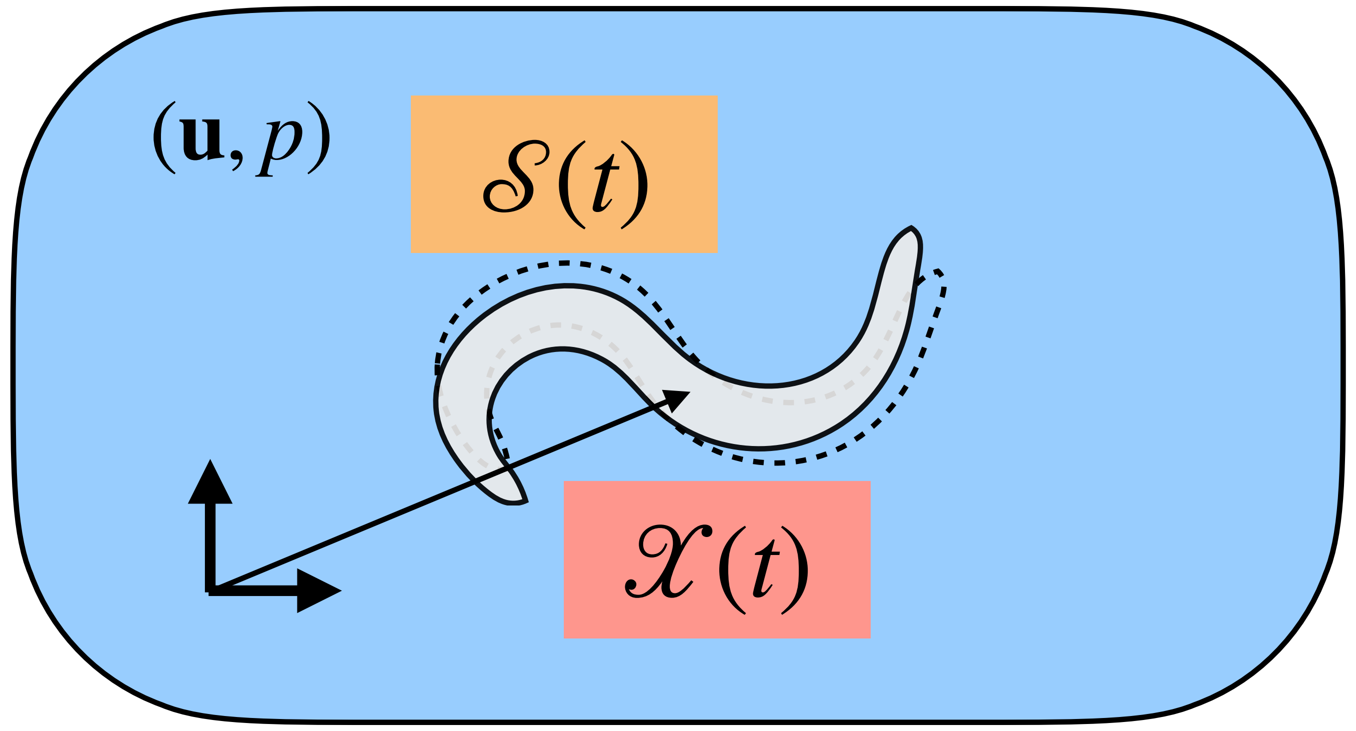


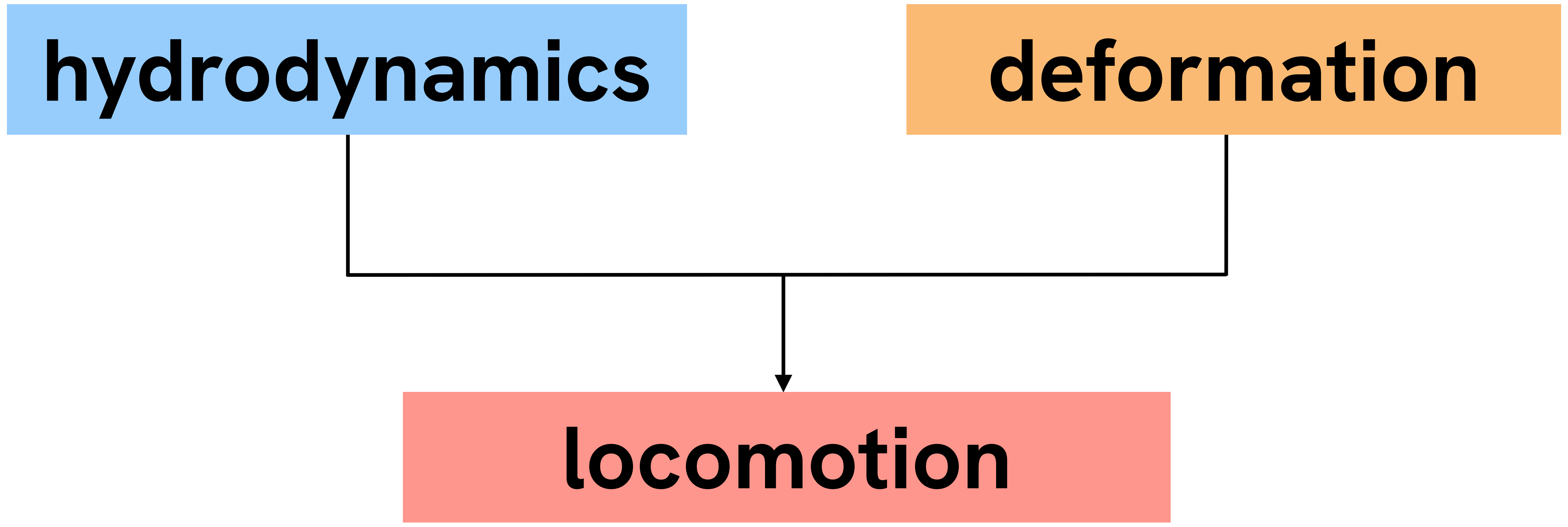
hydrodynamics

deformation

locomotion

Principles of micro-swimming





hydrodynamics

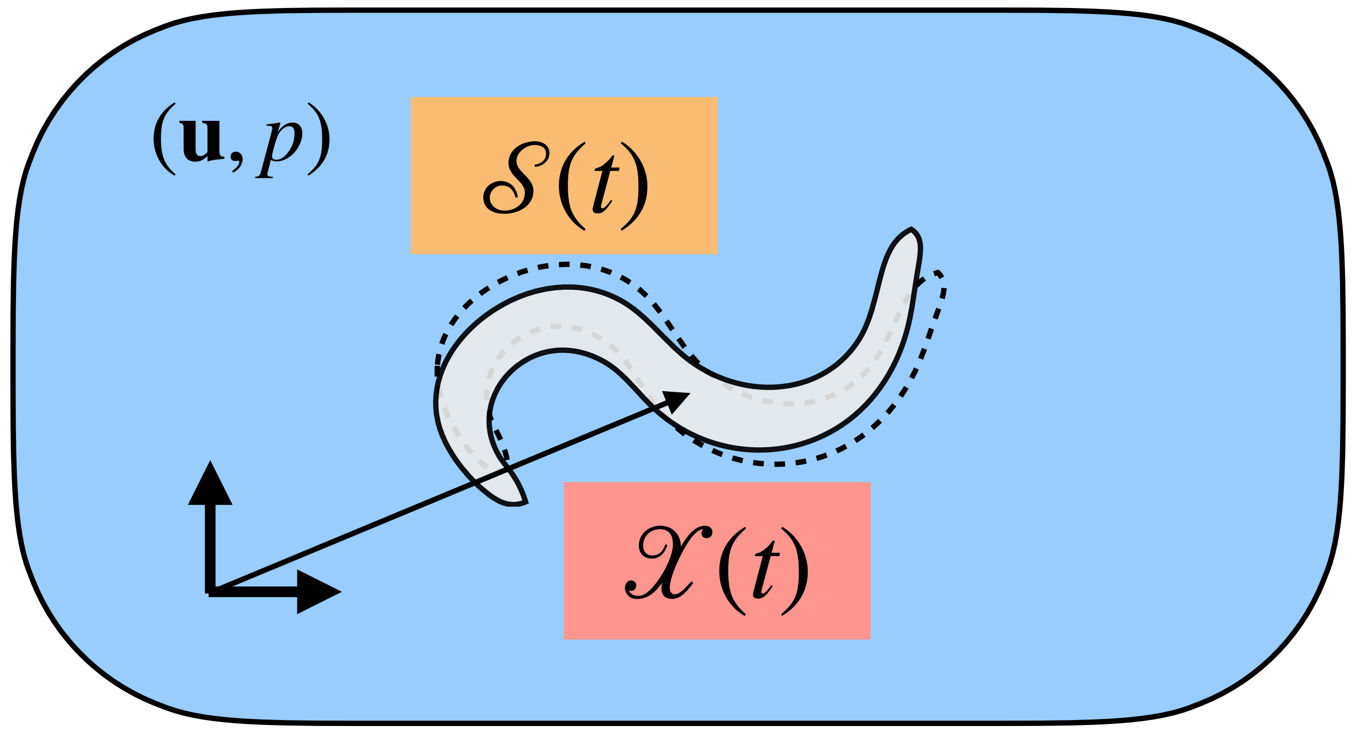
deformation

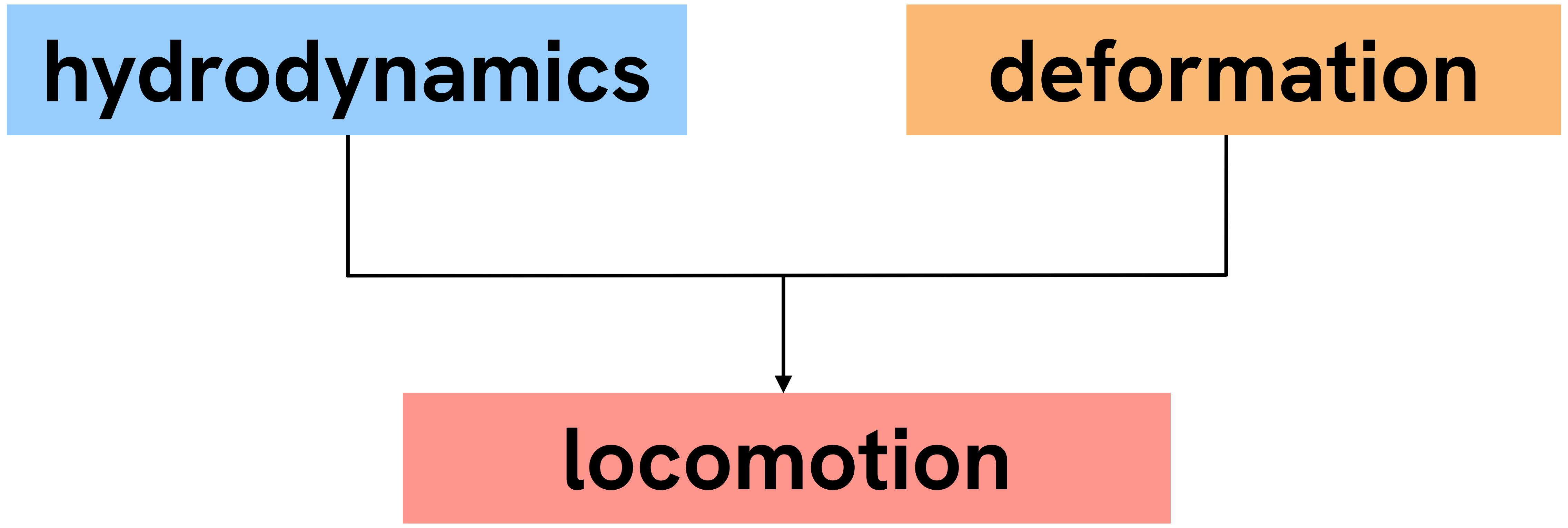
locomotion

rigid motion

$\mathcal{X}(t)$ Pose in $SE(n)$

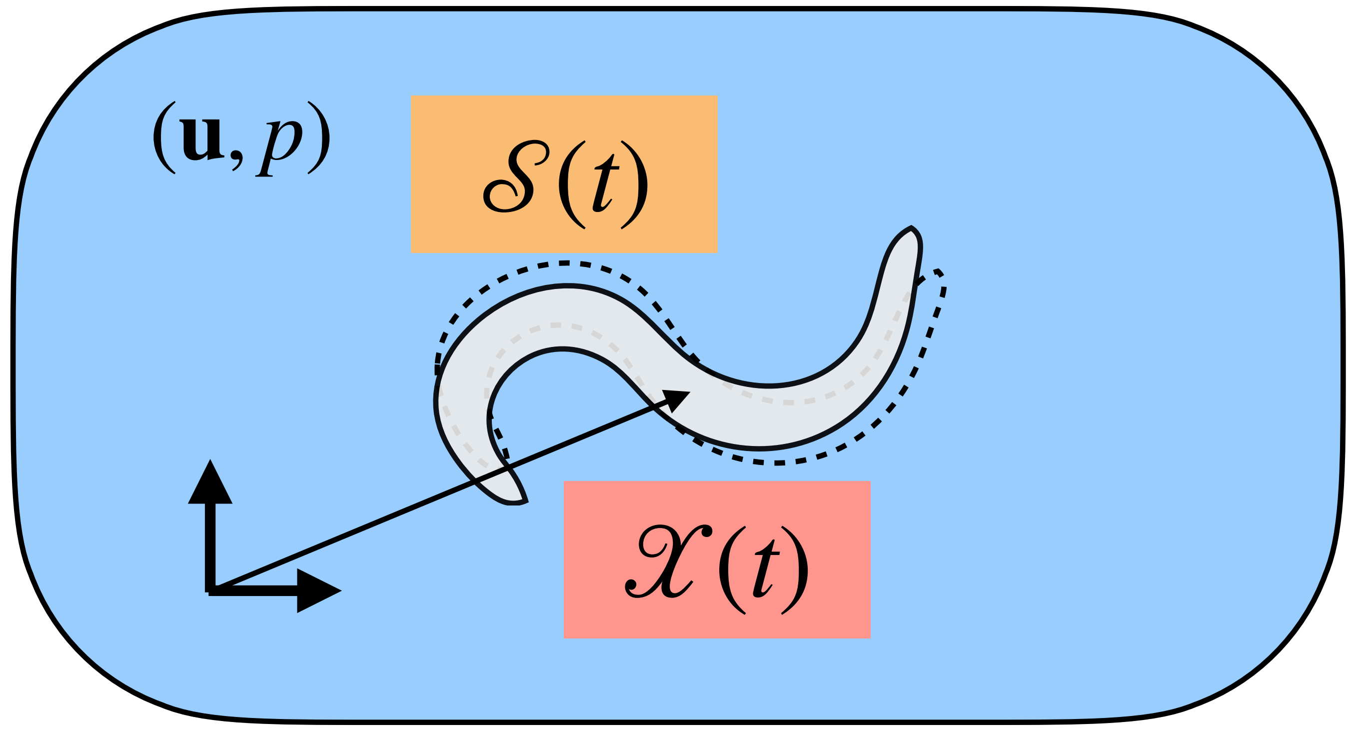
Principles of micro-swimming

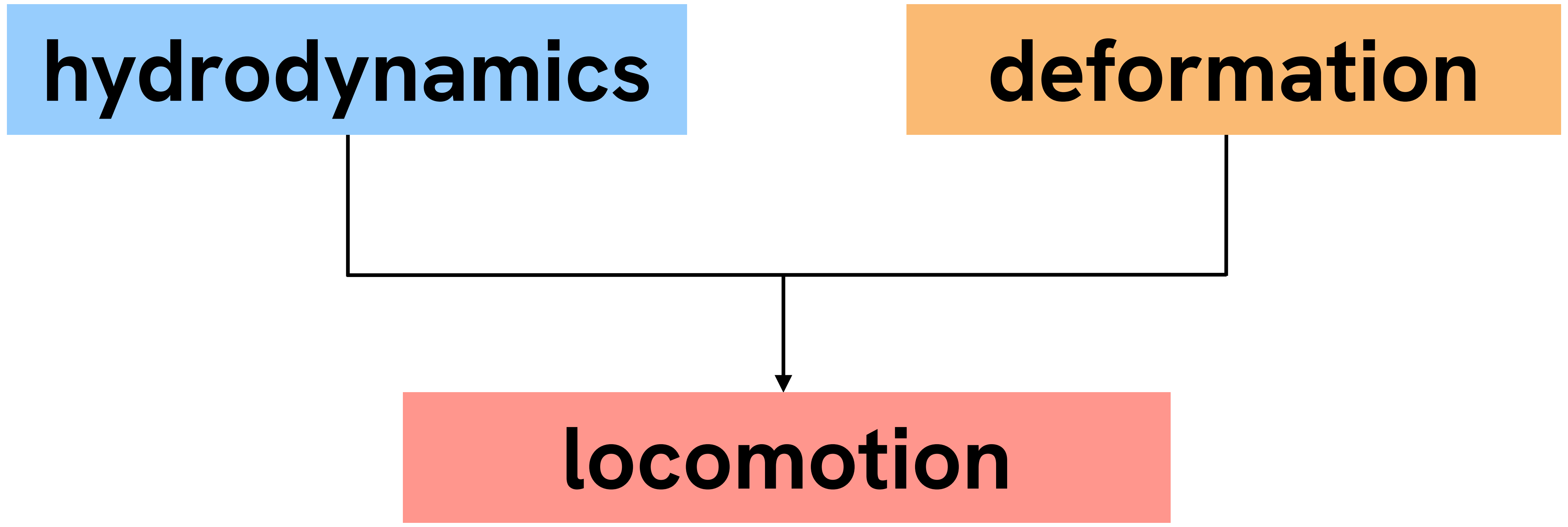




rigid motion	$\mathcal{X}(t)$	Pose in $SE(n)$
rate of deformation	$\dot{\mathcal{S}}(t)$	« Degrees of freedom », finite (or infinite) dimension

Principles of micro-swimming





hydrodynamics

deformation

locomotion

rigid motion	$\mathscr{X}(t)$	Pose in $SE(n)$
rate of deformation	$\dot{\mathscr{S}}(t)$	« Degrees of freedom », finite (or infinite) dimension

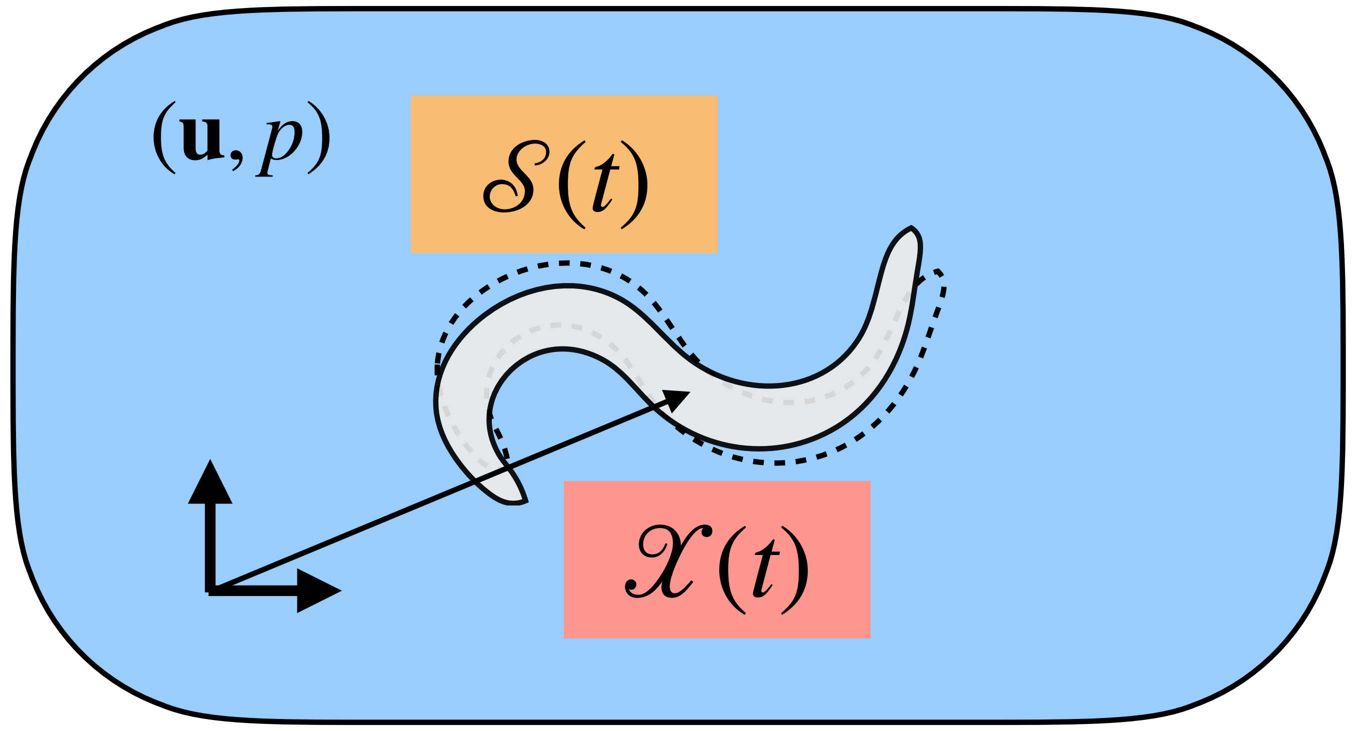
Stokes flow

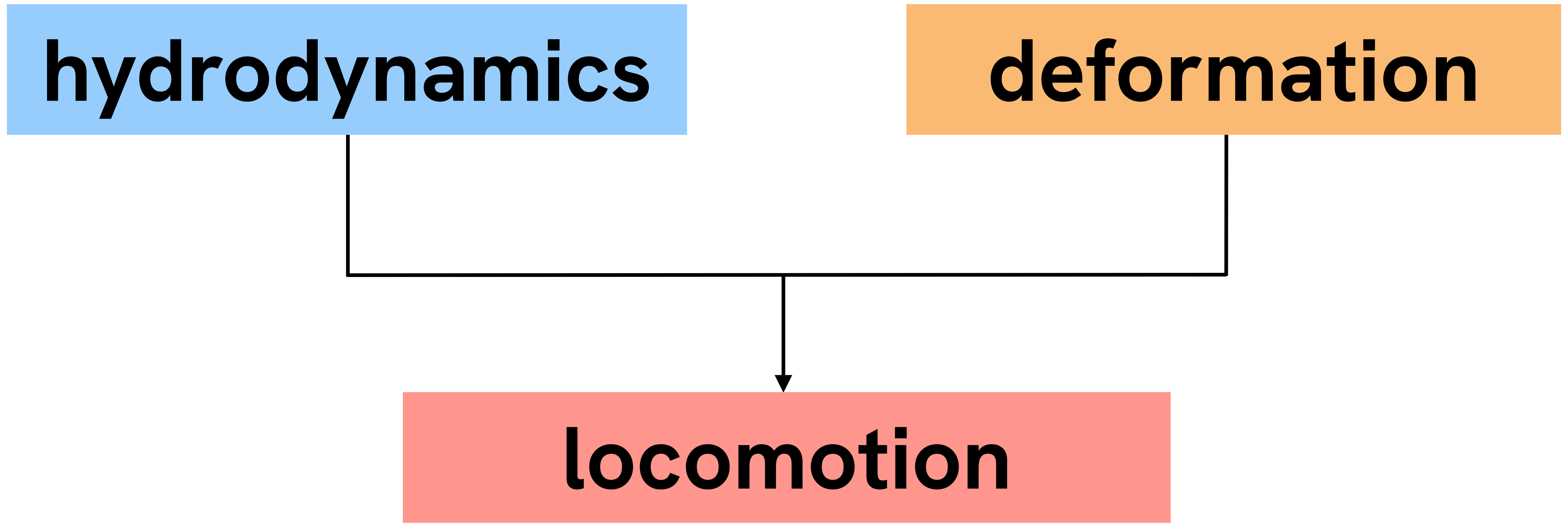
(Zero Reynolds number)

$$\Delta \mathbf{u} + \nabla p = 0, \;\; \nabla \cdot u = 0$$

B.C. : $\dot{\mathscr{X}} + \dot{\mathscr{S}}$

Principles of micro-swimming





rigid motion $\mathscr{X}(t)$ Pose in $SE(n)$

rate of deformation $\dot{\mathscr{S}}(t)$ « Degrees of freedom », finite (or infinite) dimension

Stokes flow

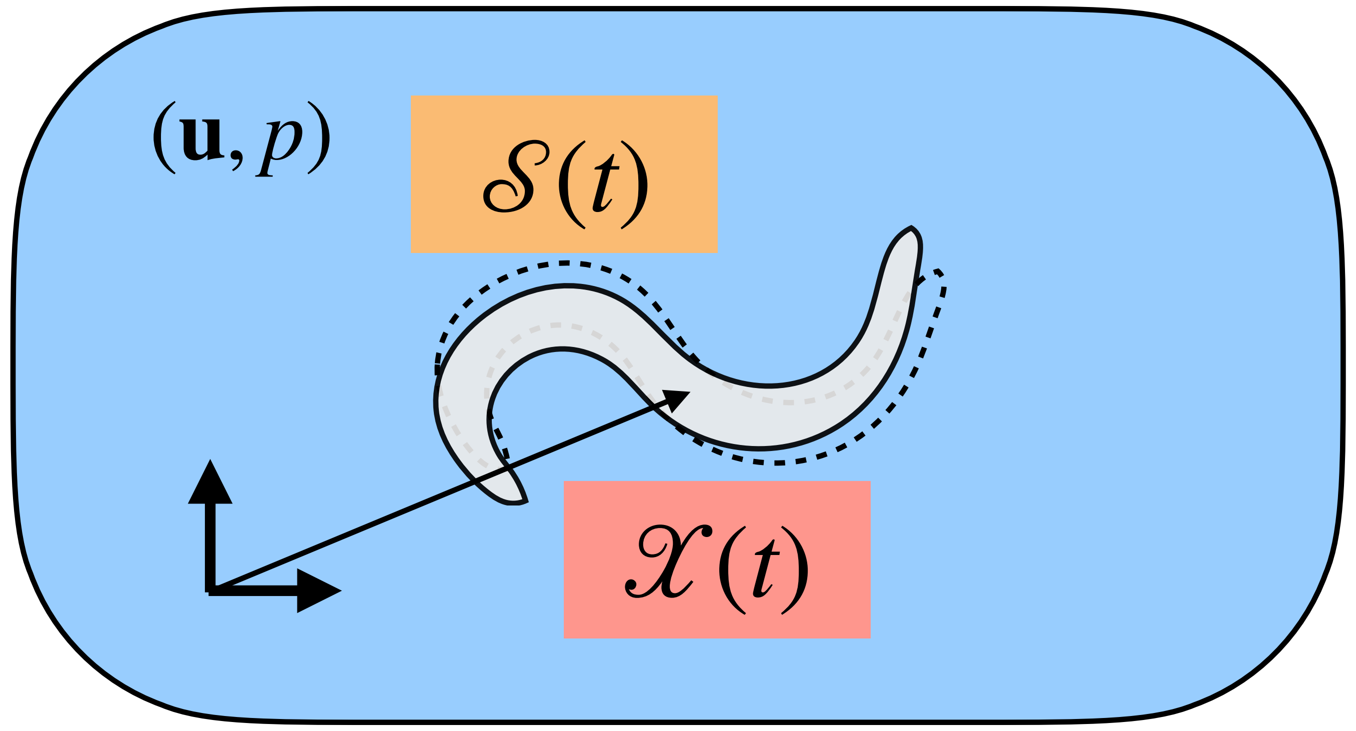
- Linearity:

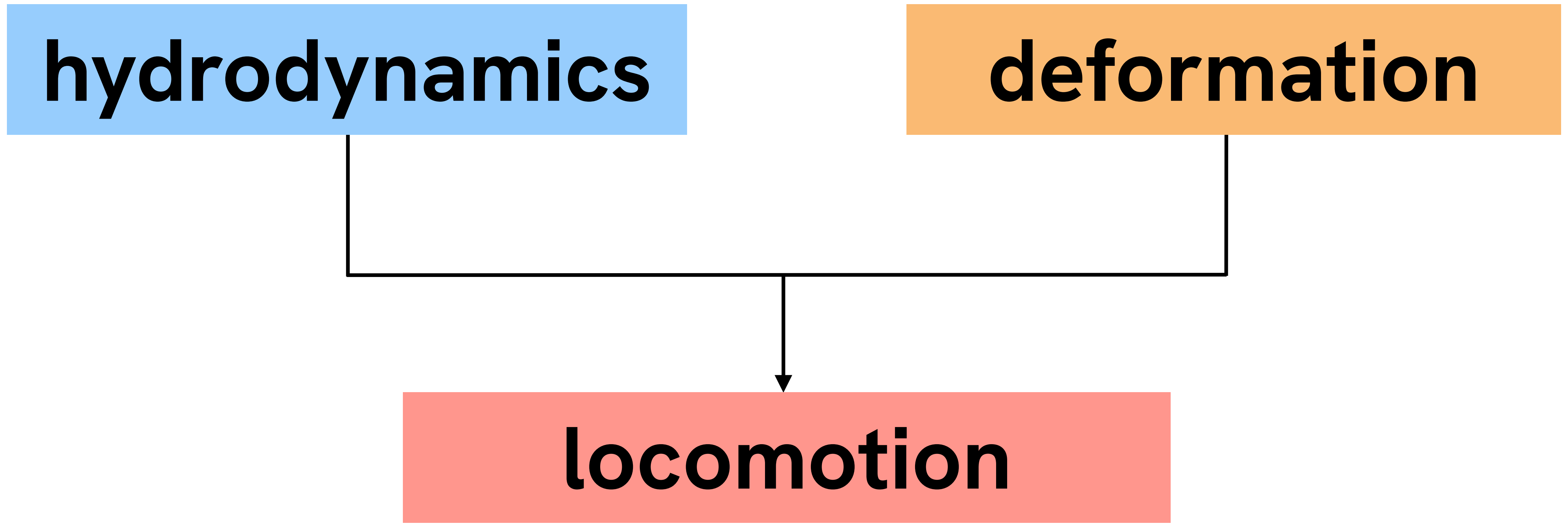
$$\mathbf{F}^{\text{hydro}} = \mathbf{Q}(\mathscr{S})(\dot{\mathscr{X}}, \dot{\mathscr{S}})^T$$

Stokes flow

$$\Delta \mathbf{u} + \nabla p = 0, \quad \nabla \cdot u = 0$$

Principles of micro-swimming




hydrodynamics

deformation

locomotion

rigid motion $\mathcal{X}(t)$ Pose in $SE(n)$

rate of deformation $\dot{\mathcal{S}}(t)$ « Degrees of freedom », finite (or infinite) dimension

Stokes flow

Stokes flow

$$\Delta \mathbf{u} + \nabla p = 0, \ \ \nabla \cdot u = 0$$

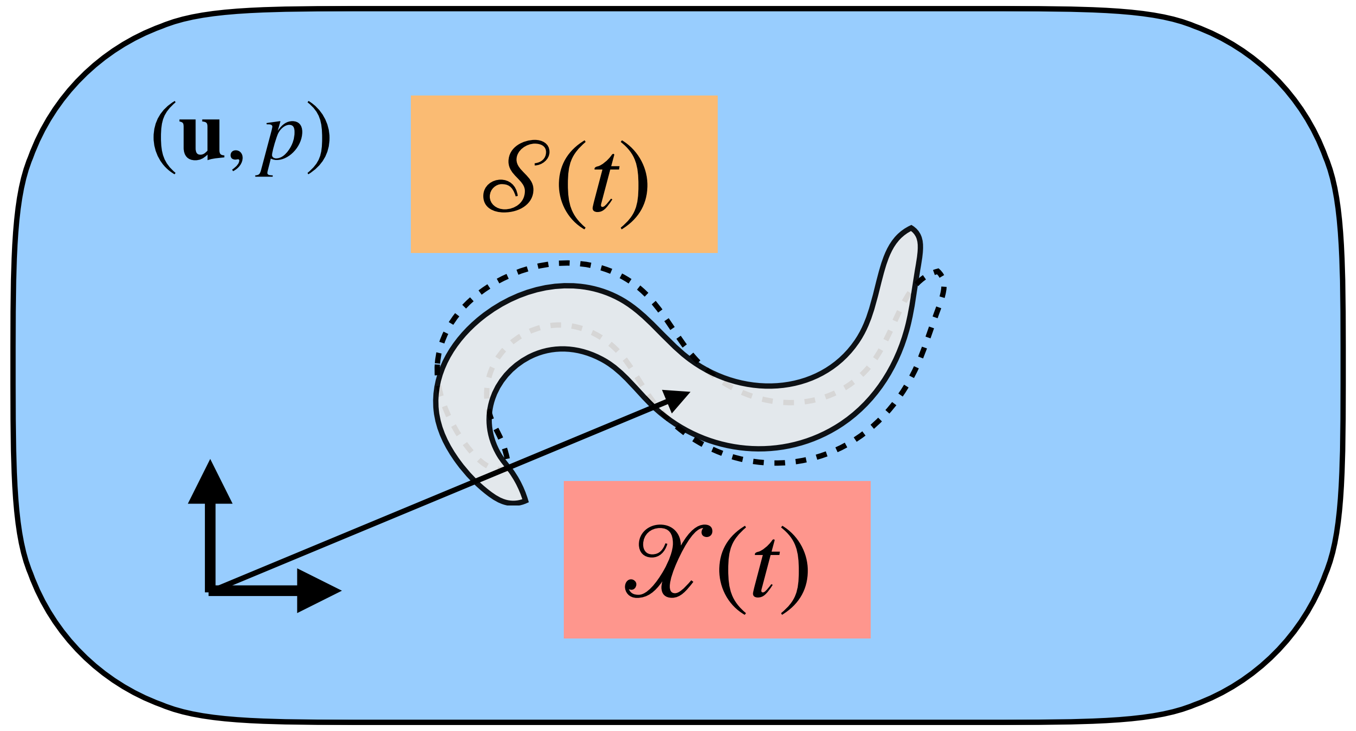
- Linearity:

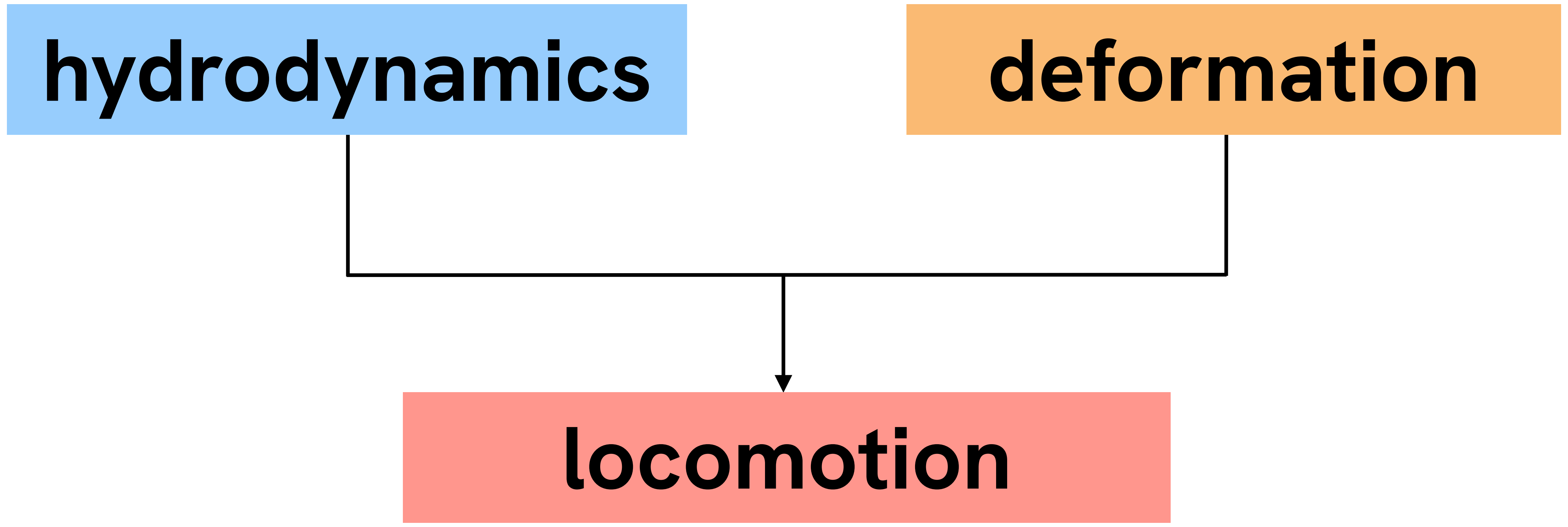
$$\mathbf{F}^{\text{hydro}} = \mathbf{Q}(\mathcal{S})(\dot{\mathcal{X}}, \dot{\mathcal{S}})^T$$

- Inertialess motion:

$$\mathbf{F}^{\text{hydro}} = 0$$

Principles of micro-swimming





hydrodynamics

deformation

locomotion

rigid motion	$\mathscr{X}(t)$	Pose in $SE(n)$
rate of deformation	$\dot{\mathscr{S}}(t)$	« Degrees of freedom », finite (or infinite) dimension

Stokes flow

Stokes flow

$$\Delta \mathbf{u} + \nabla p = 0, \;\; \nabla \cdot u = 0$$

- Linearity:

$$\mathbf{F}^{\text{hydro}} = \mathbf{Q}(\mathscr{S})(\dot{\mathscr{X}}, \dot{\mathscr{S}})^T$$

- Inertialess motion:

$$\mathbf{F}^{\text{hydro}} = 0$$

→

« Stokes connection »

$$\dot{\mathscr{X}} = \mathbf{N}(\mathscr{S})\dot{\mathscr{S}}$$

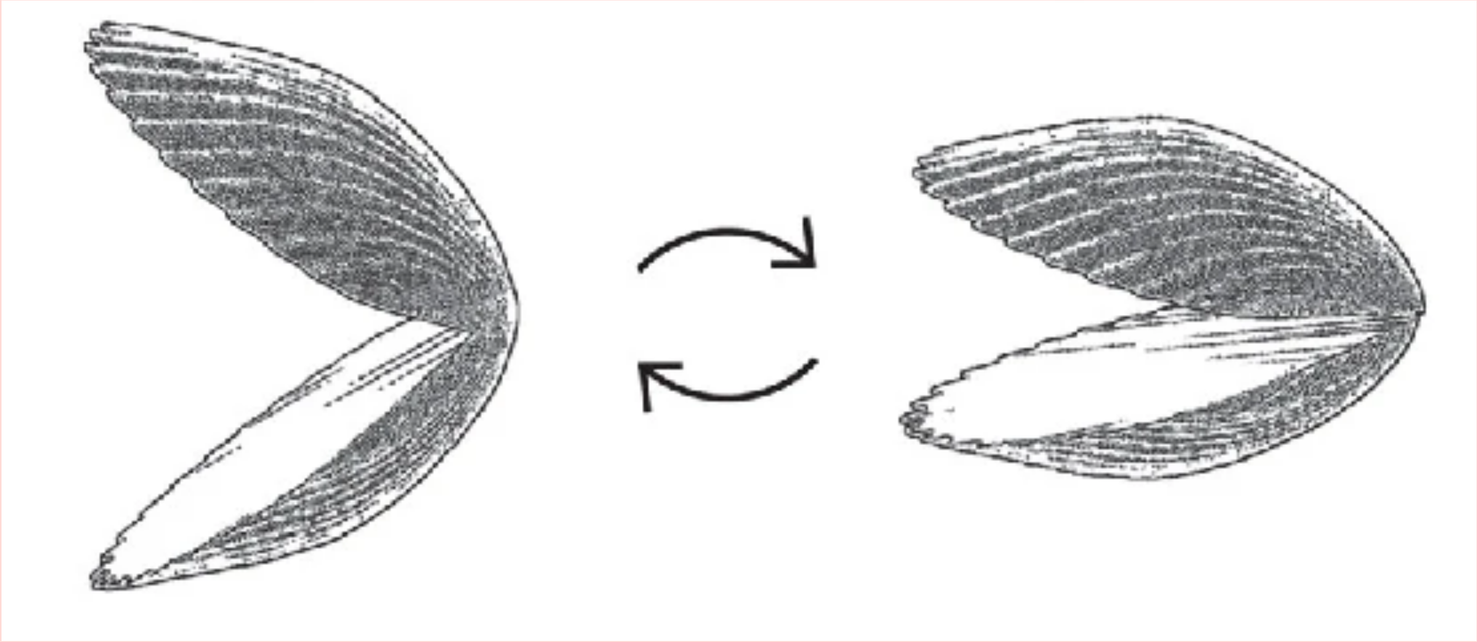
Swimming equation

Principles of microswimming

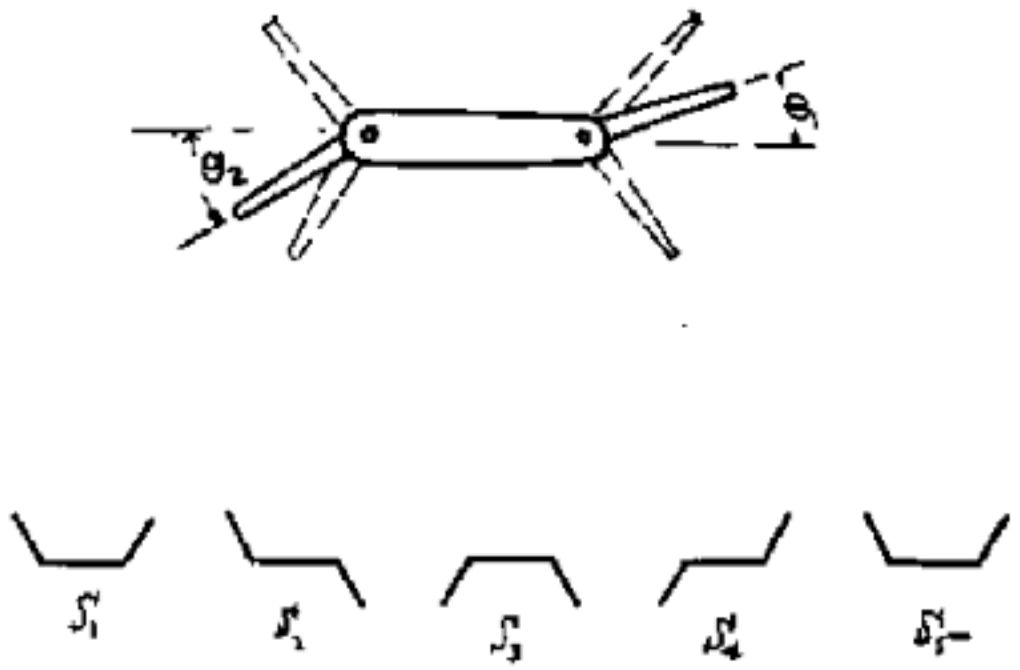
Consequence: Scallop theorem

“A swimmer with only one degree of freedom cannot produce net locomotion at low Reynold’s number”

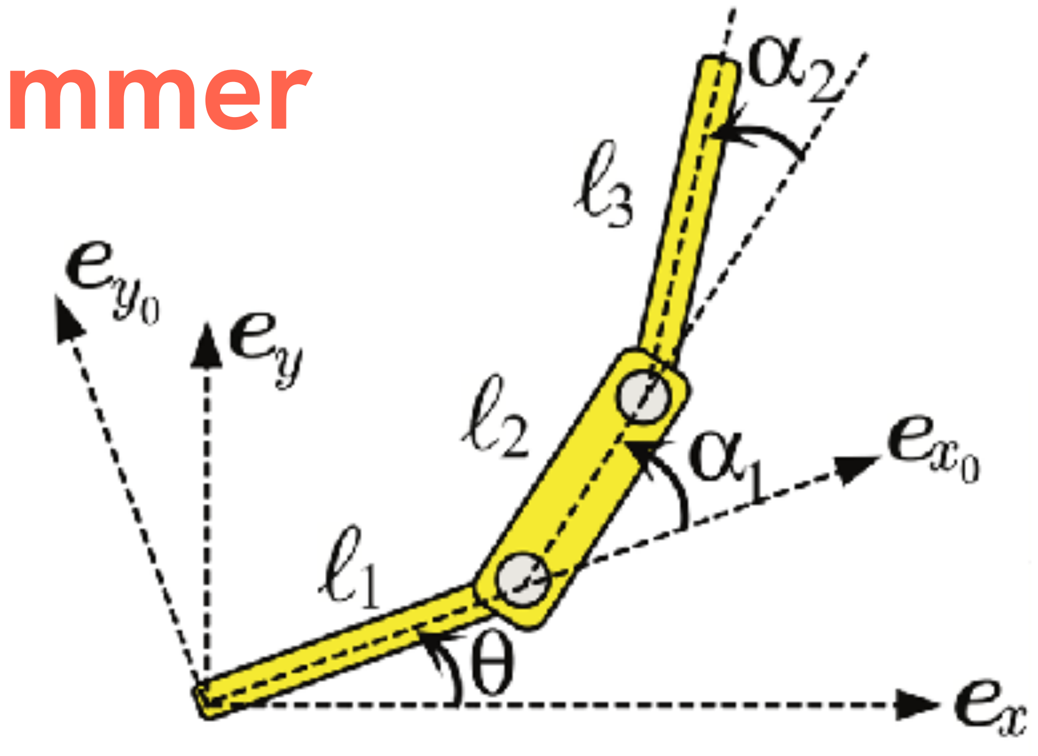
→ Non-reciprocal deformation is required



- **The famous Purcell swimmer**

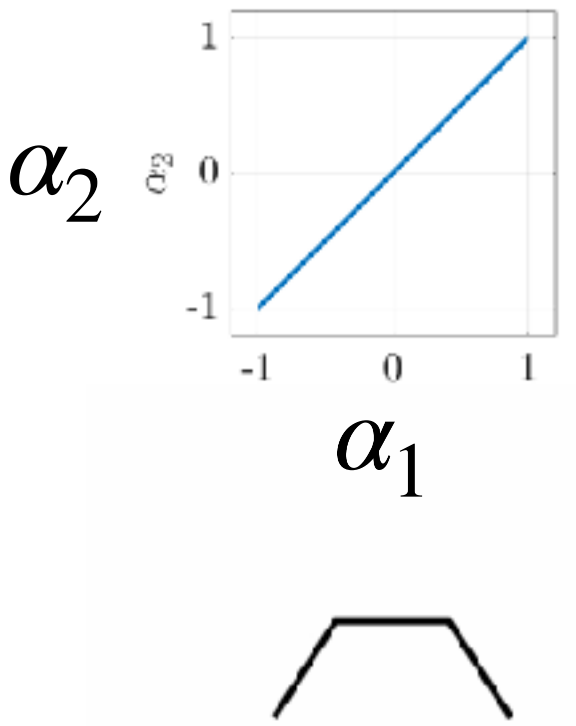


Purcell, Am. J. Phys., 1977



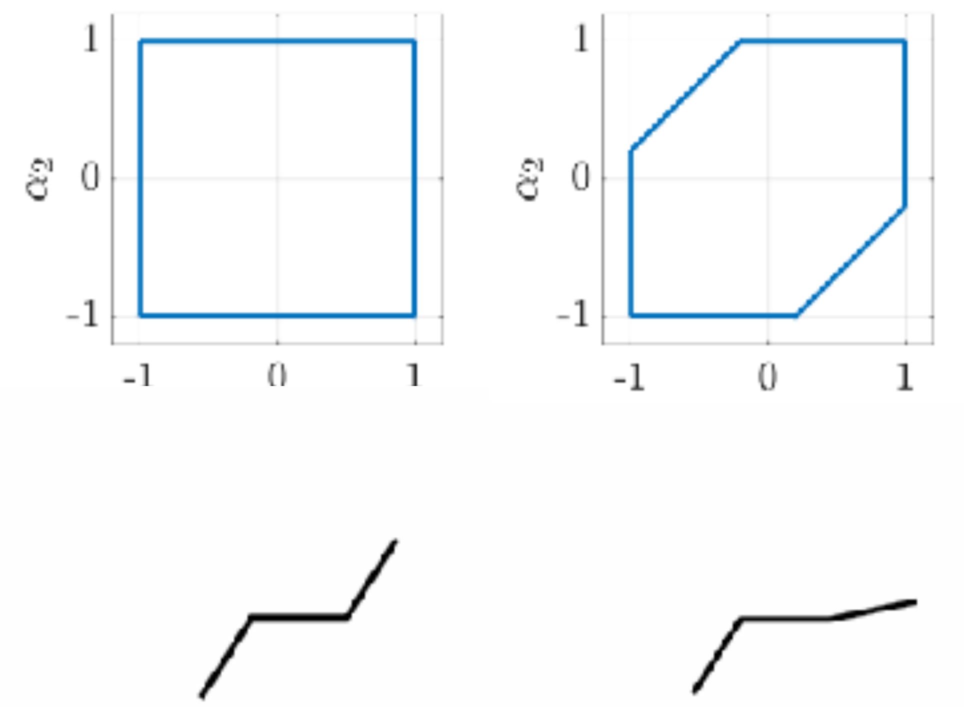
Three rigid links with flexible junctions

Reciprocal gait



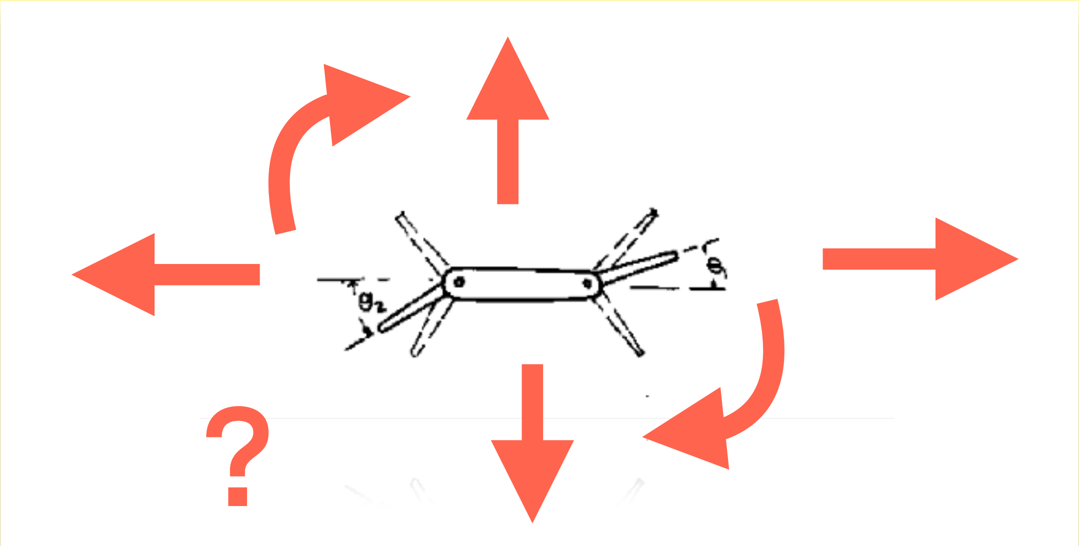
No net motion

Non-reciprocal gait



Net motion

“Non-reciprocal deformation is required for net locomotion”



- **Question 1**

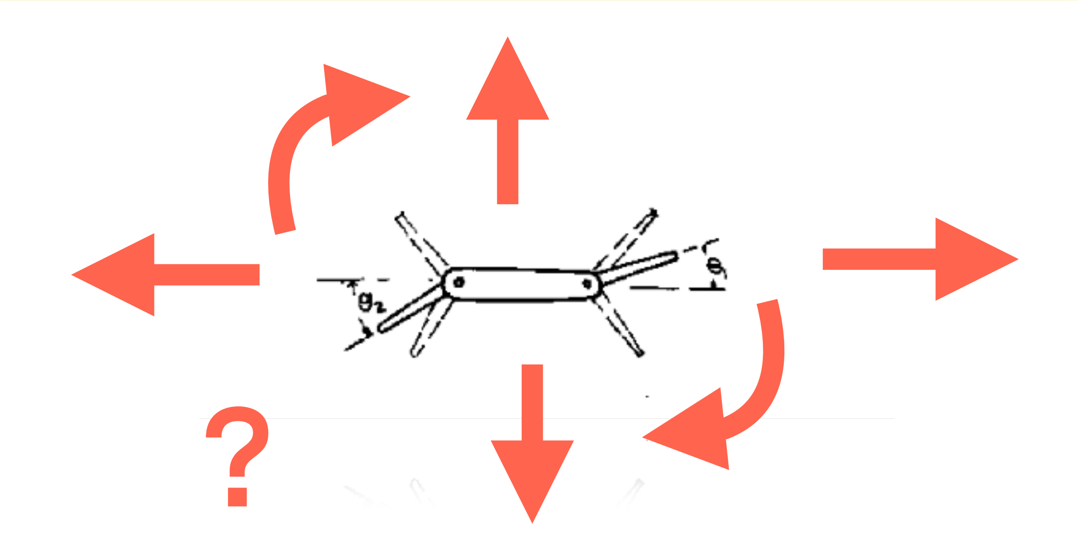
Which kind of net locomotion?
Is **every direction** reachable as soon as I can do a “nonreciprocal” deformation?

→ “Controllability”

- **Question 2**

Can I generate nonreciprocal deformation (do I preserve controllability) if I **don’t control the shape directly** and if so, how?

“Non-reciprocal deformation is required for net locomotion”



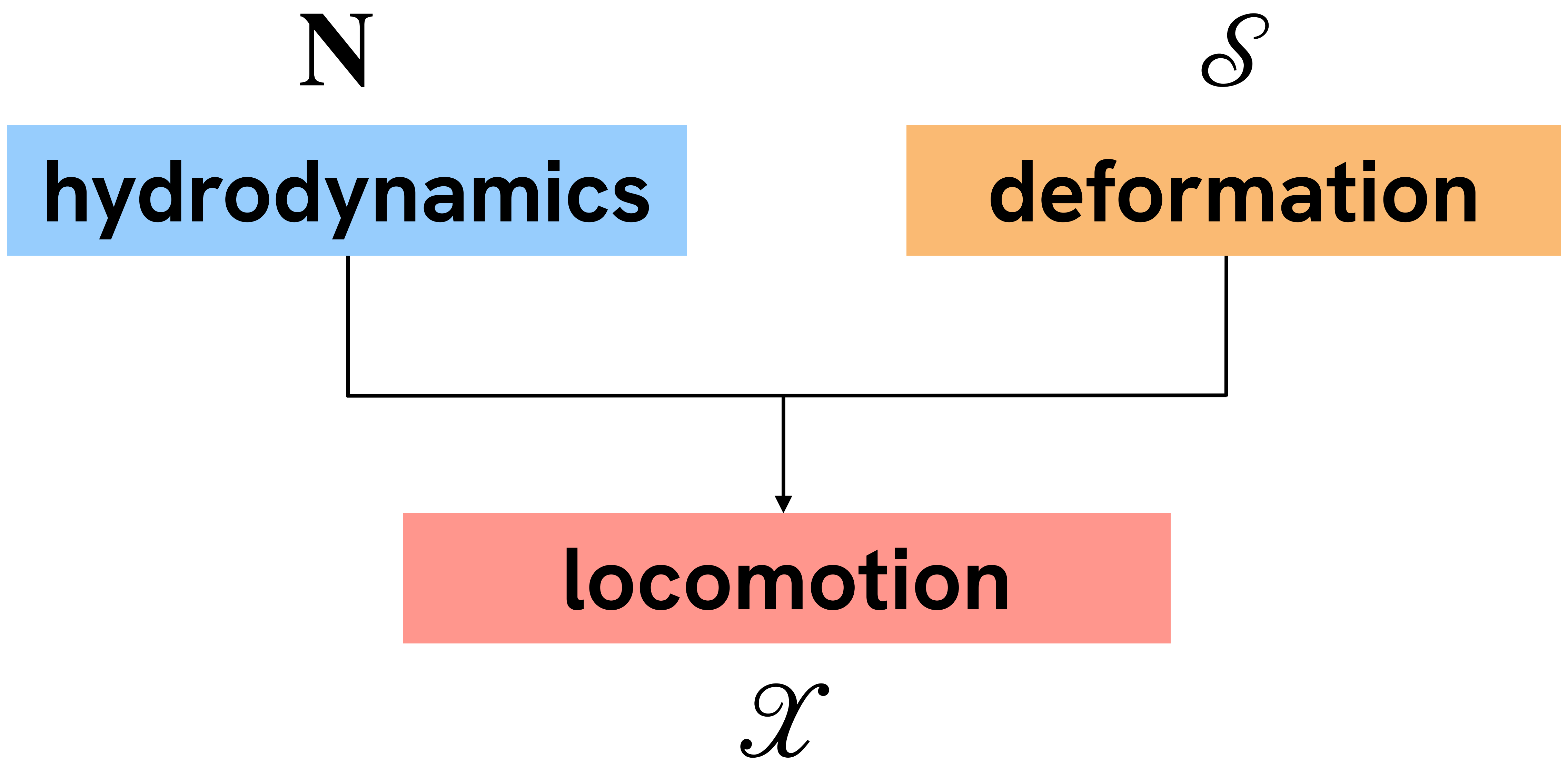
Question:

Which kind of net locomotion?

Is **every direction** reachable as soon as I can do a “nonreciprocal” deformation?

→ **controllability**

Swimming is a driftless control-affine system



$\mathbf{N}$

hydrodynamics

$\mathcal{S}$

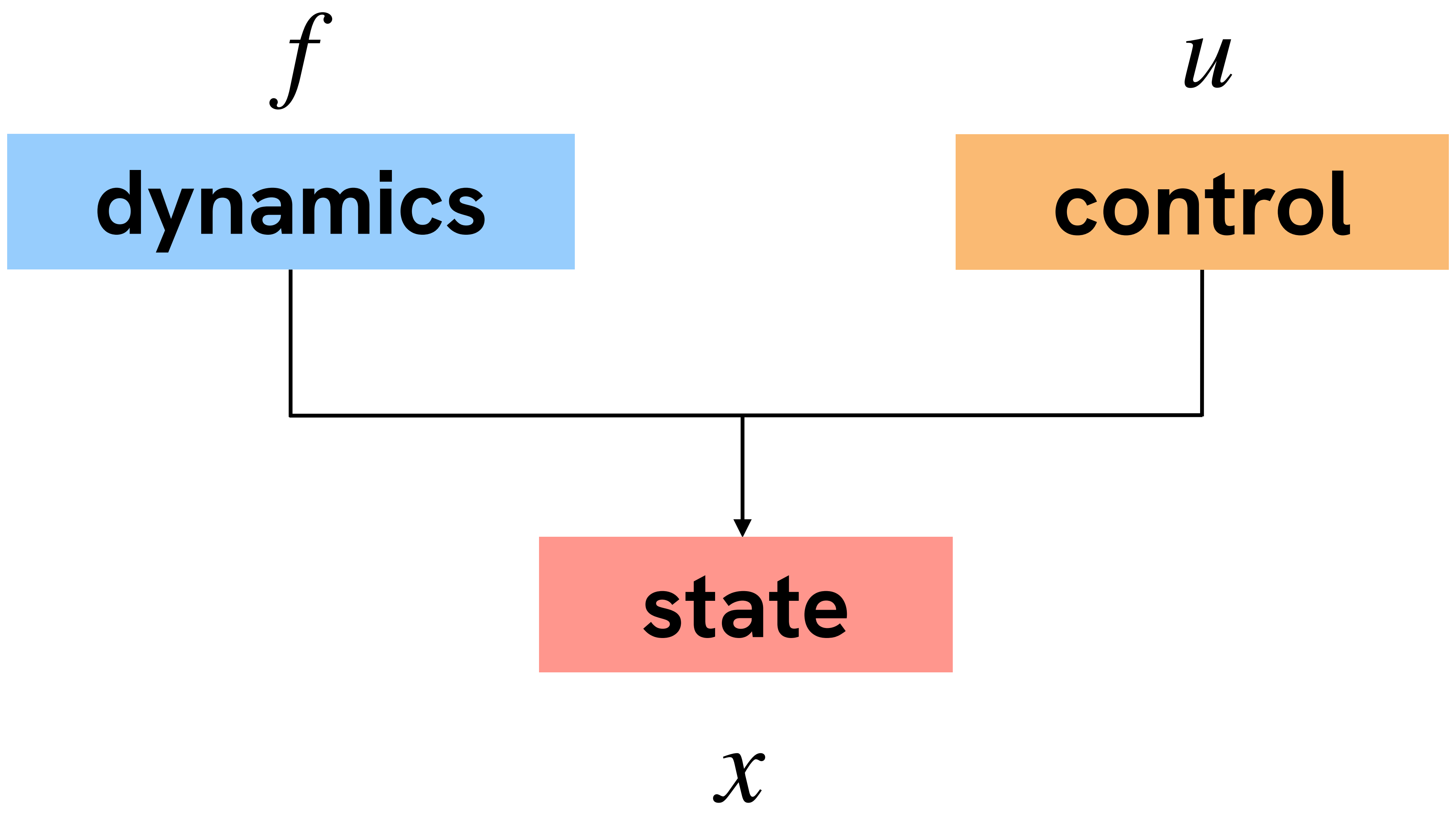
deformation

locomotion

$\mathcal{X}$

$$\dot{\mathcal{X}} = \mathbf{N}(\mathcal{S})\dot{\mathcal{S}}$$

Swimming equation



f

dynamics

u

control

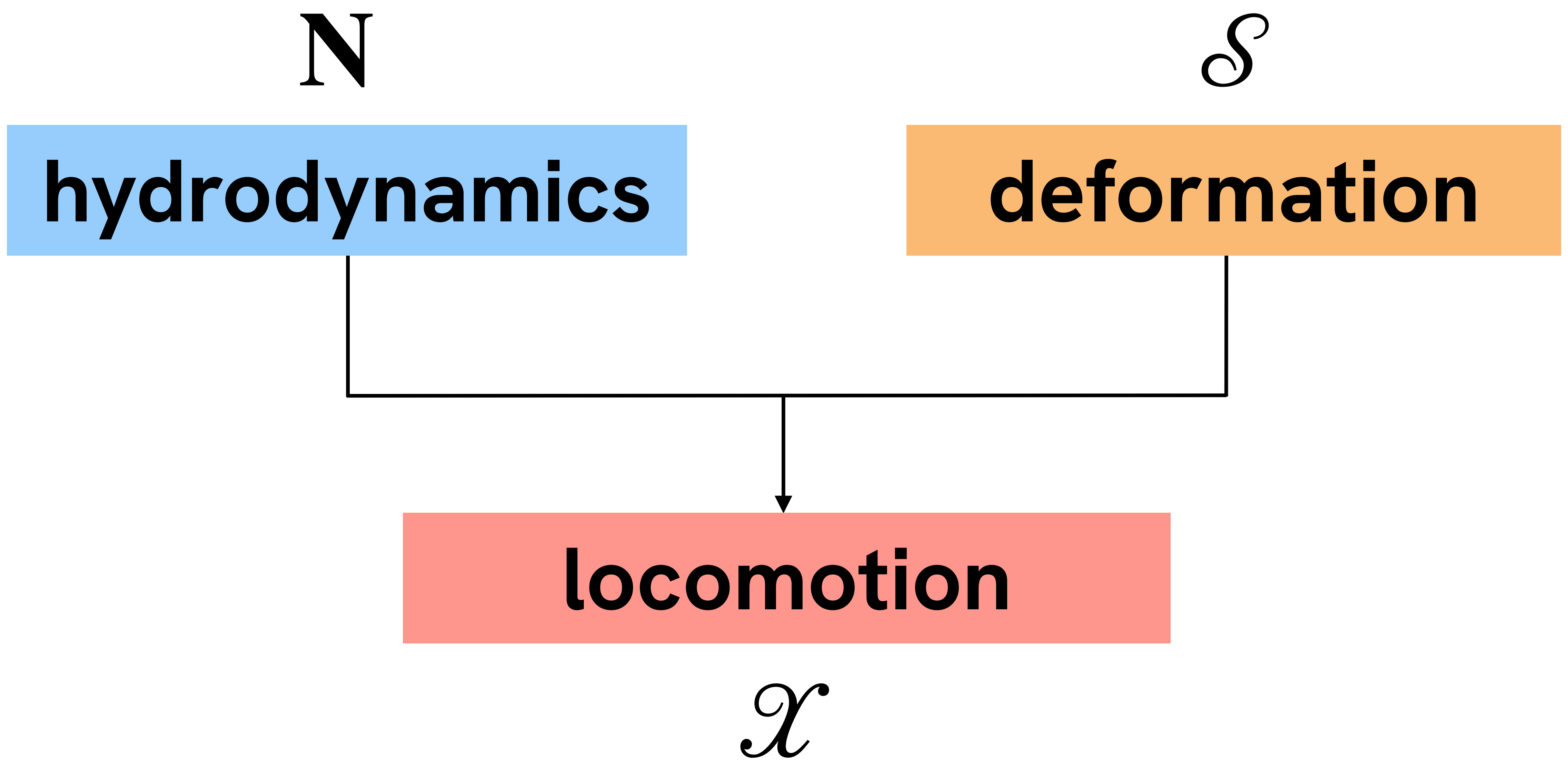
state

x

$$\dot{x} = \sum_{i=1}^{m} f_i(x)u_i$$

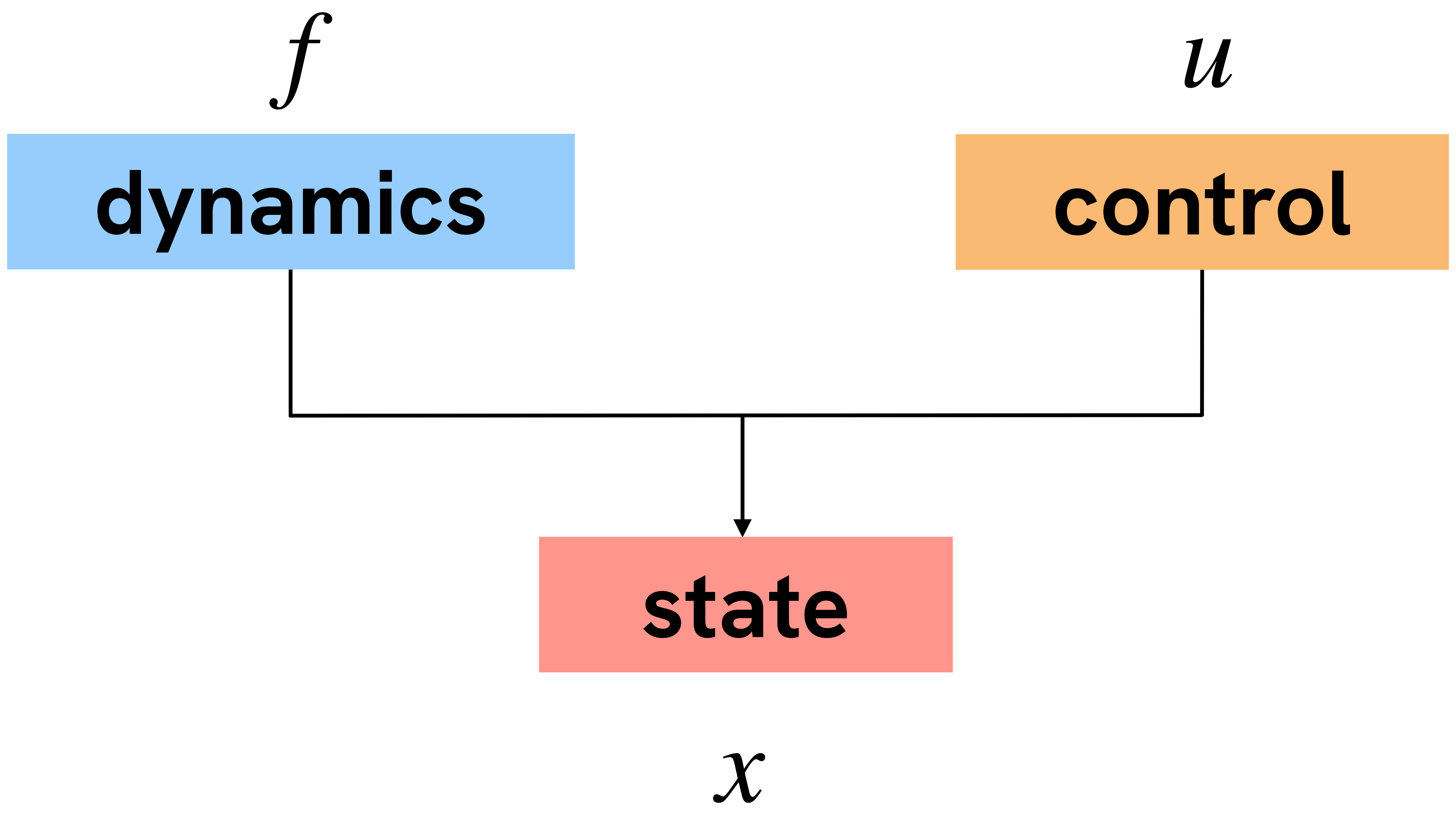
Control-affine system

Swimming is a driftless control-affine system



$$\begin{pmatrix}\dot{\mathscr{X}}\\ \dot{\mathscr{S}}\end{pmatrix} = \begin{pmatrix}\mathbf{N}(\mathscr{S})\\ \mathrm{Id}\end{pmatrix}\dot{\mathscr{S}}$$

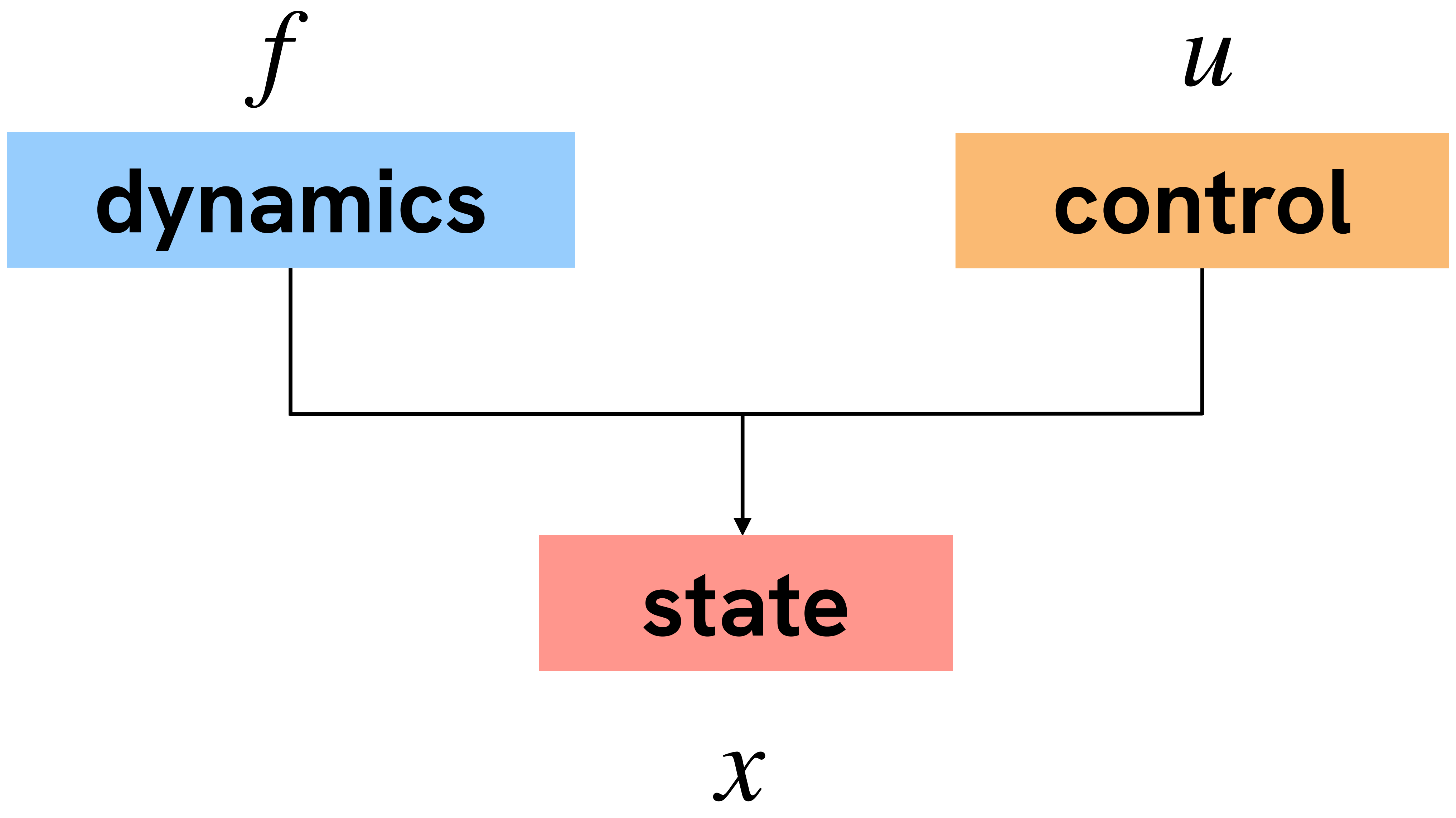
Swimming equation



$$\dot{x} = \sum_{i=1}^{m} f_i(x)u_i$$

Control-affine system

Swimming is a driftless control-affine system



f

dynamics

u

control

state

x

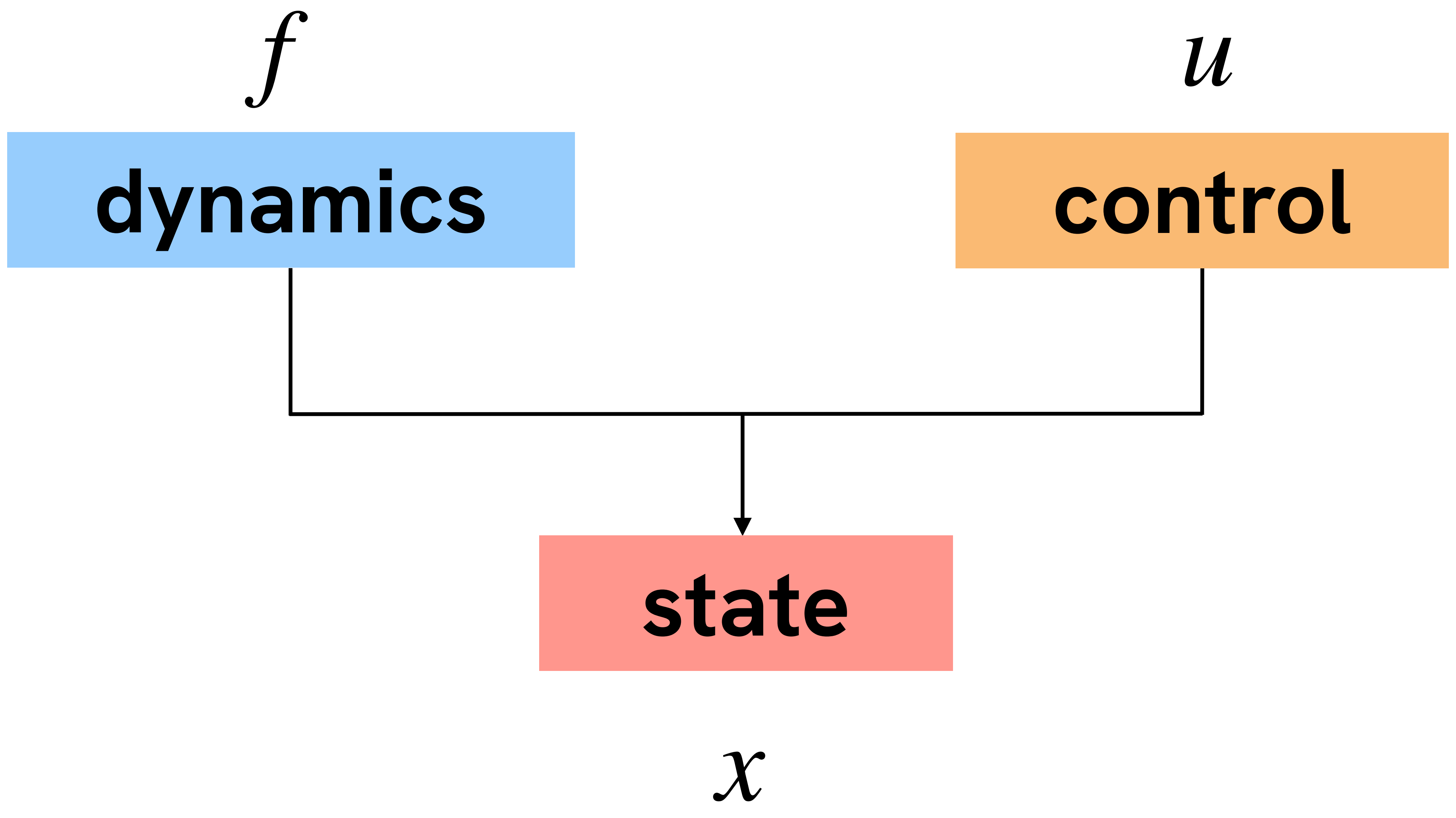
$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i$$

Control-affine system

Assumption

f_i **real analytic** vector fields

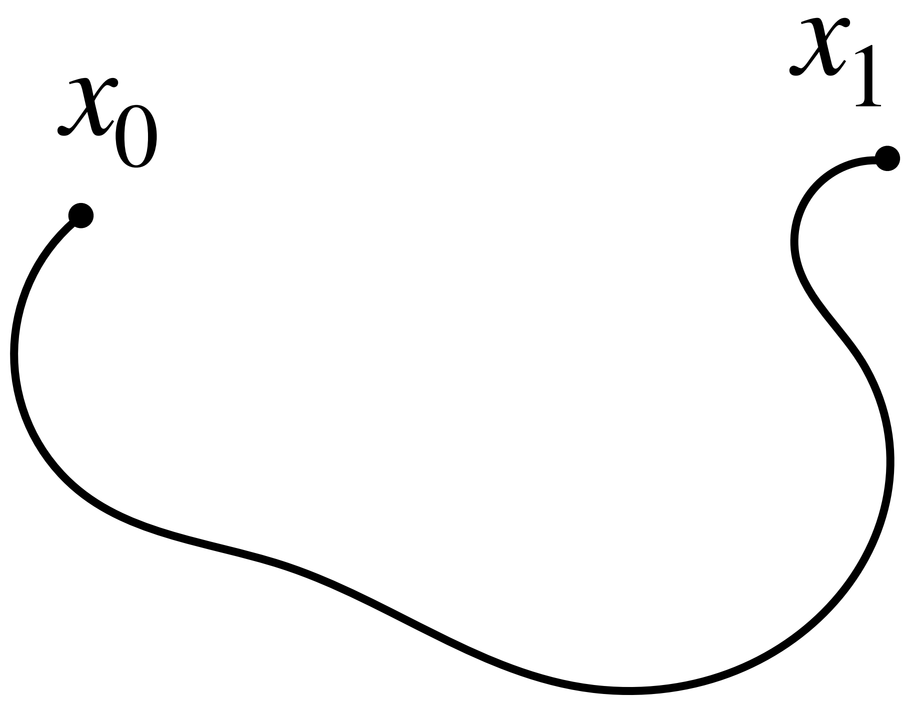
Controllability



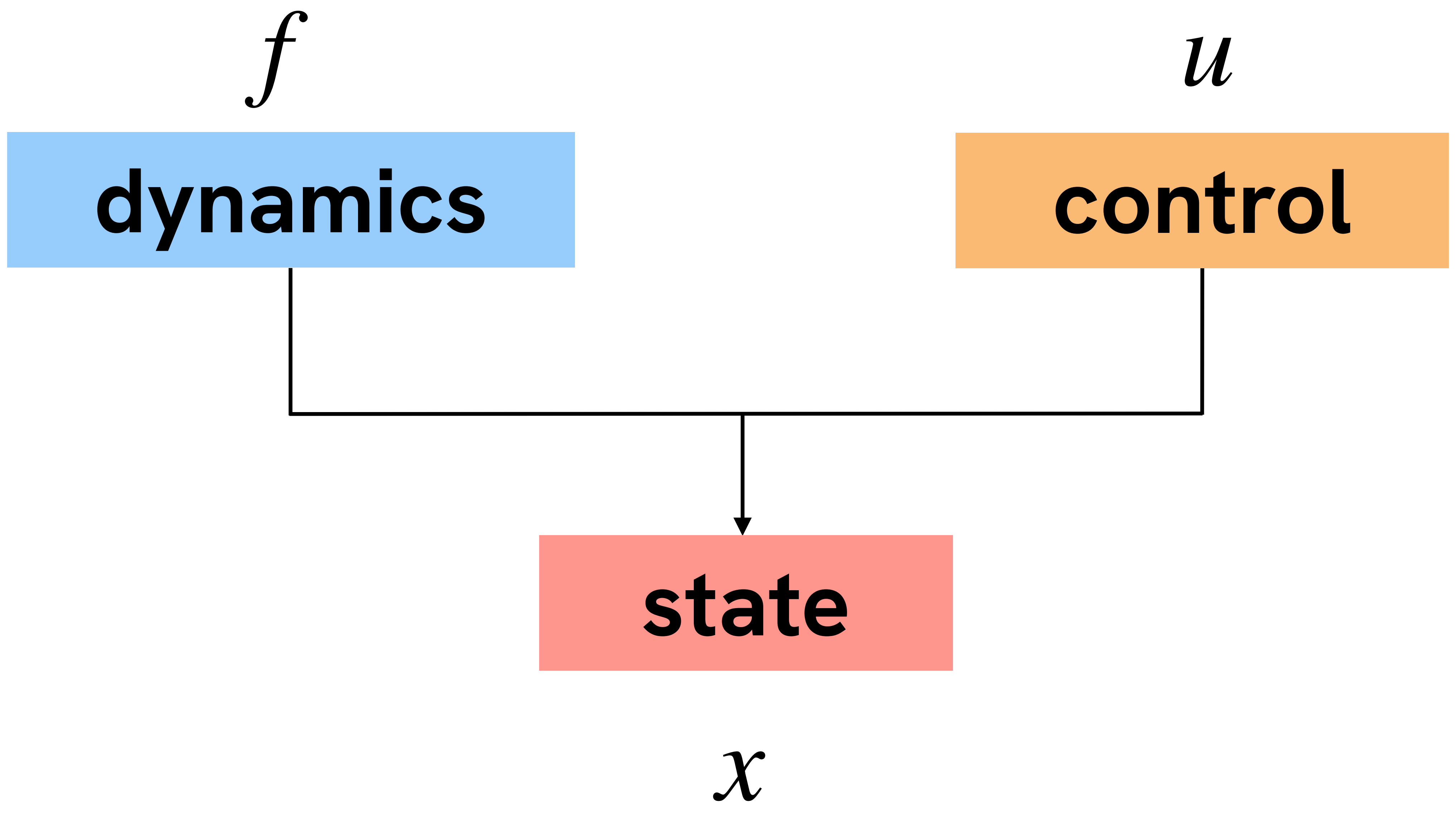
$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i$$

Control-affine system

Controllability: it is possible to go from x_0 to x_1 (for all x_0, x_1)



Controllability



f

dynamics

u

control

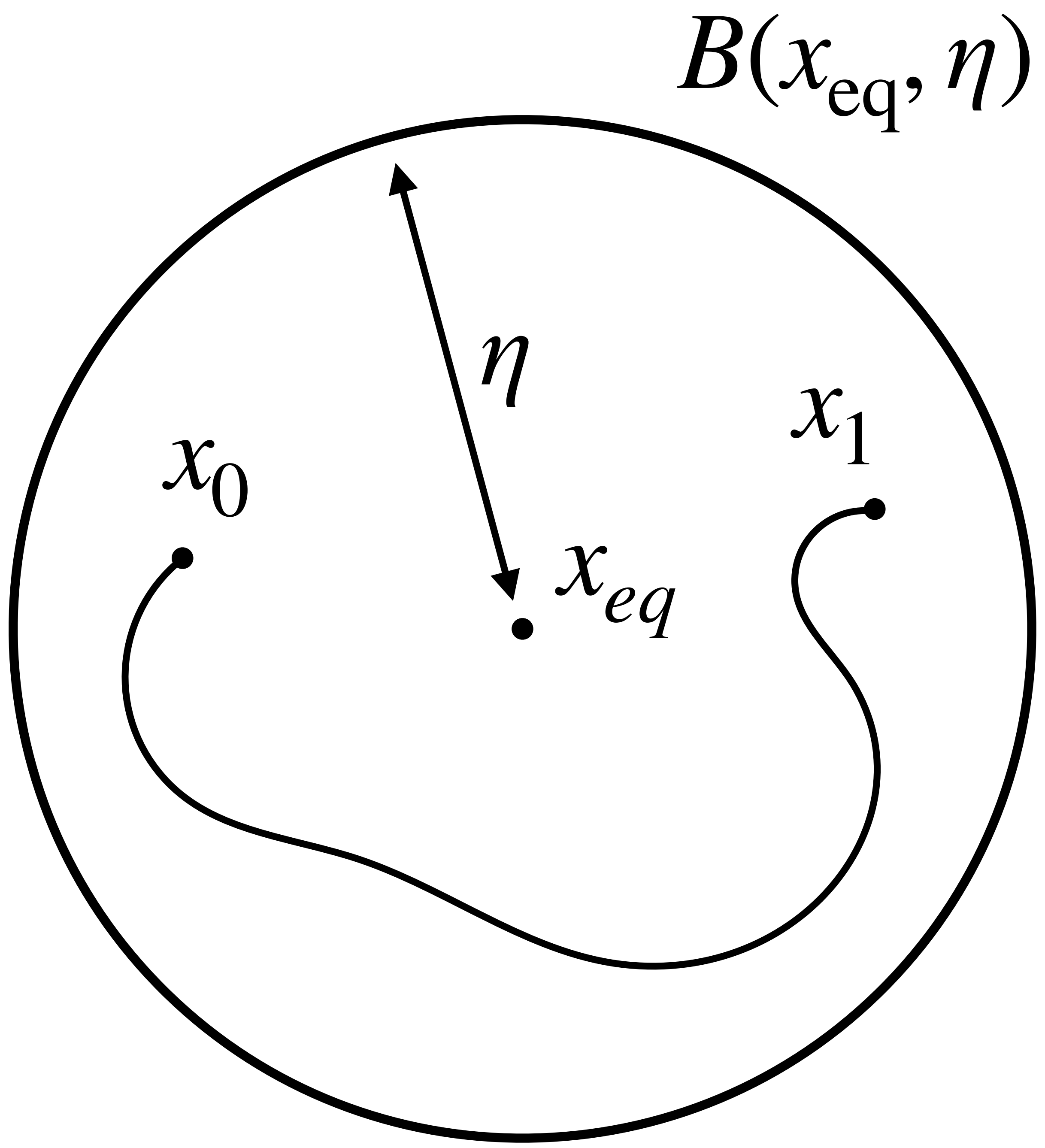
state

x

$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i$$

Control-affine system

Local controllability: it is possible to go from x_0 to x_1 **in a neighbourhood of an equilibrium**



STLC: in **small time** and with **small controls**

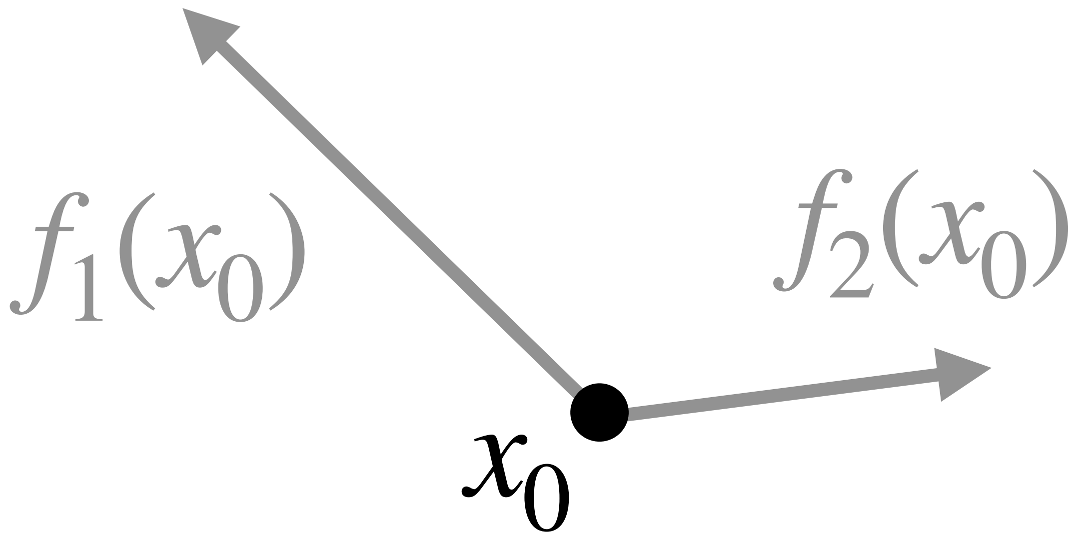
B-STLC: in **small time** and with **bounded controls**

Lie brackets: formal non-reciprocity

$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i$$

$$\searrow \; m = 2$$

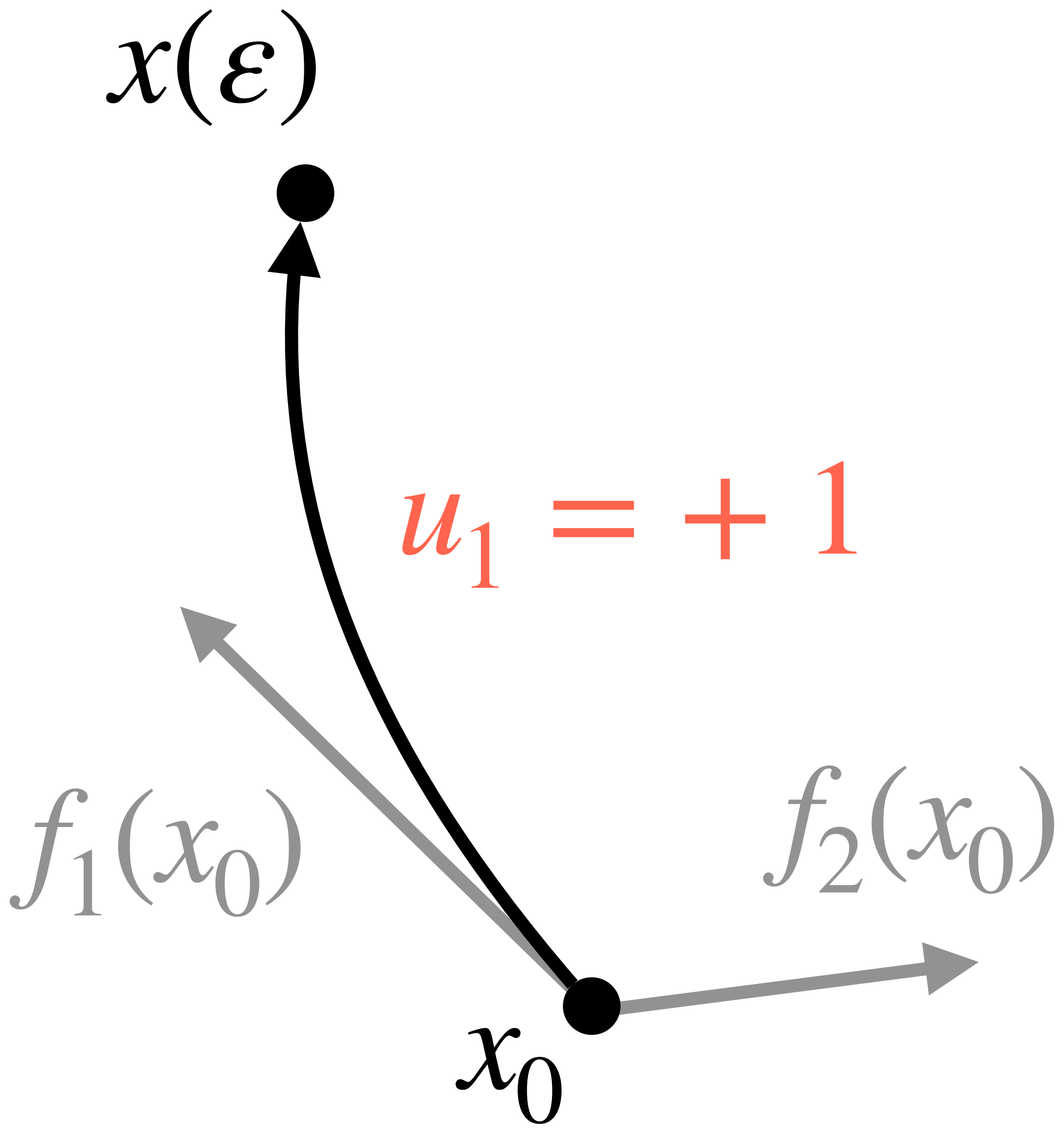
$$\dot{x} = f_1(x)u_1 + f_2(x)u_2$$



Lie brackets: formal non-reciprocity

$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i \quad \xrightarrow{m = 2}$$

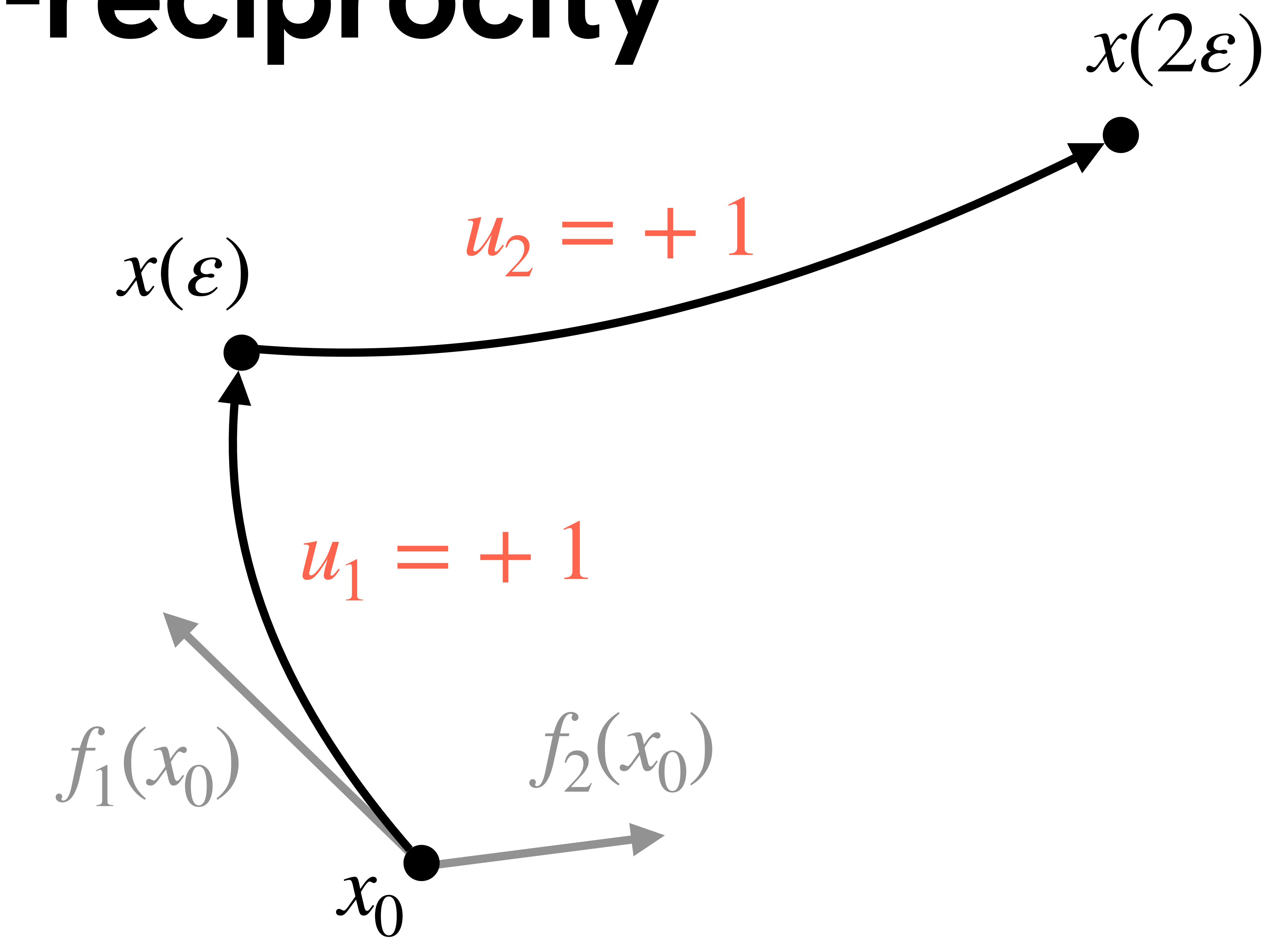
$$\dot{x} = f_1(x)u_1 + f_2(x)u_2$$



Lie brackets: formal non-reciprocity

$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i \searrow m = 2$$

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2$$

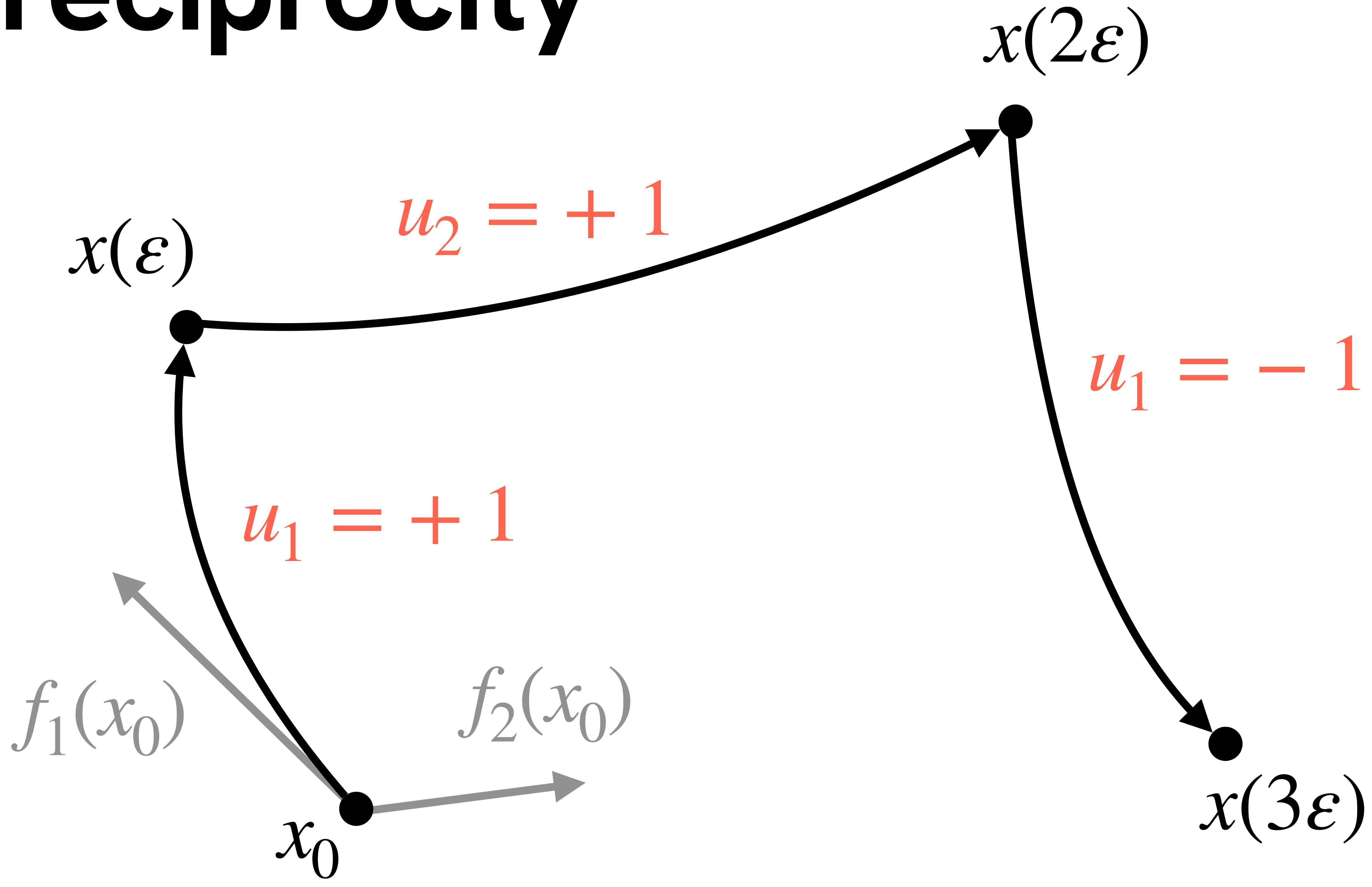


Lie brackets: formal non-reciprocity

$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i$$

$$\searrow m = 2$$

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2$$

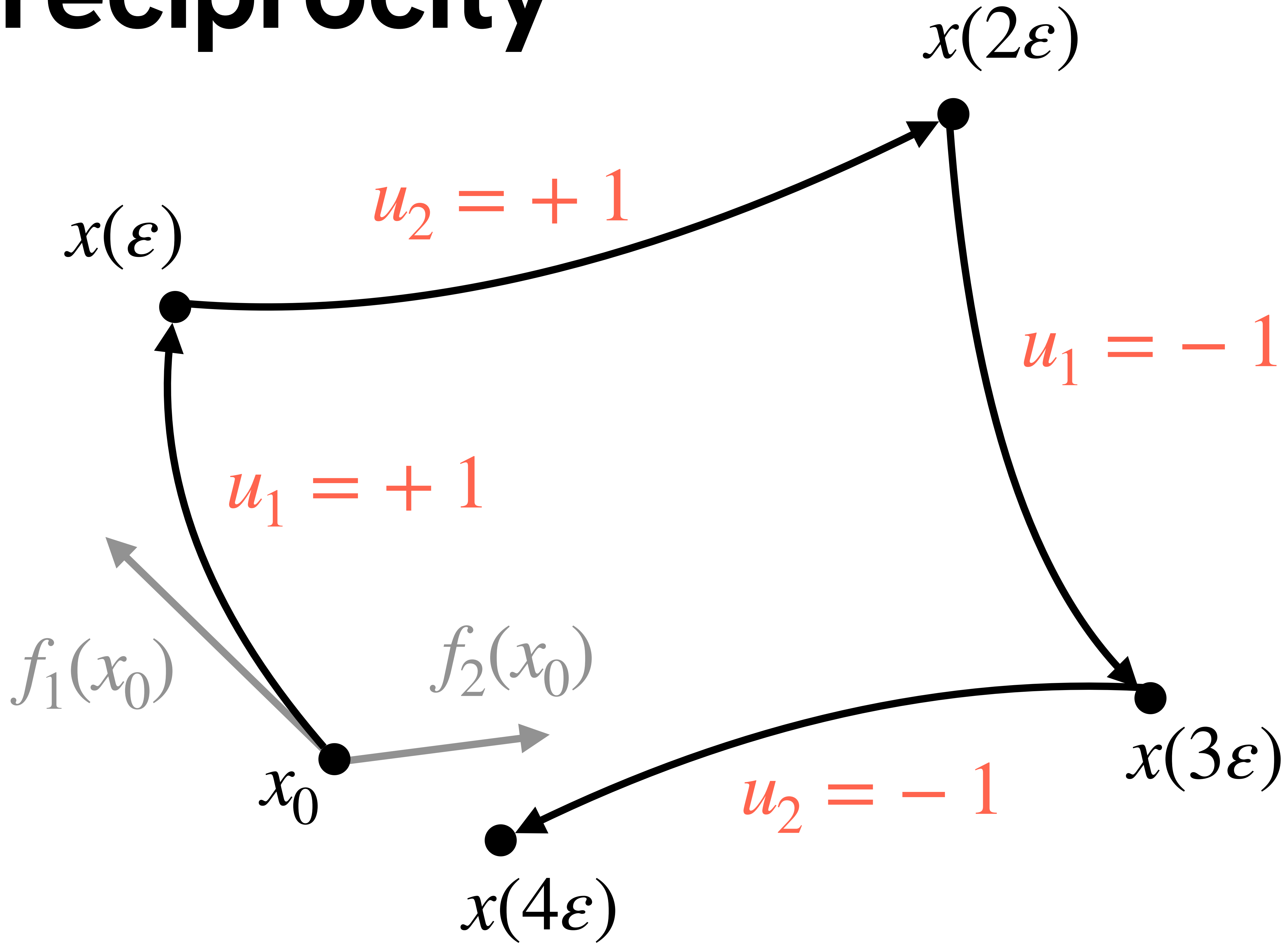


Lie brackets: formal non-reciprocity

$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i$$

$\searrow$ $m = 2$

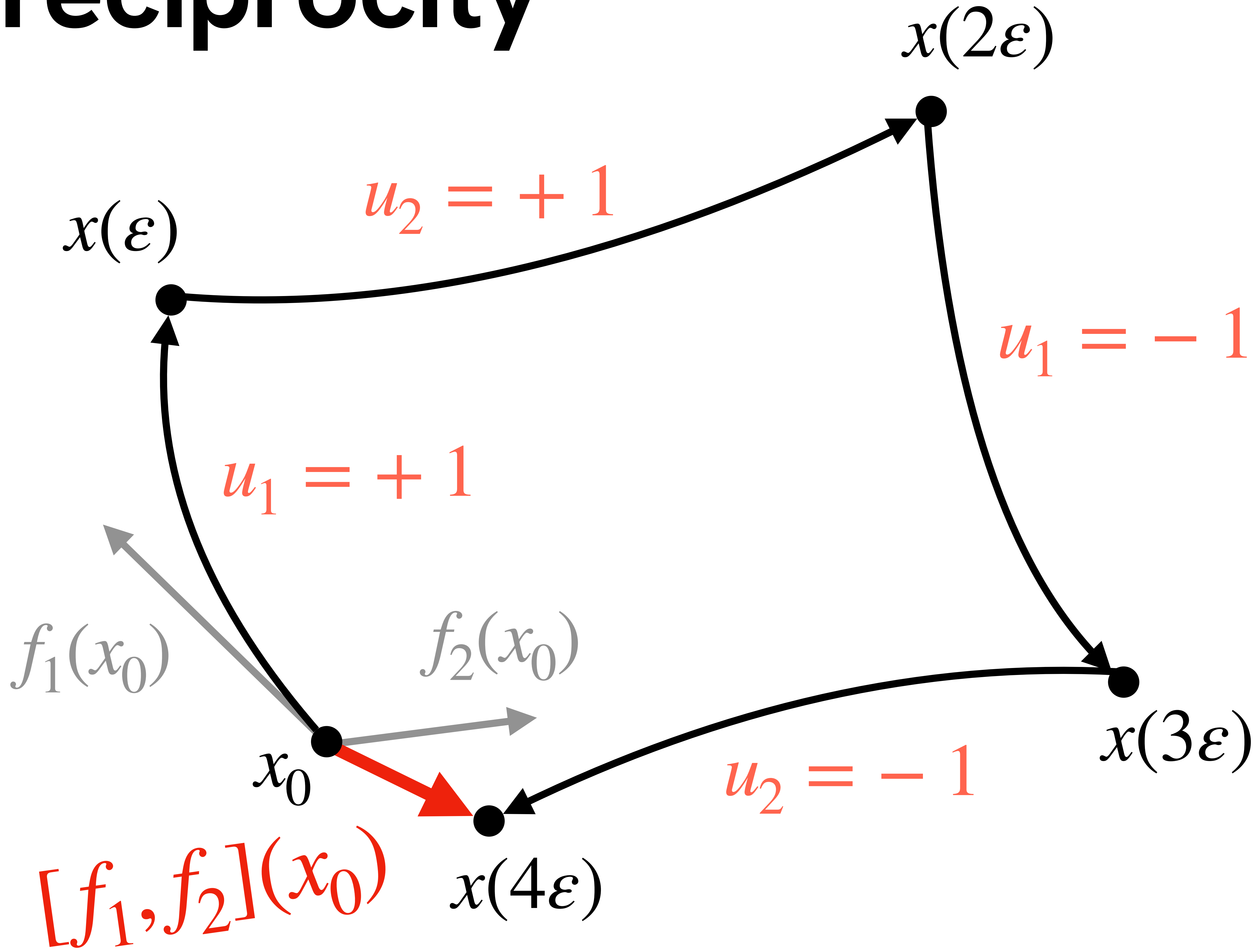
$$\dot{x} = f_1(x)u_1 + f_2(x)u_2$$



Lie brackets: formal non-reciprocity

$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i \quad \searrow \; m = 2$$

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2$$

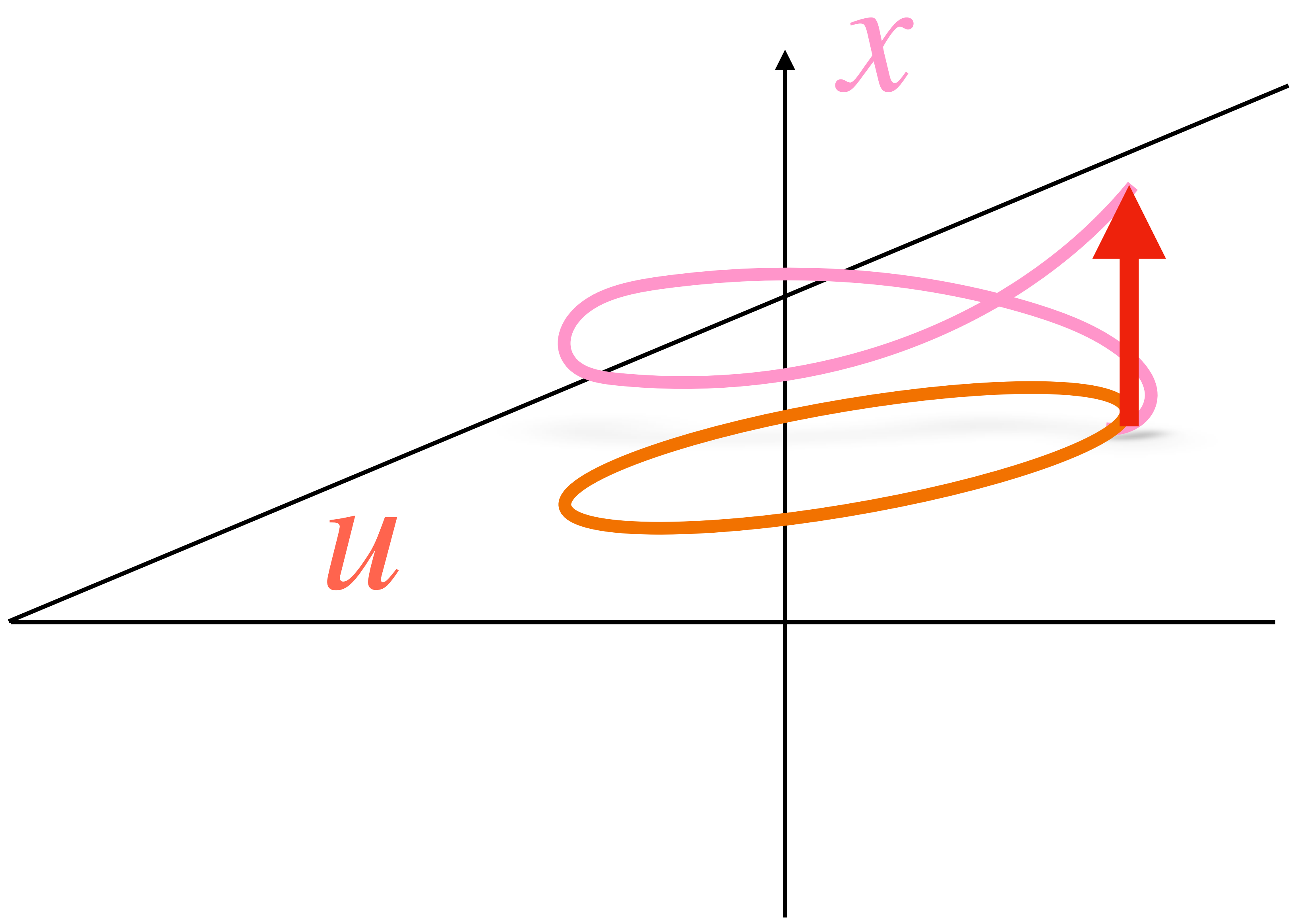


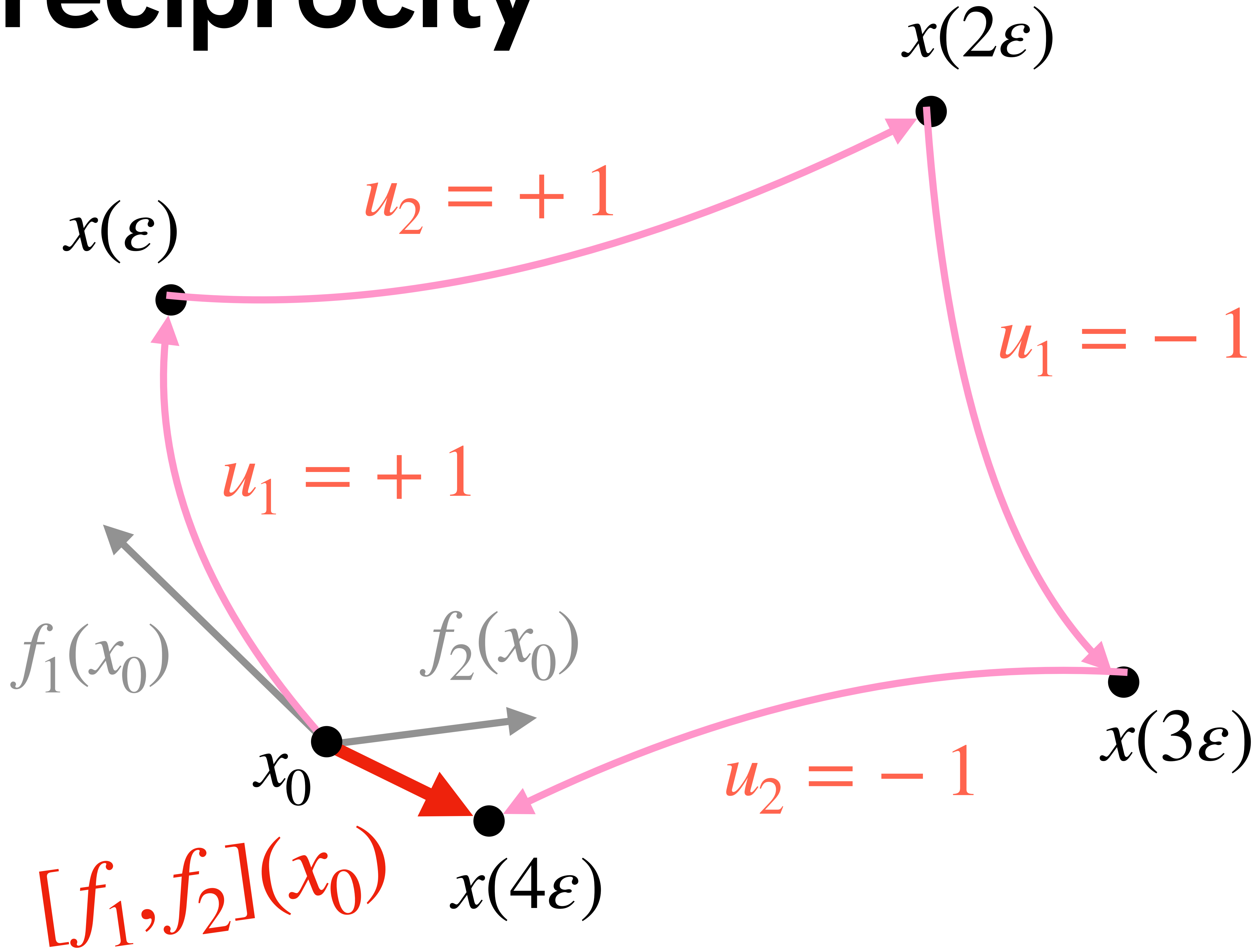
Lie brackets: formal non-reciprocity

$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i$$

$$\searrow \; m = 2$$

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2$$



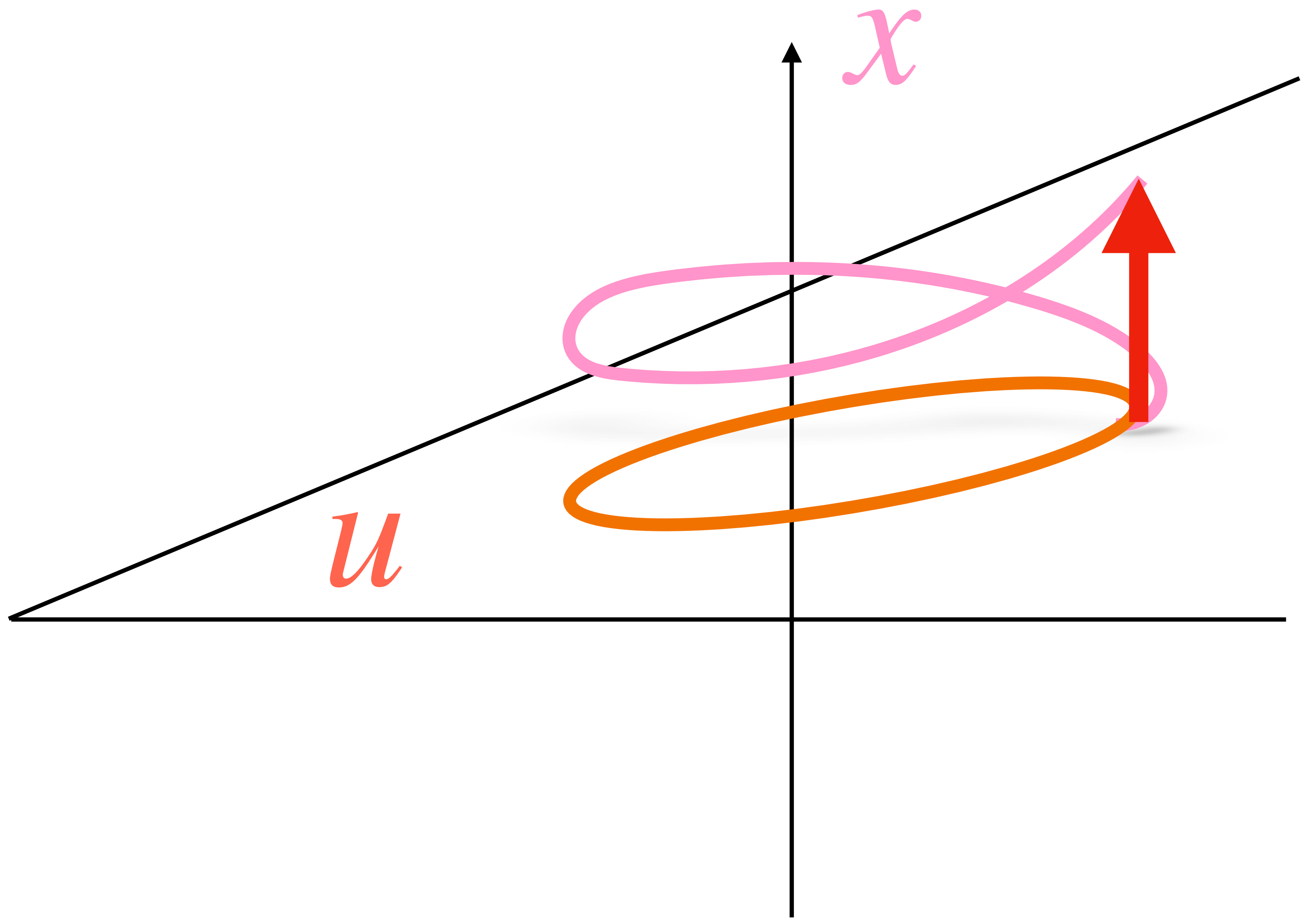


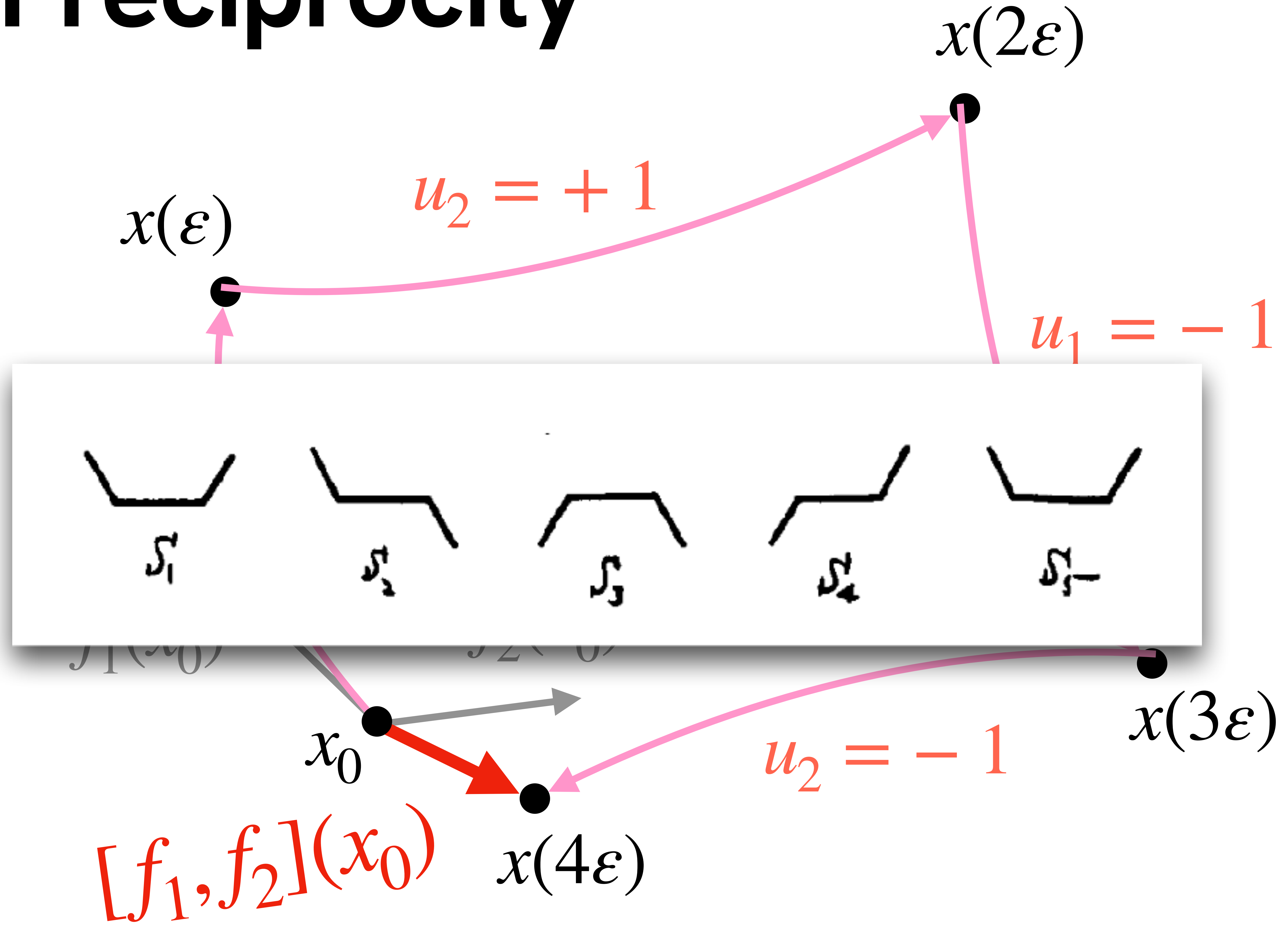
Lie brackets: formal non-reciprocity

$$\dot{x} = \sum_{i=1}^{m} f_i(x)u_i$$

$\searrow$ $m = 2$

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2$$





Lie brackets: formal non-reciprocity

$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i$$

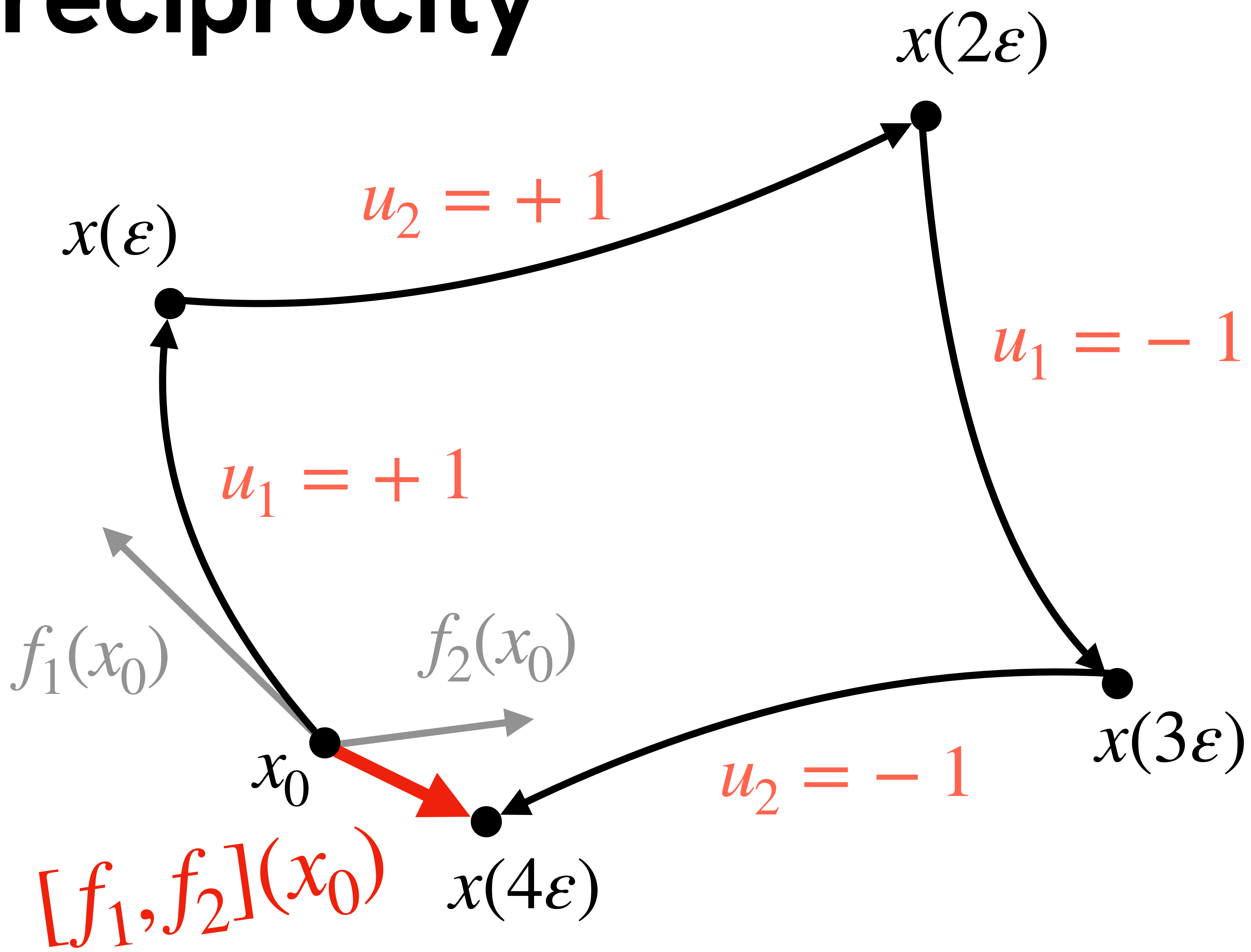
$\searrow$ $m = 2$

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2$$

Analysis viewpoint

$$x(4\varepsilon) = x_0 + \varepsilon^2 [f_1, f_2](x_0) + o(\varepsilon^2)$$

$$[f_1, f_2](x) = \nabla f_2(x) f_1(x) - \nabla f_1(x) f_2(x)$$



Lie brackets: formal non-reciprocity

$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i$$

$\searrow$ $m = 2$

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2$$

Analysis viewpoint

$$x(4\varepsilon) = x_0 + \varepsilon^2[f_1, f_2](x_0) + o(\varepsilon^2)$$

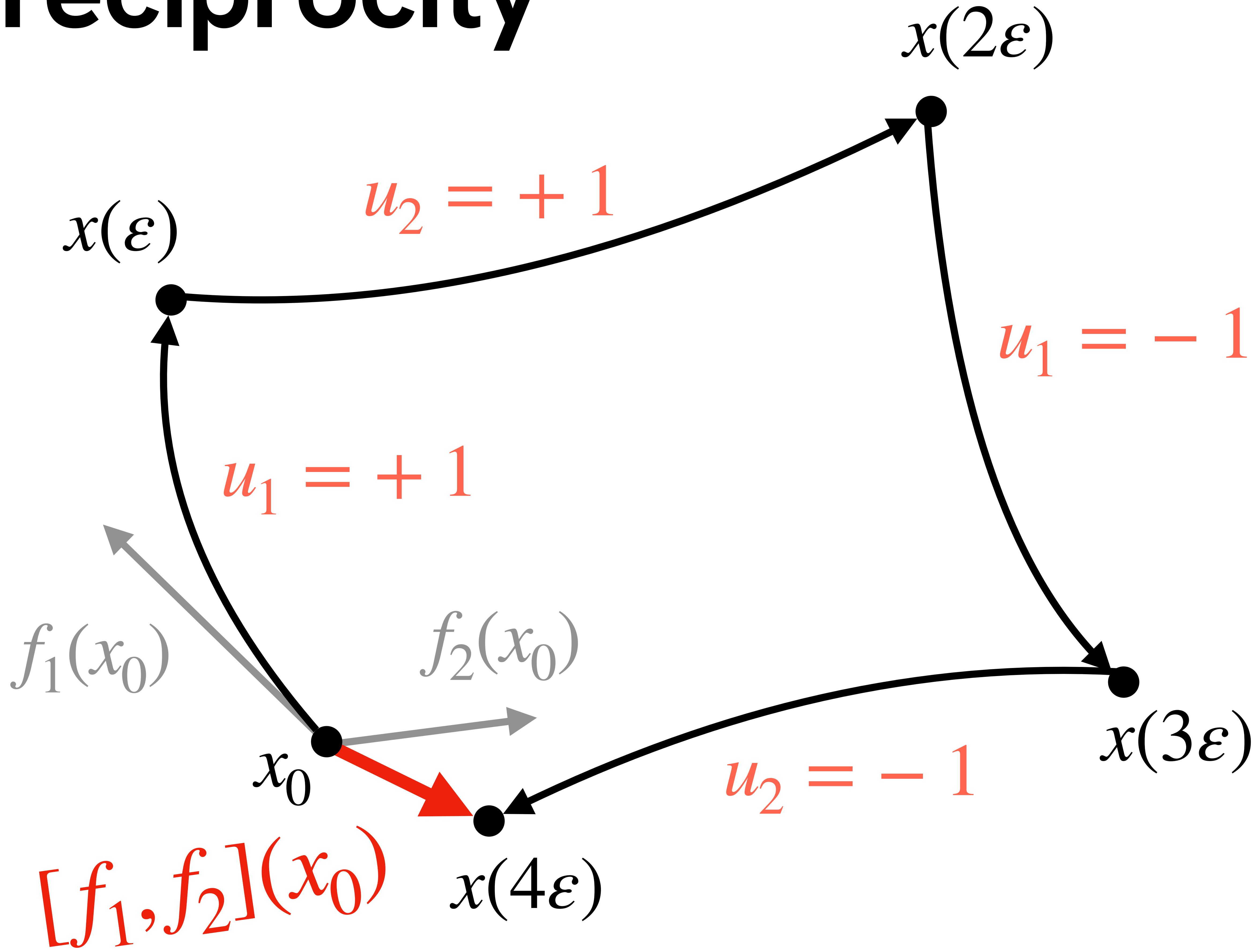
$$[f_1, f_2](x) = \nabla f_2(x) f_1(x) - \nabla f_1(x) f_2(x)$$

Differential geometry viewpoint

$$f_1 = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$$

$$f_2 = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}$$

$$f_1 f_2 = \sum_{j=1}^{n} \sum_{i=1}^{n} a_j b_i \frac{\partial^2}{\partial x_i x_j} + \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} a_j \frac{\partial b_i}{\partial x_j}\Big) \frac{\partial}{\partial x_i} \rightarrow [f_1, f_2] = f_1 f_2 - f_2 f_1 = \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} a_j \frac{\partial b_i}{\partial x_j} - \sum_{i=1}^{n} b_j \frac{\partial a_i}{\partial x_j}\Big) \frac{\partial}{\partial x_i}$$



Lie brackets: formal non-reciprocity

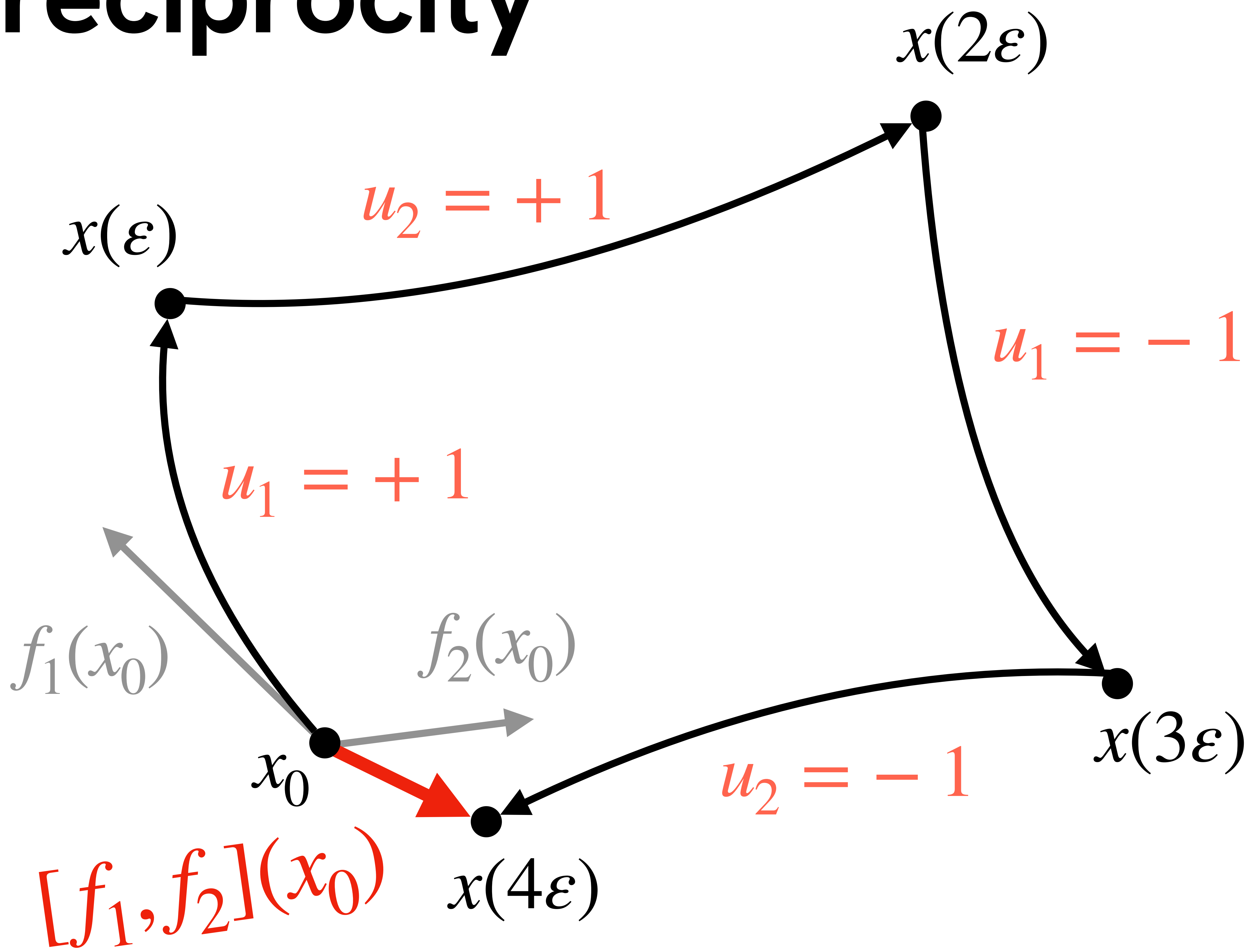
$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i$$

$\searrow$ $m = 2$

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2$$

$$x(4\varepsilon) = x_0 + \varepsilon^2 [f_1, f_2](x_0) + o(\varepsilon^2)$$

$$[f_1, f_2](x) = \nabla f_2(x) f_1(x) - \nabla f_1(x) f_2(x)$$



→ *We can generate new reachable directions given by the Lie brackets of the* f_i

→ of course, this can be iterated: $\big[[f_0, f_1], [f_0, [f_1, f_2]]\big]$, etc.

Conditions for controllability

$$\dot{x} = \sum_{i=1}^{m} f_i(x)u_i$$

Definition

Let $x_0 \in \mathbb{R}^n$. Let $\mathrm{Lie}(f_0, \ldots, f_m)$ be the set with all the iterated Lie brackets:

$$\mathrm{Lie}(f_0, \ldots, f_m) = \left\{f_0, \ldots f_m, [f_0, f_1], \ldots, [f_0, f_m], \ldots, [[f_0, [f_0, f_i]], [f_j, f_k]], \ldots\right\}$$

Lie Algebra Rank Condition in the point x_0:

$$\mathrm{Span}\{g(x_0), g \in \mathrm{Lie}(f_1, \ldots, f_m)\} = \mathbb{R}^n.$$

→ *"The LARC is satisfied if Lie brackets generate all possible directions"*

Conditions for controllability

$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i$$

Definition

Let $x_0 \in \mathbb{R}^n$. Let $\mathrm{Lie}(f_0, \ldots, f_m)$ be the set with all the iterated Lie brackets:

$$\mathrm{Lie}(f_0, \ldots, f_m) = \left\{ f_0, \ldots f_m, [f_0, f_1], \ldots, [f_0, f_m], \ldots, [[f_0, [f_0, f_i]], [f_j, f_k]], \ldots \right\}$$

Lie Algebra Rank Condition in the point x_0:

$$\mathrm{Span}\{g(x_0), g \in \mathrm{Lie}(f_1, \ldots, f_m)\} = \mathbb{R}^n.$$

→ *"The LARC is satisfied if Lie brackets generate all possible directions"*

Theorem (Rashevski-Chow)

Lie Algebra Rank Condition at x_0 ⇔ local controllability at x_0

LARC for all x_0 in $\mathbb{R}^n$ ⇔ **controllable everywhere** in time T, for all T

Conditions for controllability

$$\dot{x} = \sum_{i=1}^{m} f_i(x)u_i$$

Definition

Let $x_0 \in \mathbb{R}^n$. Let $\mathrm{Lie}(f_0, \ldots, f_m)$ be the set with all the iterated Lie brackets:

$$\mathrm{Lie}(f_0, \ldots, f_m) = \left\{f_0, \ldots f_m, [f_0, f_1], \ldots, [f_0, f_m], \ldots, [[f_0, [f_0, f_i]], [f_j, f_k]], \ldots\right\}$$

Lie Algebra Rank Condition in the point x_0:

$$\mathrm{Span}\{g(x_0), g \in \mathrm{Lie}(f_1, \ldots, f_m)\} = \mathbb{R}^n.$$

→ *"The LARC is satisfied if Lie brackets generate all possible directions"*

Theorem (Rashevski-Chow)

Lie Algebra Rank Condition at x_0 ⇔ local controllability at x_0

LARC for all x_0 in $\mathbb{R}^n$ ⇔ **controllable everywhere** in time T, for all T

→ **Scallop theorem :** $m = 1$**...**

Shape-controlled swimmers

hydrodynamics

rigid motion

deformation

$$\dot{\mathscr{X}} = \mathbf{N}(\mathscr{S})\dot{\mathscr{S}}$$

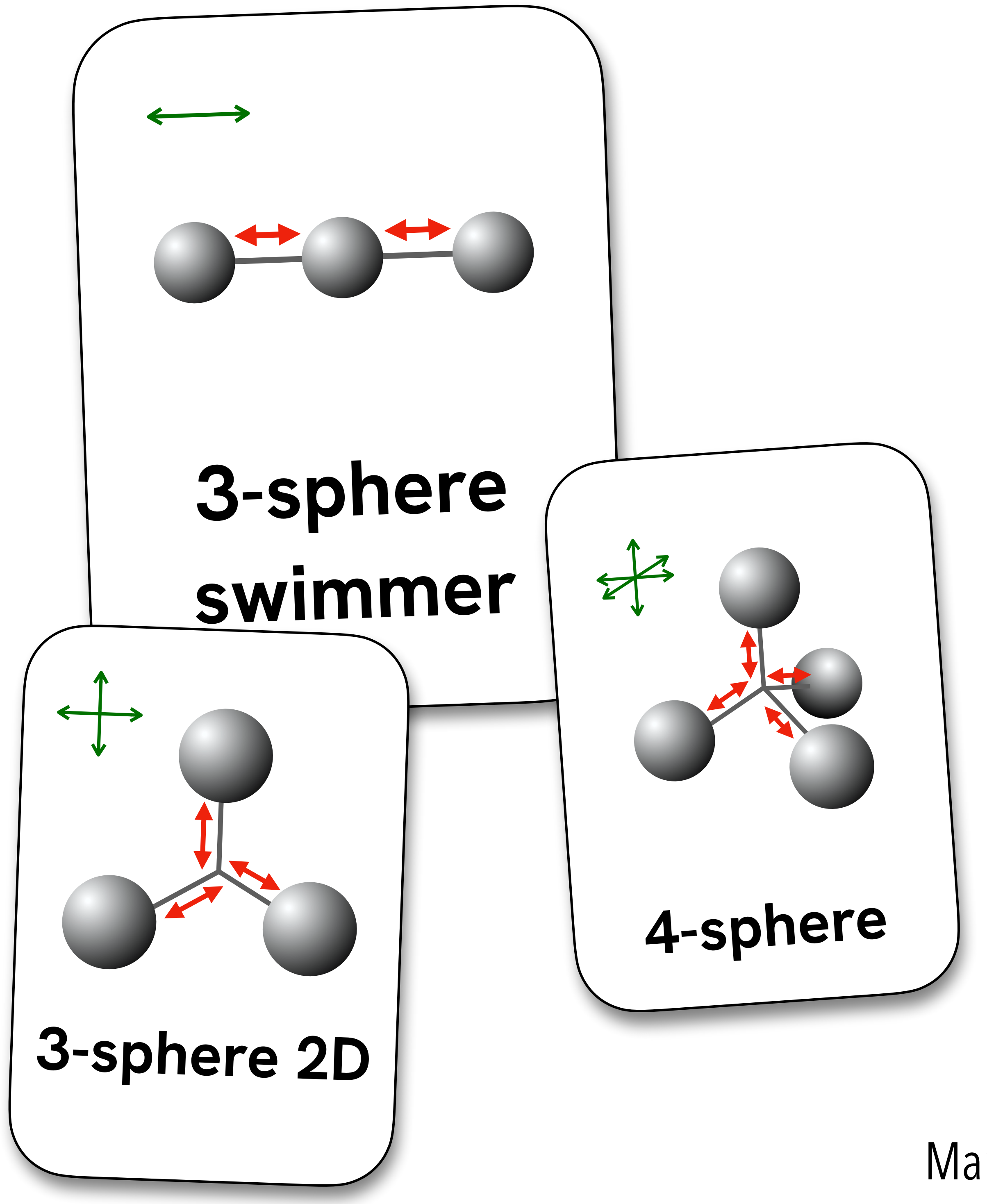
The swimming equation is a **driftless control-affine system**

→ Rashveski-Chow theorem

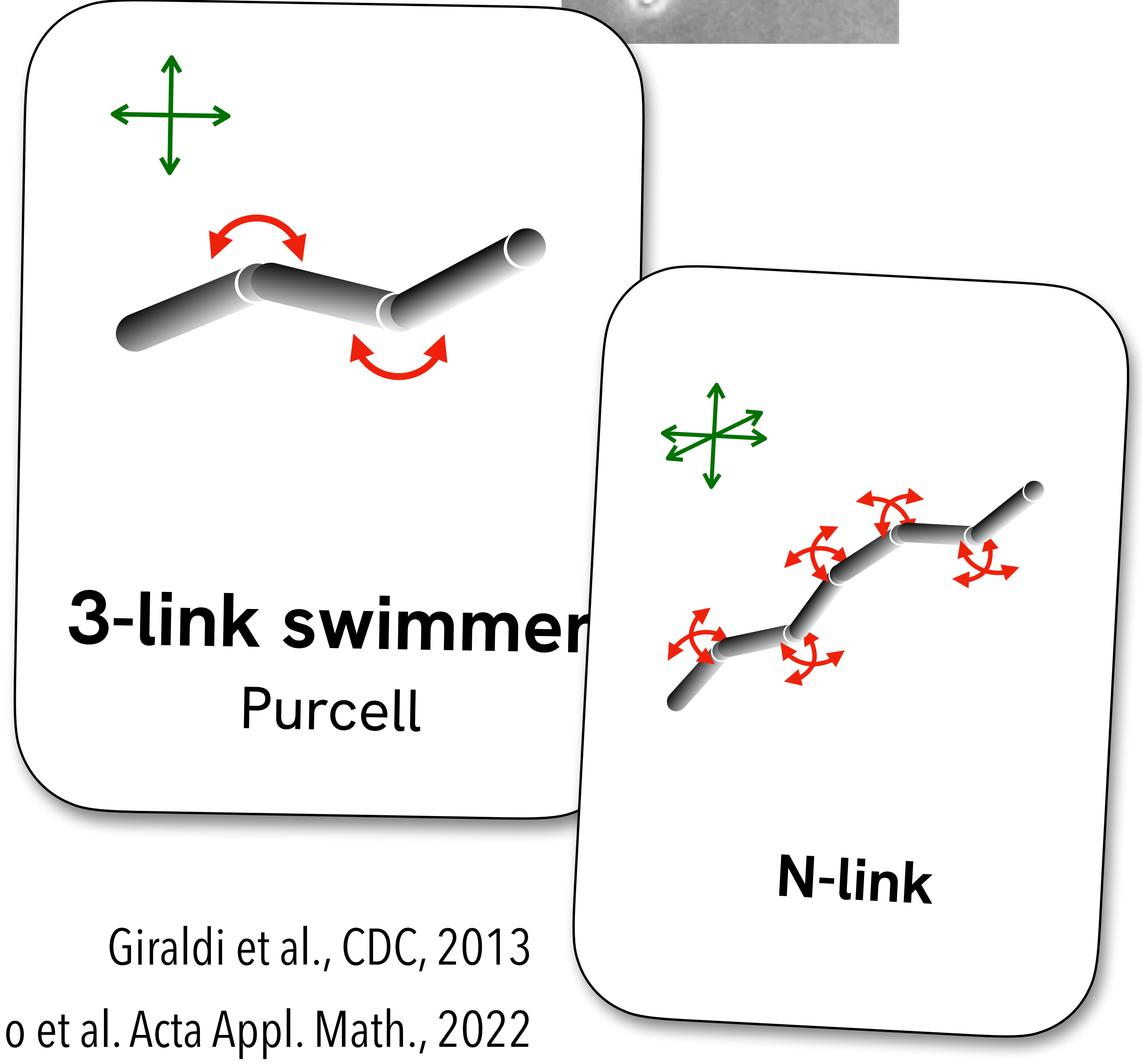
→ many results of controllability

minimal swimmers

Alouges et al., Discrete Contin. Dyn. Syst. 2013



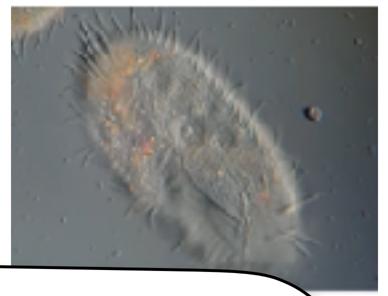
flagellar motion

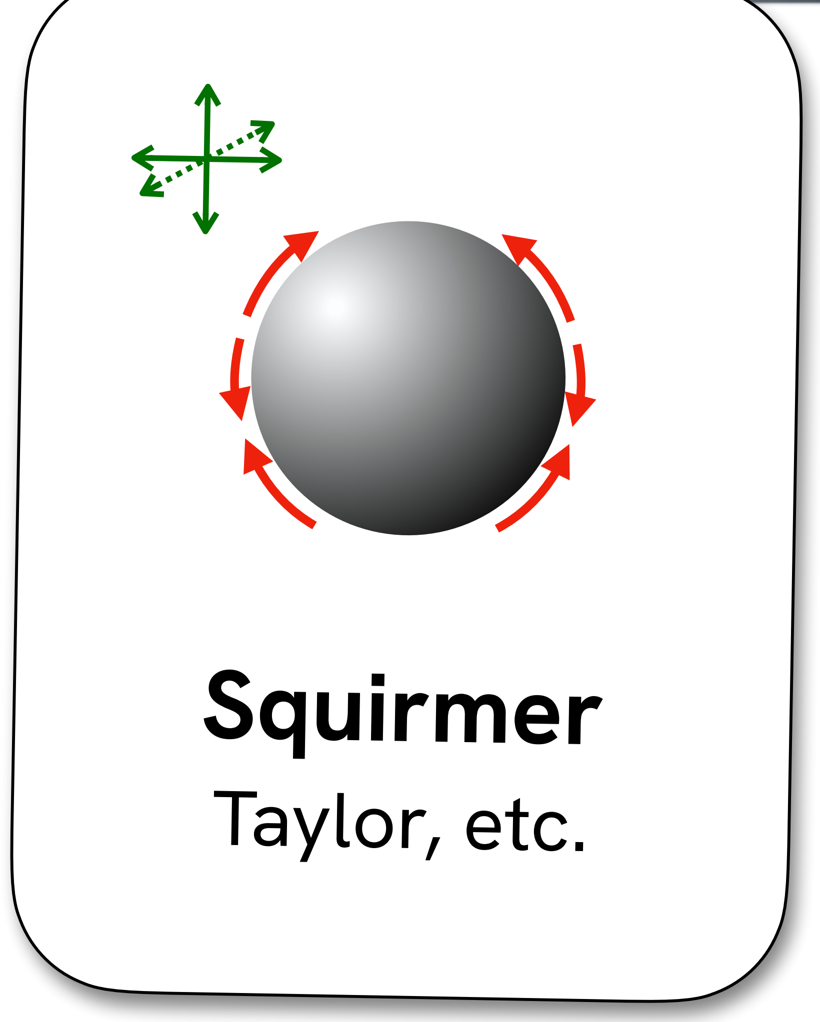


Giraldi et al., CDC, 2013

Marchello et al. Acta Appl. Math., 2022

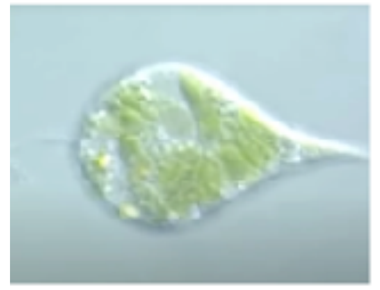
ciliate motion



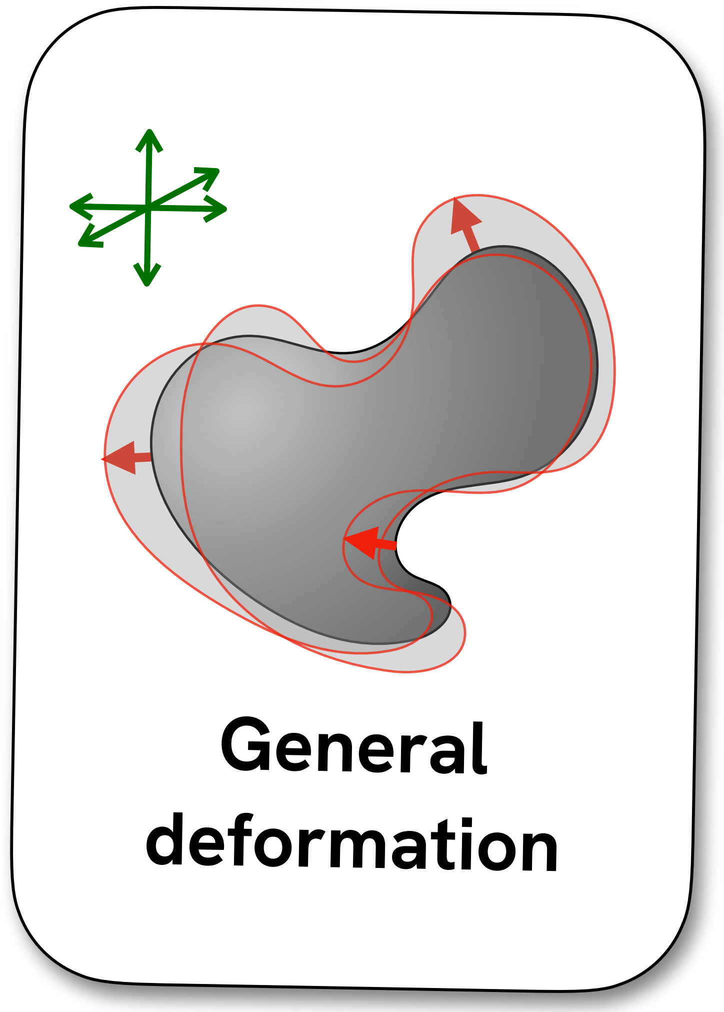


Lohéac, Takahashi, ESAIM:COCV, 2020

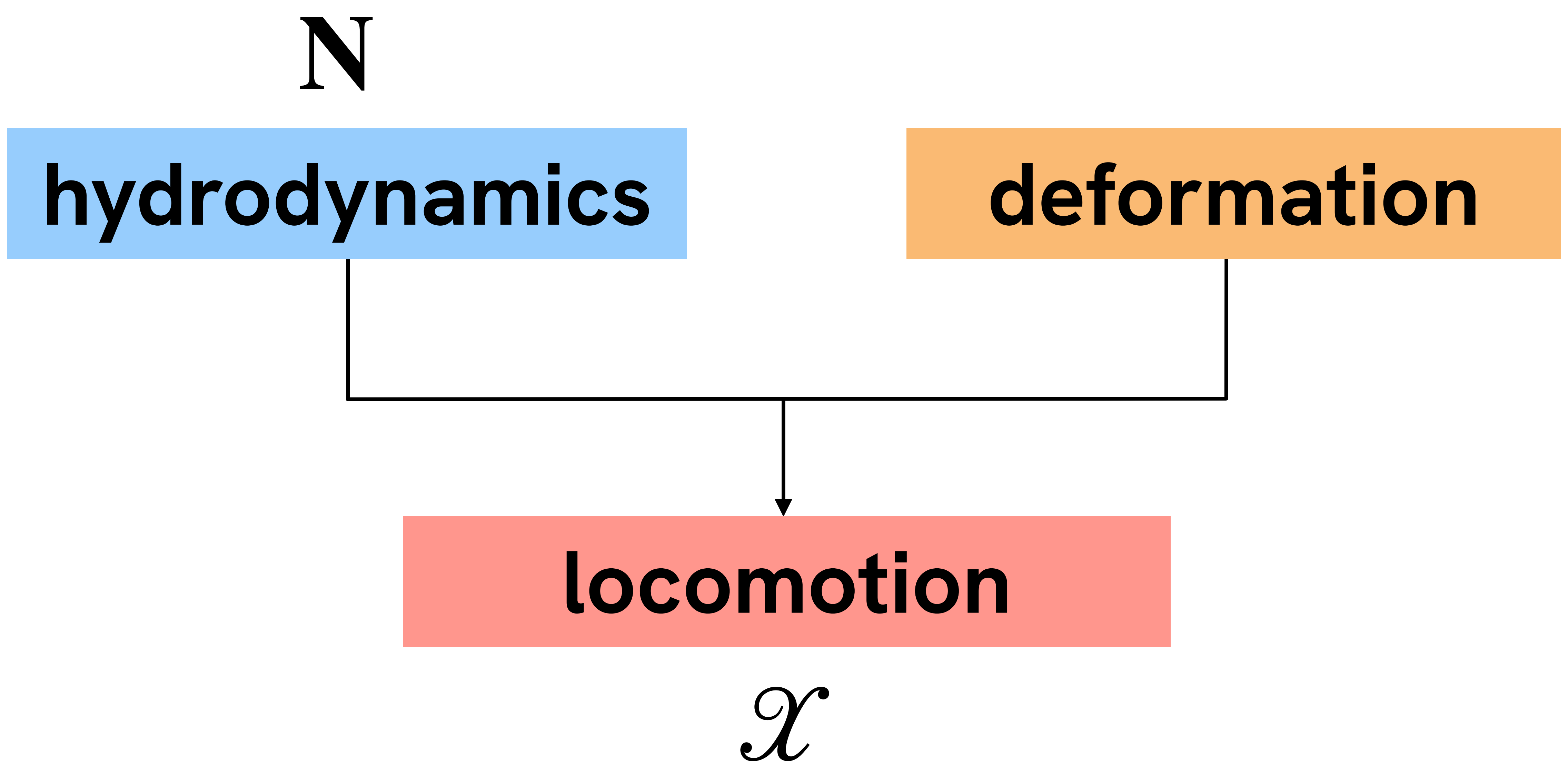
euglenoid motion



Lohéac, Munnier, ESAIM:COCV, 2014



But... one does not always control the deformation



$\mathbf{N}$

hydrodynamics

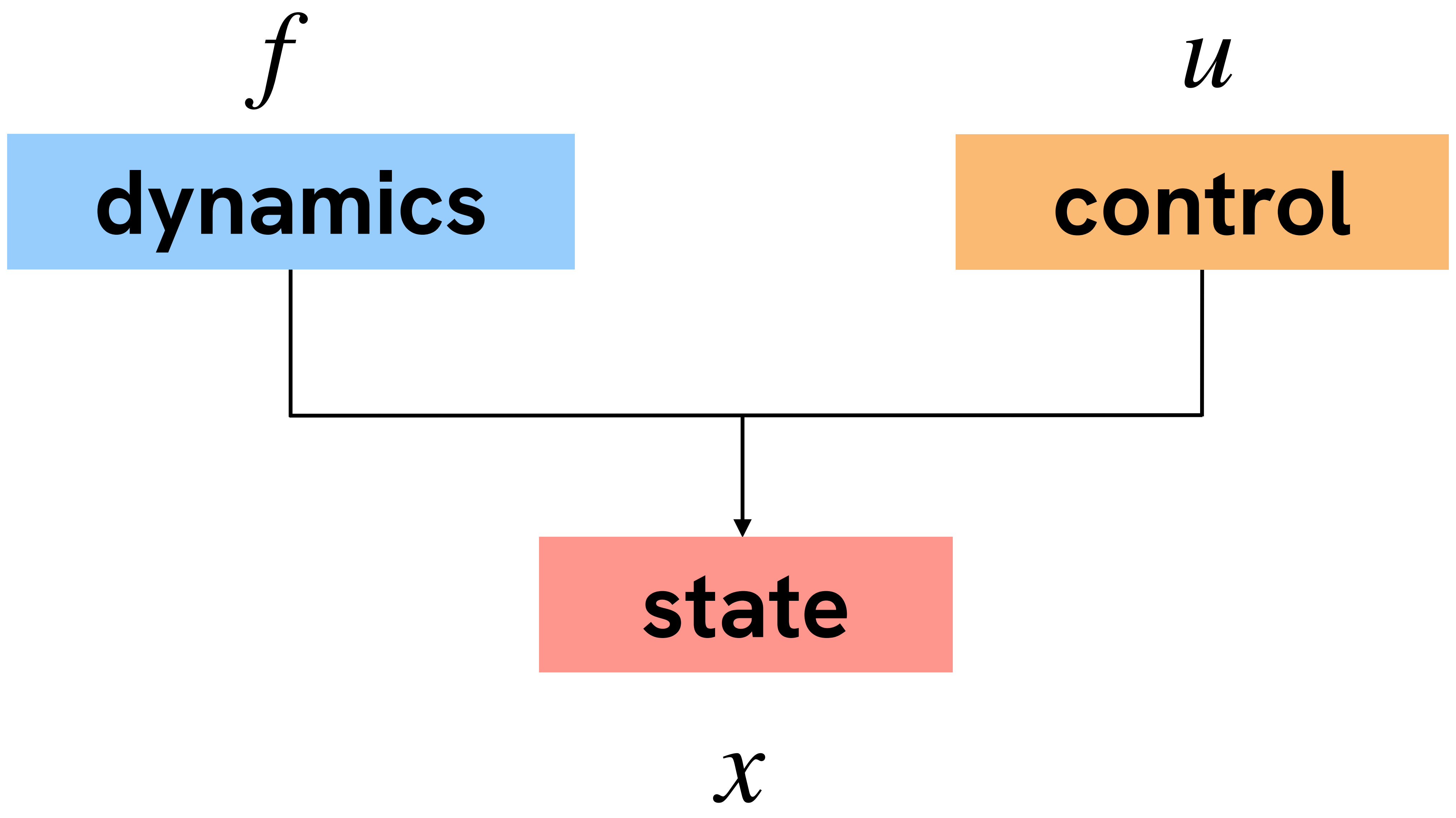
deformation

locomotion

$\mathscr{X}$

$$\dot{\mathscr{X}} = \mathbf{N}(\mathscr{S})\dot{\mathscr{S}}$$

Swimming equation



f

dynamics

u

control

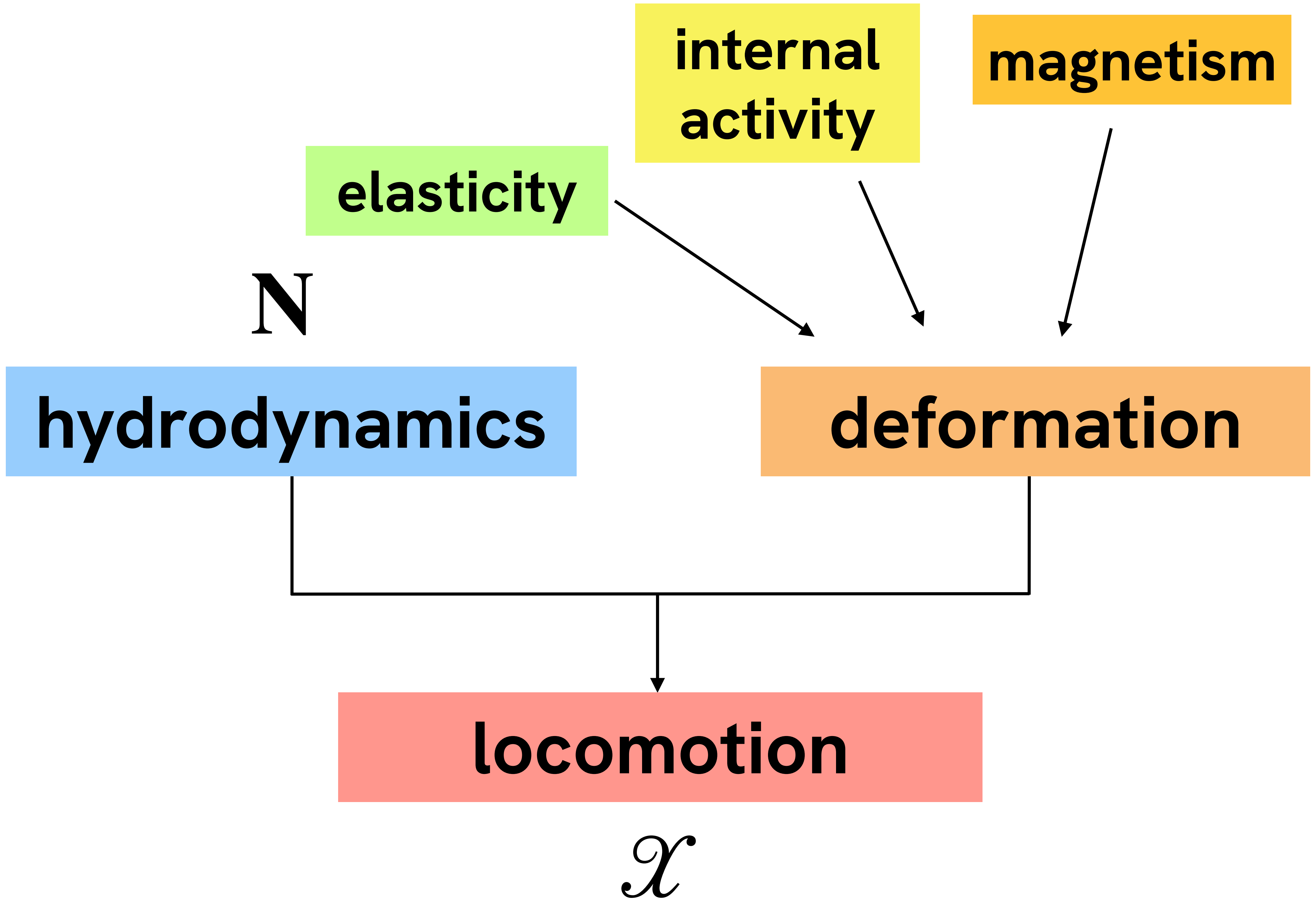
state

x

$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i$$

Control-affine system

But… one does not always control the deformation



internal activity

magnetism

elasticity

$\mathbf{N}$

hydrodynamics

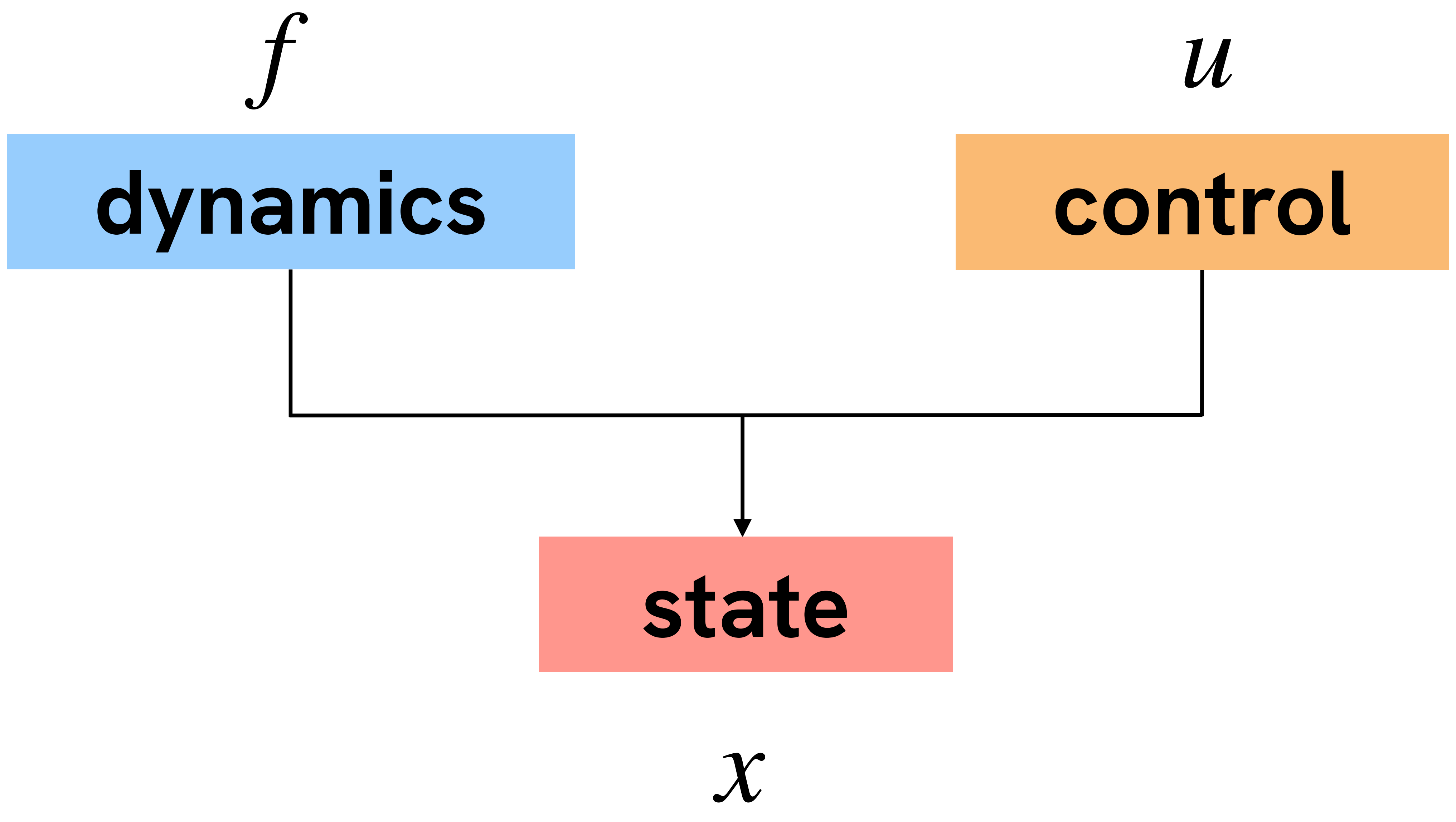
deformation

locomotion

$\mathscr{X}$

$$\xcancel{\dot{\mathscr{X}} = \mathbf{N}(\mathscr{S})\dot{\mathscr{S}}}$$

Swimming equation



f

u

dynamics

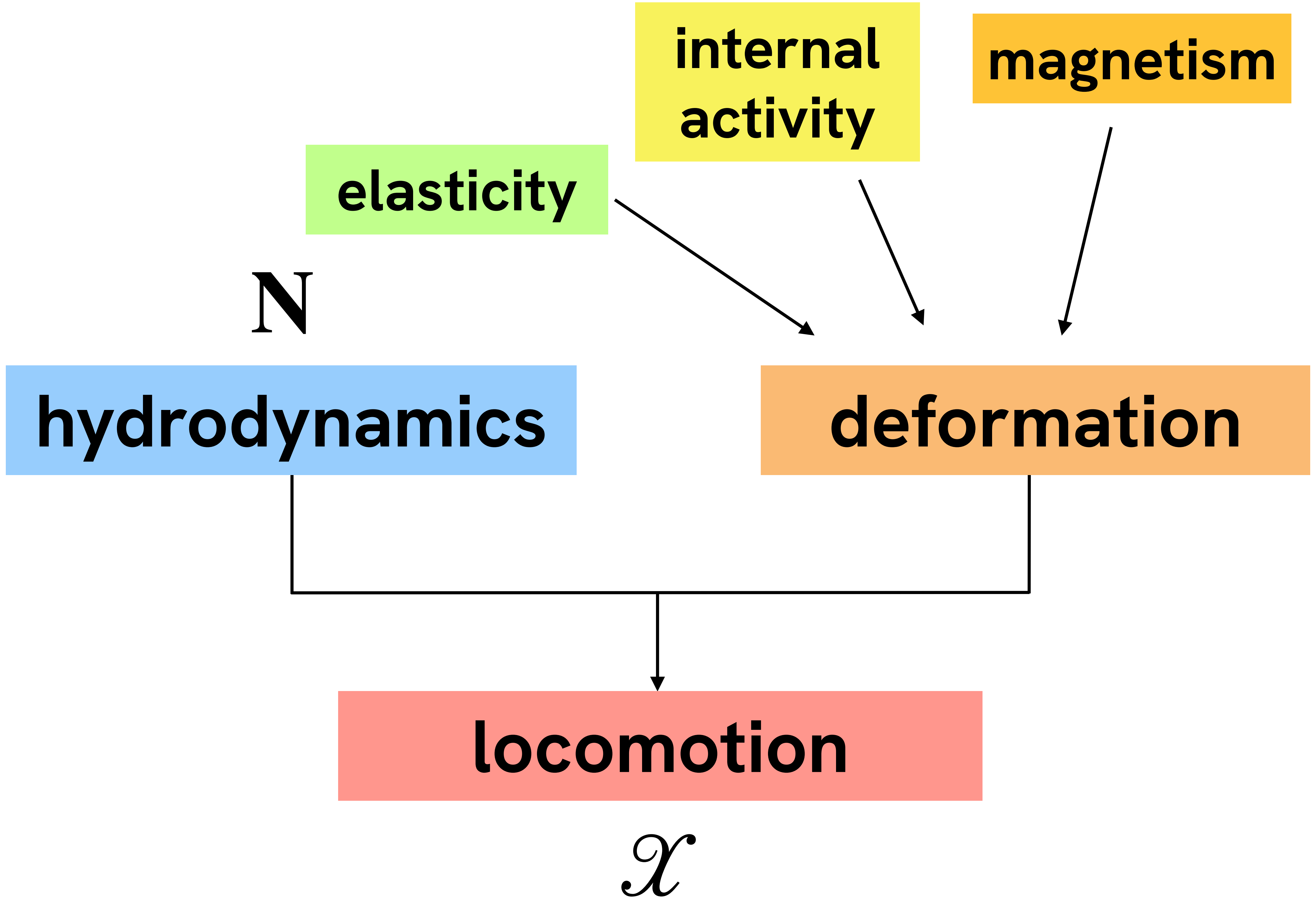
control

state

x

$$\xcancel{\dot{x} = \sum_{i=1}^{m} f_i(x)u_i}$$

But... one does not always control the deformation



elasticity

internal activity

magnetism

$\mathbf{N}$

hydrodynamics

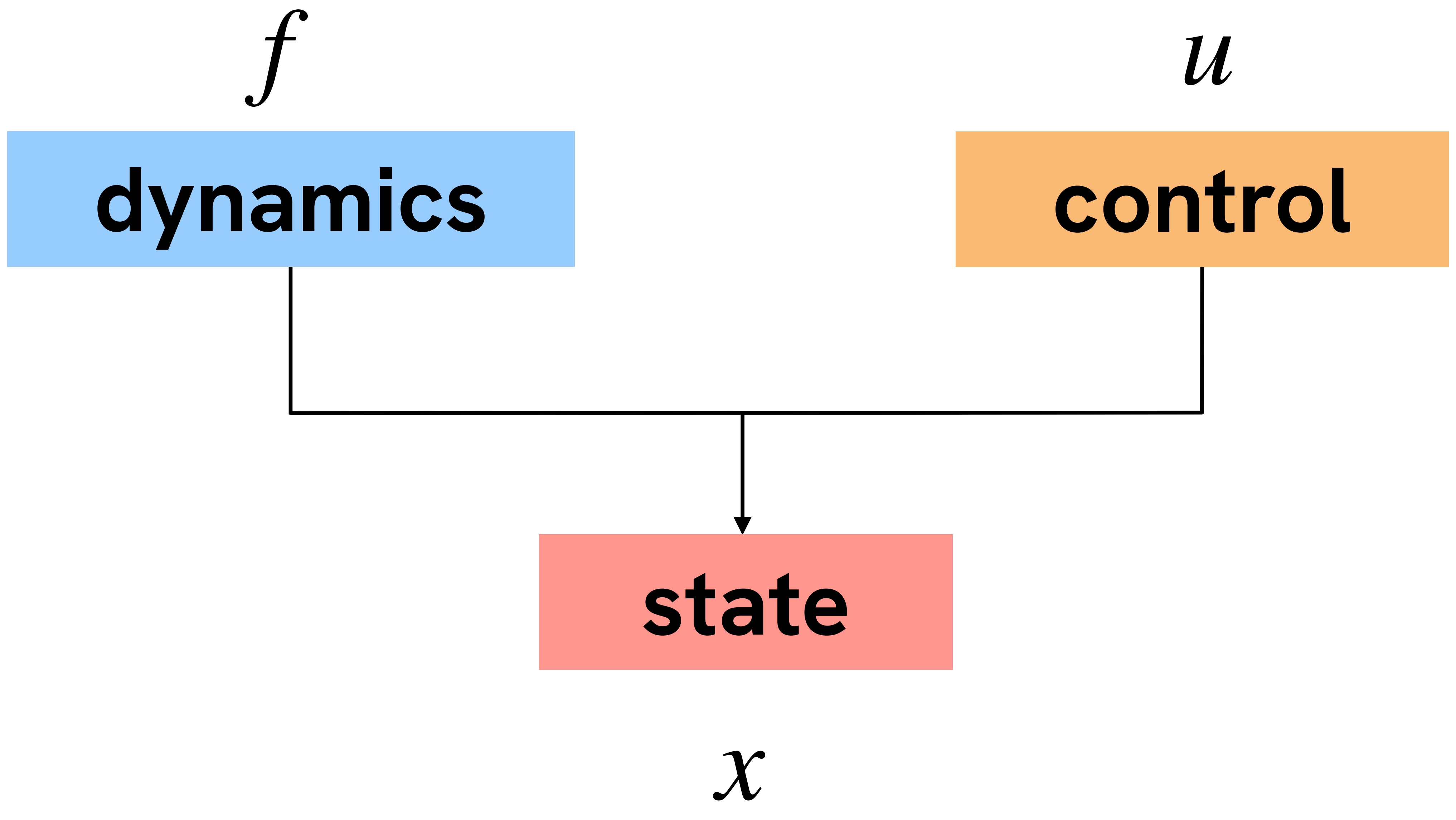
deformation

locomotion

$\mathscr{X}$

$$\cancel{\dot{\mathscr{X}} = \mathbf{N}(\mathscr{S})\dot{\mathscr{S}}} \qquad \dot{\mathscr{X}} = \mathbf{N}(\mathscr{S})\dot{\mathscr{S}} + \mathbf{M}(\mathscr{S})$$

Swimming equation



f

dynamics

u

control

state

x

$$\cancel{\dot{x} = \sum_{i=1}^{m} f_i(x)u_i} \qquad \dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x)u_i$$

Control-affine system *with a drift*

Conditions for controllability: with a drift

$$(\text{Aff}) : \dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x) u_i$$

- More complicated when $f_0 \neq 0$: the LARC is only necessary. Counterexample:

$$\begin{cases} \dot{x}_1 = x_2^2 \\ \dot{x}_2 = u \end{cases}$$

- **Heuristics:** ***not all non-reciprocities are equal*** “some brackets are good and some are bad”; brackets with an even number of times f_i are « bad » …

No NS condition for STLC is known!

Scalar-input case

$$\dot{x} = f_0(x) + f_1(x)u$$

- **Sufficient conditions:** Sussmann 1986

$$\text{LARC} \ \& \ (\forall k, S_{2k} \subset S_{2k-1})$$

- **Necessary conditions:** Sussmann, Kawski, Krastanov, Beauchard, Marbach, etc.

$$S_2 \not\subset S_1 ?$$

$$[f_1, [f_0, f_1]]$$

Two-control case

$$\dot{x} = f_0(x) + f_1(x)u_1 + f_2(x)u_2$$

- **Sufficient condition**: Sussmann « $S(\theta)$ », with « **good** » **and** « **bad** » **brackets**
- **Necessary conditions:** none…! until recently

Two-control case

$$\dot{x} = f_0(x) + f_1(x)u_1 + f_2(x)u_2$$

- **Sufficient condition**: Sussmann « $S(\theta)$ »
- **Necessary conditions**: none…! until recently
  - 2024: **one result coming from a micro-swimmer model**!
  - Update 2025: now additional necessary conditions! (Gherdaoui, Beauchard, Marbach)

Two-control case

$$\dot{x} = f_0(x) + f_1(x)u_1 + f_2(x)u_2$$

- **Sufficient condition**: Sussmann « $S(\theta)$ »
- **Necessary conditions**: none…! until recently
  - 2024: **one result coming from a micro-swimmer model**!
  - Update 2025: now additional necessary conditions! (Gherdaoui, Beauchard, Marbach)

$$R_2 \not\subset R_1 ?$$

- **Particular case:** $f_2(0) = 0$

$[f_1, [f_0, f_1]]$ 🟥

$[f_1, [f_2, f_1]]$ 🟩

Two-control case

$$\dot{x} = f_0(x) + f_1(x)u_1 + f_2(x)u_2$$

- **Sufficient condition**: Sussmann « $S(\theta)$ »
- **Necessary conditions**: none!
  - 2024: **one result coming from a micro-swimmer model**!
  - Update 2025: now additional necessary conditions! (Gherdaoui)

Theorem (Giraldi, Lissy, M., Pomet, ESAIM:COCV 2024)

Assume $f_2(0) = 0$ and $f_{101}(0) \notin R_1$.

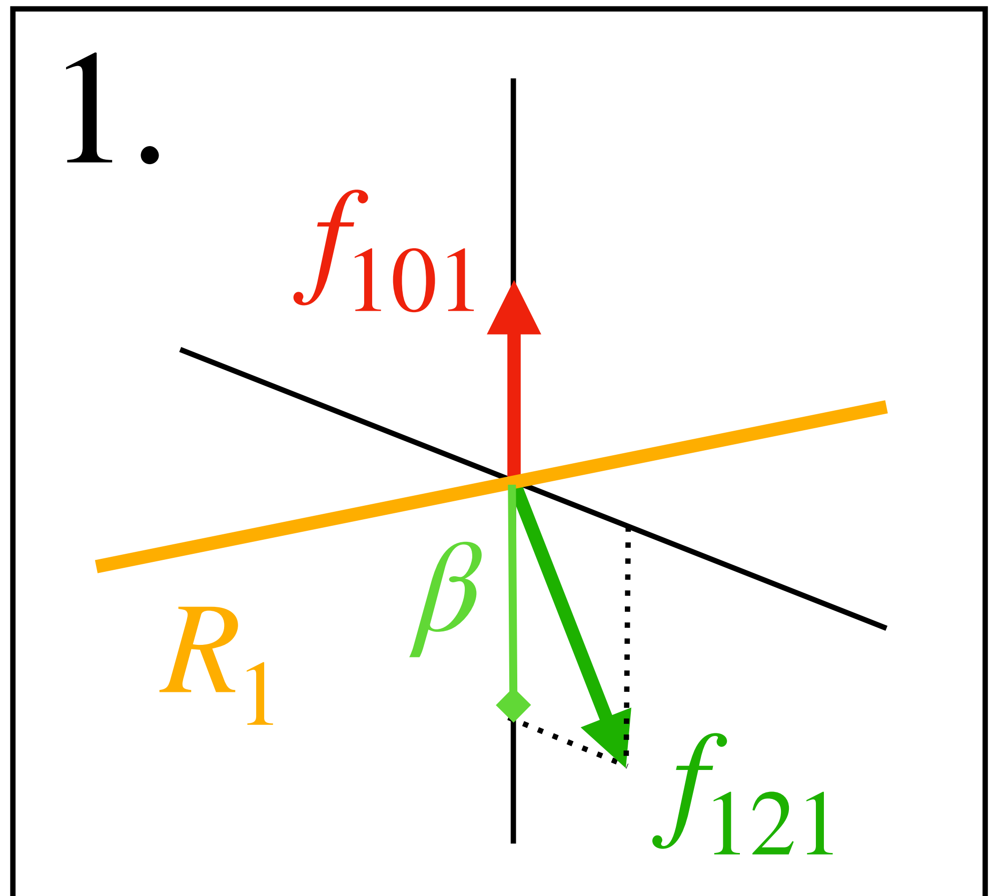
1. If $f_{101}(0) \in R_1 + \mathrm{Span}(f_{121}(0))$, let $\beta \in \mathbb{R}$ such that $f_{101}(0) + \beta f_{121}(0) \in R_1$.
Then, for all $u_{\mathrm{eq},2} \in \mathbb{R}$ such that $u_{\mathrm{eq},2} \neq \beta$, the system is **not STLC** at $(0,(0,u_{\mathrm{eq},2}))$.

2. If $f_{101}(0) \notin R_1 + \mathrm{Span}(f_{121}(0))$, then, for all $u_{\mathrm{eq},2} \in \mathbb{R}$, the system is **not B-STLC** at $(0,(0,u_{\mathrm{eq},2}))$.

- **One may neutralize the bad bracket f_{101} with f_{121}**

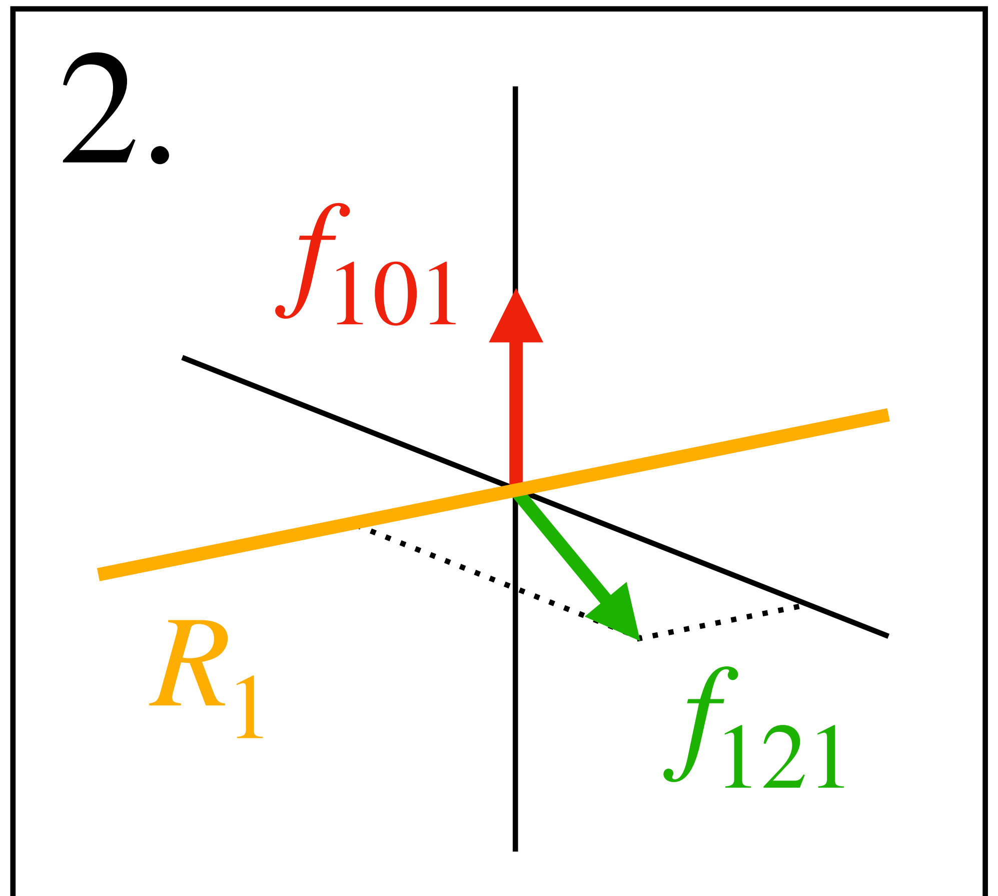
Two-control case

$$\dot{x} = f_0(x) + f_1(x)u_1 + f_2(x)u_2$$



$f_{101} + \beta f_{121} \in R_1$

« maybe » STLC with $u_{\text{eq}} = \beta$



$f_{101} \notin R_1 + \text{Span}(f_{121})$

not STLC

Theorem (Giraldi, Lissy, M., Pomet, ESAIM:COCV 2024)

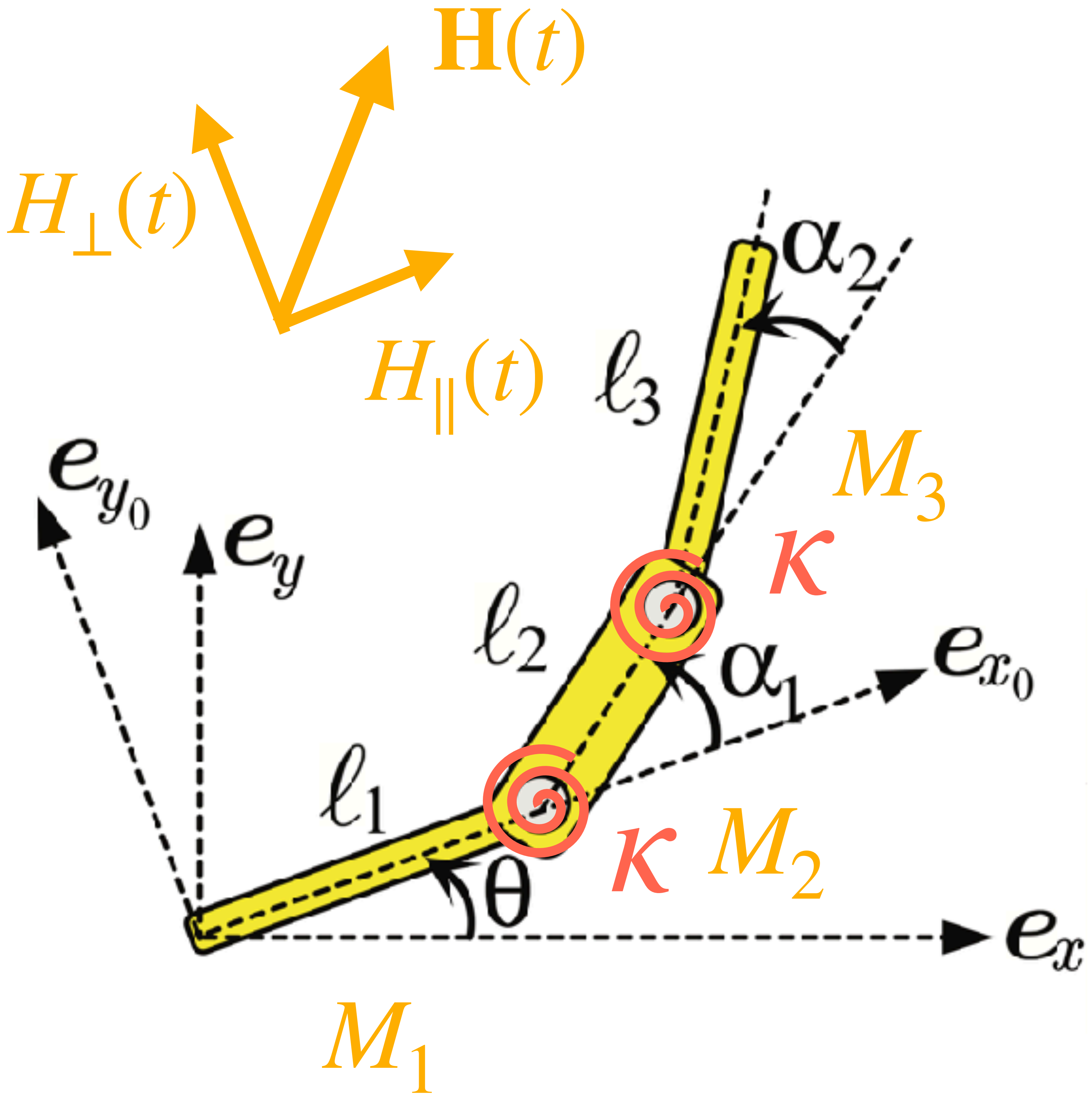
Assume $f_2(0) = 0$ and $f_{101}(0) \notin R_1$.

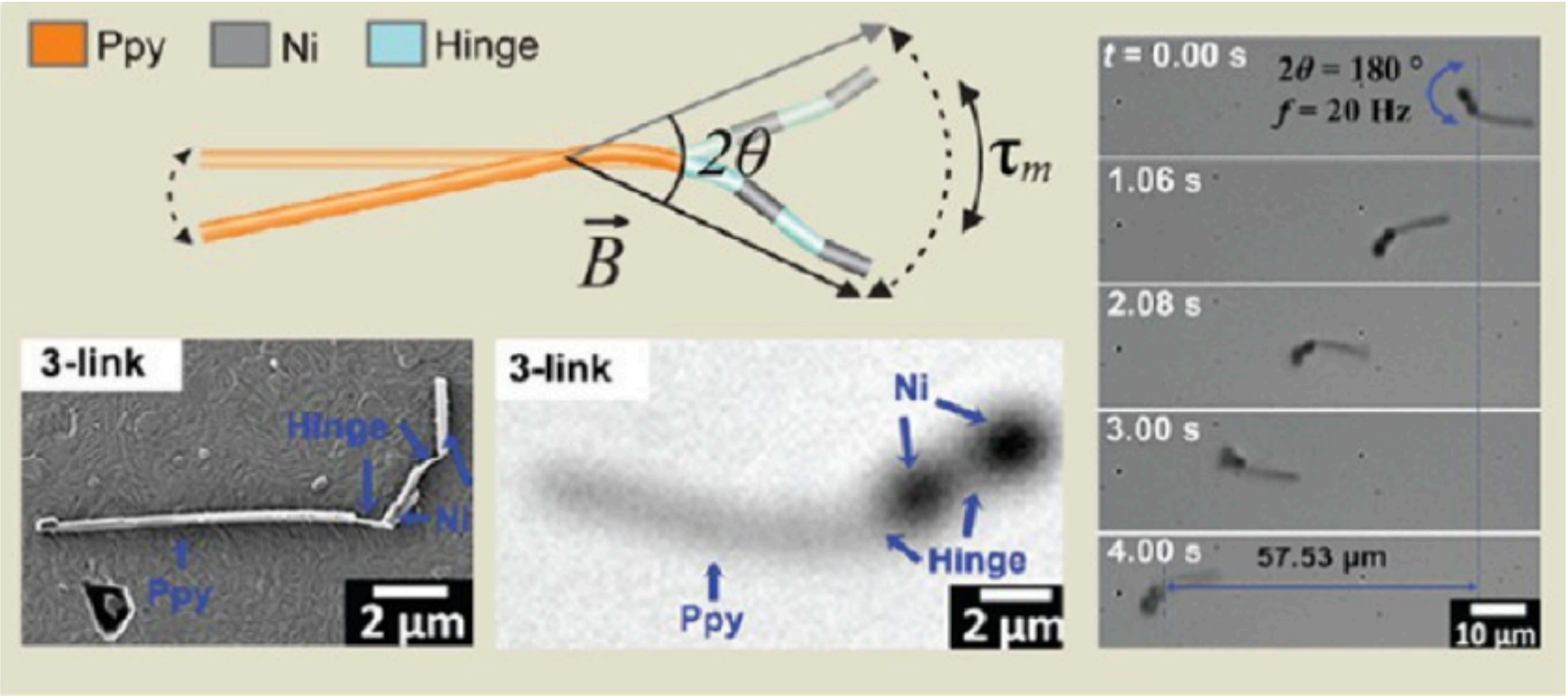
1. If $f_{101}(0) \in R_1 + \text{Span}(f_{121}(0))$ let $\beta \in \mathbb{R}$ such that $f_{101}(0) + \beta f_{121}(0) \in R_1$.
Then, for all $u_{\text{eq},2} \in \mathbb{R}$ such that $u_{\text{eq},2} \neq \beta$, the system is **not STLC** at $(0,(0,u_{\text{eq},2}))$.

2. If $f_{101}(0) \notin R_1 + \text{Span}(f_{121}(0))$, then, for all $u_{\text{eq},2} \in \mathbb{R}$, the system is **not B-STLC** at $(0,(0,u_{\text{eq},2}))$.

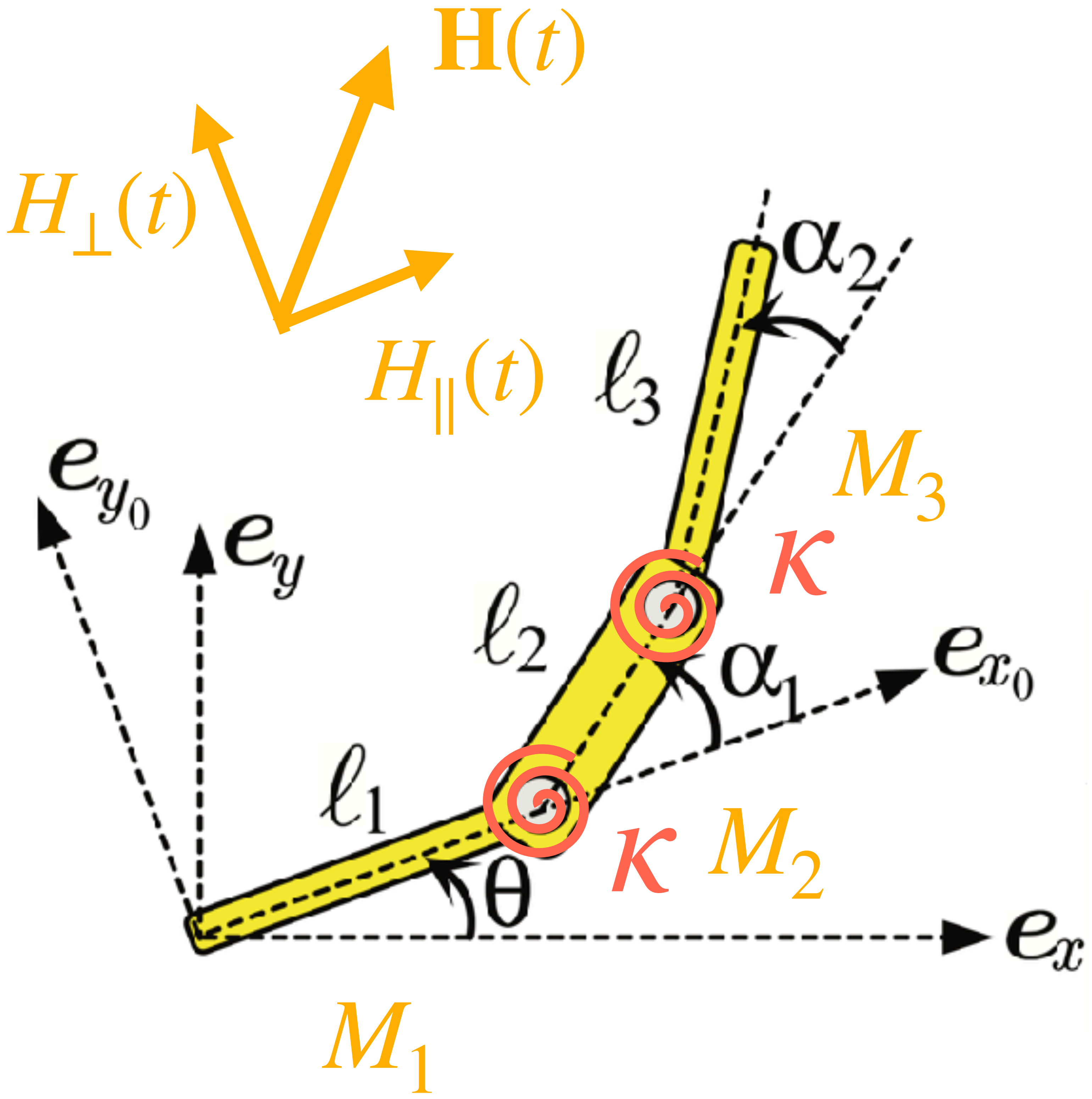
- **One may neutralize the bad bracket f_{101} with f_{121}**

Drift system: magneto-elastic Purcell swimmer

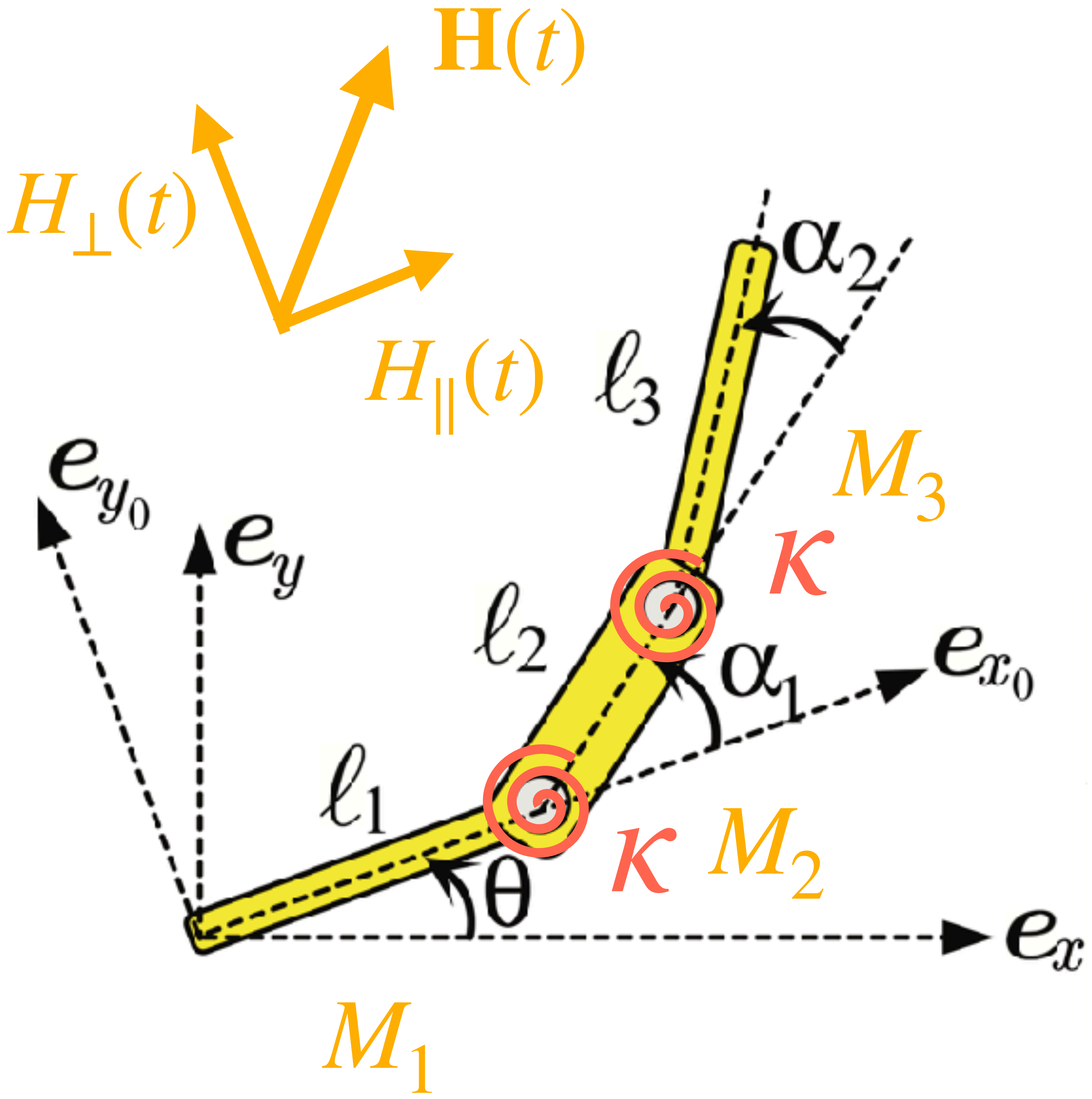




Drift system: magneto-elastic Purcell swimmer

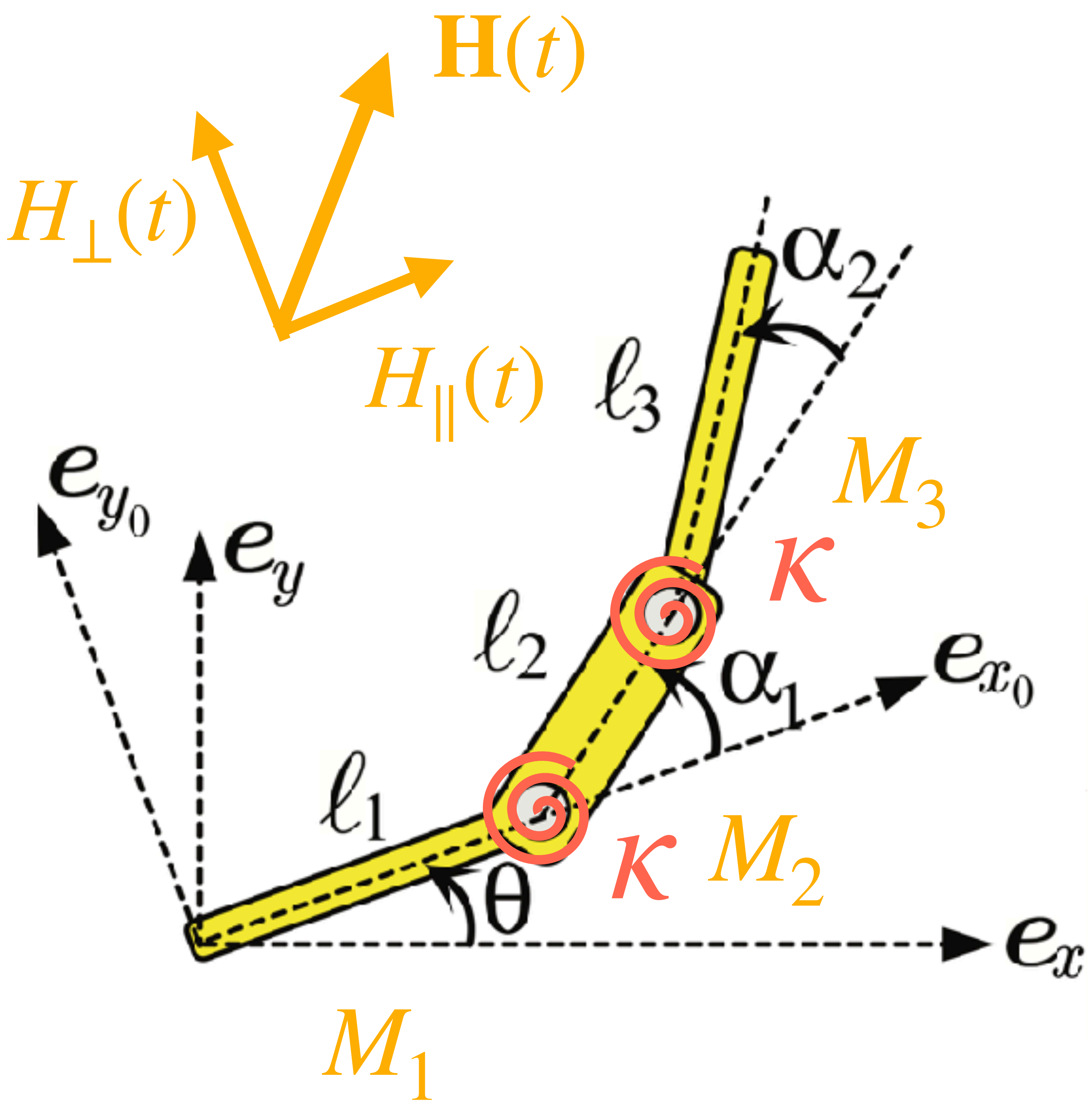


Drift system: magneto-elastic Purcell swimmer



- State: $\mathbf{z} = (x, y, \theta, \alpha_1, \alpha_2)$
- Controls: $(H_\parallel, H_\perp)$

Drift system: magneto-elastic Purcell swimmer

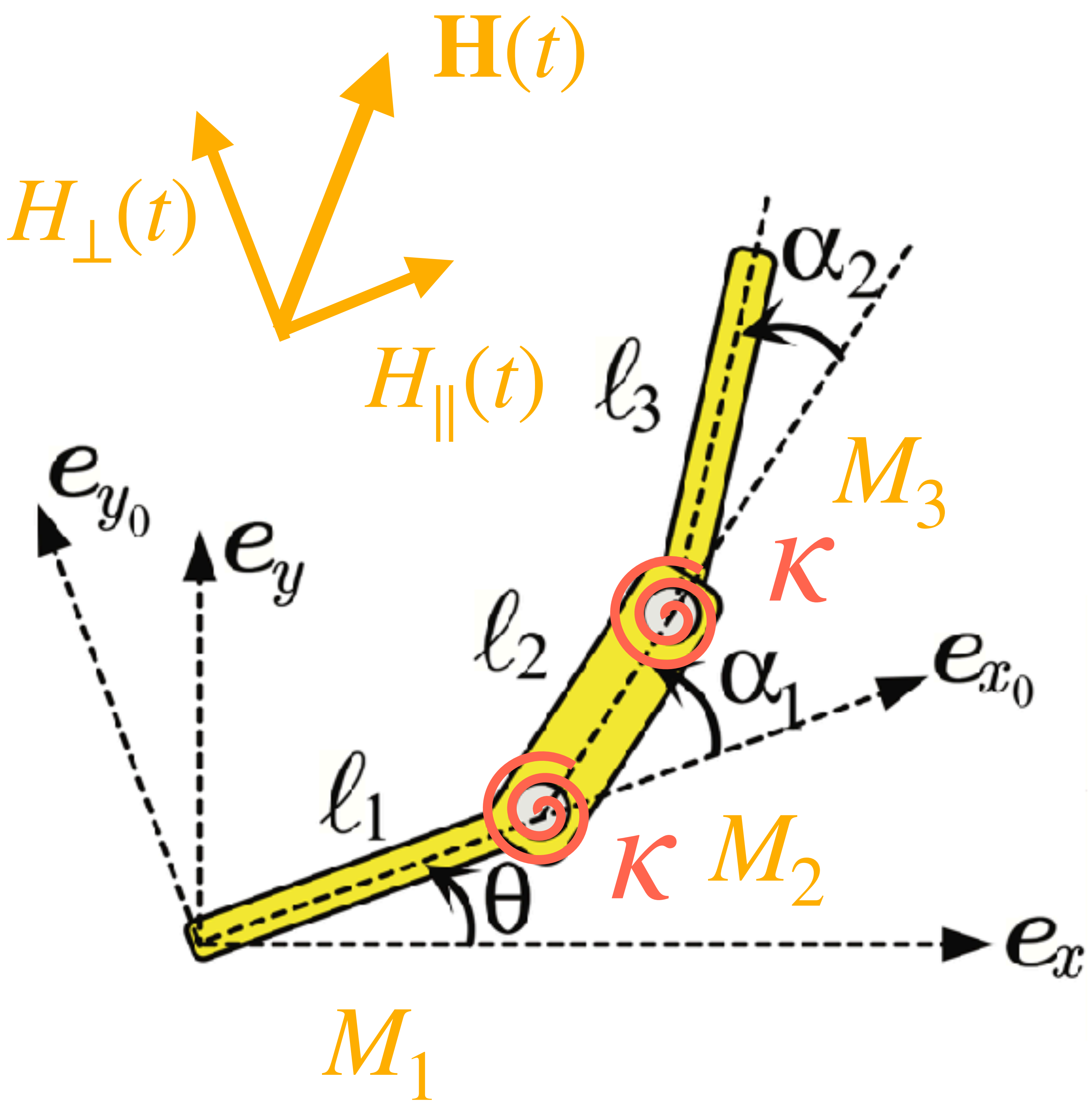


$$\begin{cases} \mathbf{f}_1^h + \mathbf{f}_2^h + \mathbf{f}_3^h = 0 \\ m_1^h + m_2^h + m_3^h = m_1^{mag} + m_2^{mag} + m_3^{mag} \\ m_2^h + m_3^h = m_1^{el} + m_2^{mag} + m_3^{mag} \\ m_3^h = m_2^{el} + m_3^{mag} \end{cases}$$

Equations of motion

- State: $\mathbf{z} = (x, y, \theta, \alpha_1, \alpha_2)$
- Controls: $(H_\parallel, H_\perp)$

Drift system: magneto-elastic Purcell swimmer



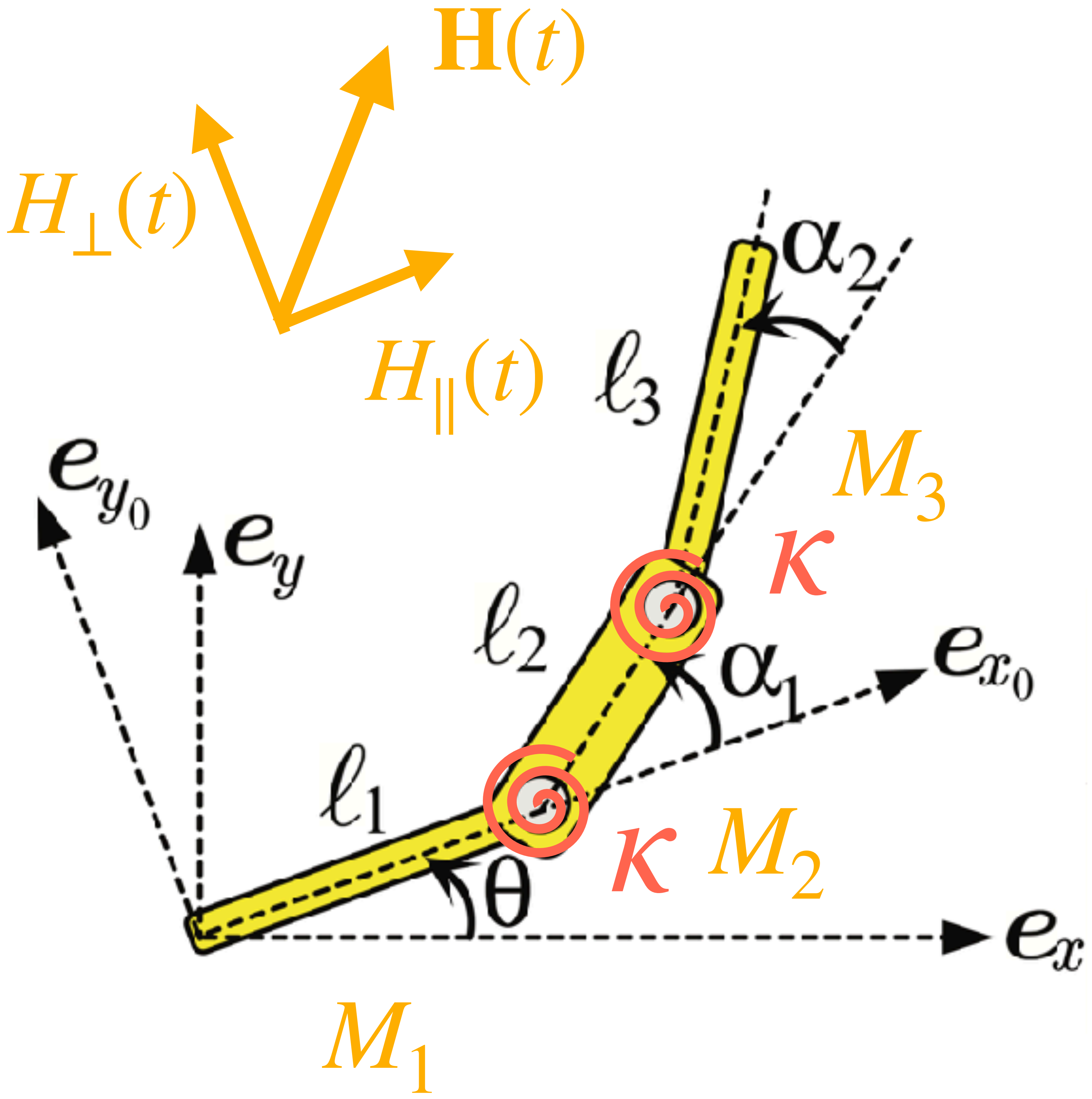
- State: $\mathbf{z} = (x, y, \theta, \alpha_1, \alpha_2)$
- Controls: $(H_\parallel, H_\perp)$

$$\left\{ \begin{aligned} \mathbf{f}_1^h + \mathbf{f}_2^h + \mathbf{f}_3^h &= 0 \\ m_1^h + m_2^h + m_3^h &= m_1^{mag} + m_2^{mag} + m_3^{mag} \\ m_2^h + m_3^h &= m_1^{el} + m_2^{mag} + m_3^{mag} \\ m_3^h &= m_2^{el} + m_3^{mag} \end{aligned} \right.$$

Hydrodynamics Elasticity Magnetism

Equations of motion

Drift system: magneto-elastic Purcell swimmer

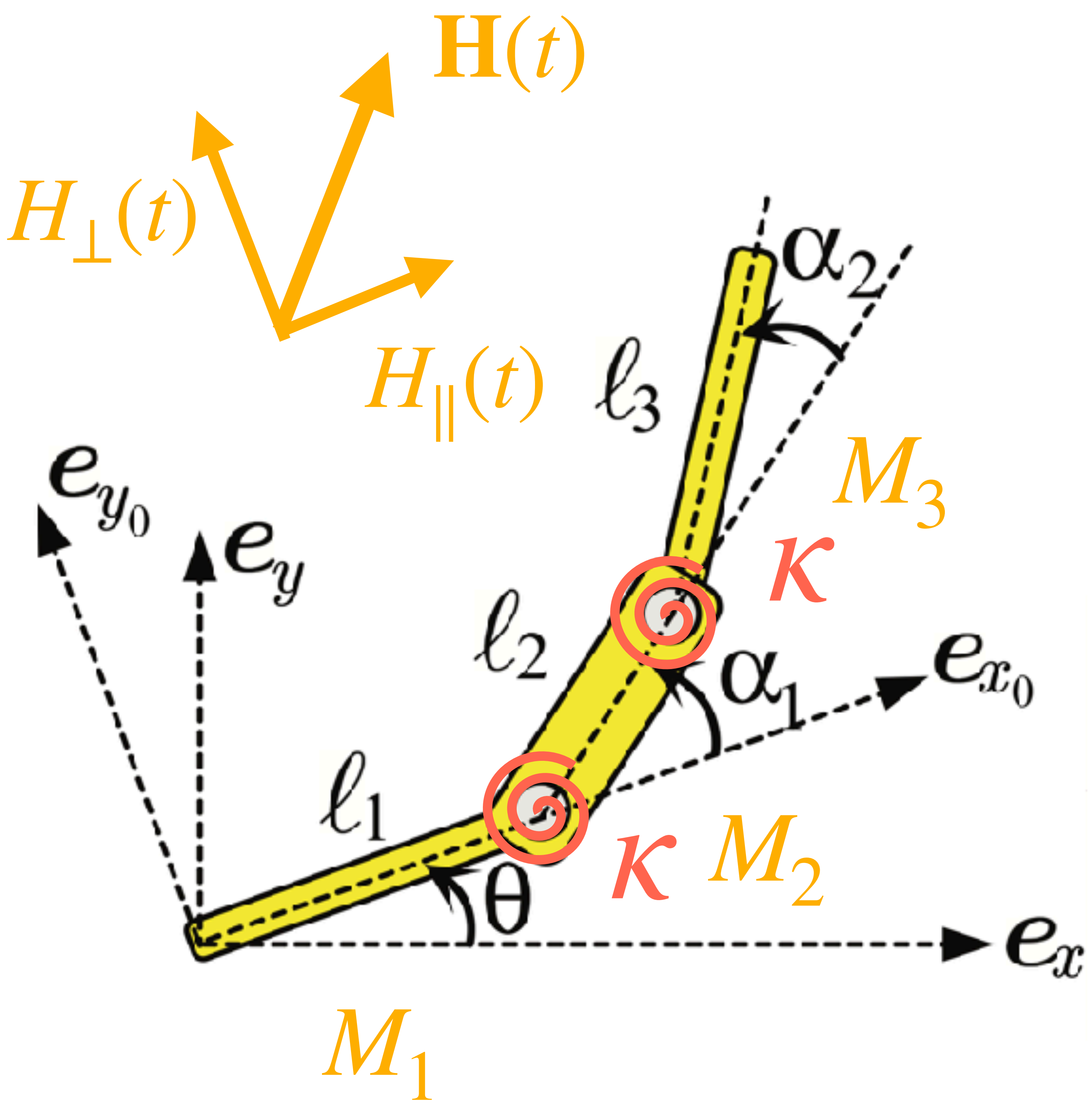


$$\begin{cases} \mathbf{f}_1^h + \mathbf{f}_2^h + \mathbf{f}_3^h = 0 \\ m_1^h + m_2^h + m_3^h = m_1^{mag} + m_2^{mag} + m_3^{mag} \\ m_2^h + m_3^h = m_1^{el} + m_2^{mag} + m_3^{mag} \\ m_3^h = m_2^{el} + m_3^{mag} \end{cases}$$

$$m_i^{\mathrm{mag}} = M_i \mathbf{H} \times \mathbf{e}_{i,\parallel}$$

- State: $\mathbf{z} = (x, y, \theta, \alpha_1, \alpha_2)$
- Controls: $(H_\parallel, H_\perp)$

Drift system: magneto-elastic Purcell swimmer

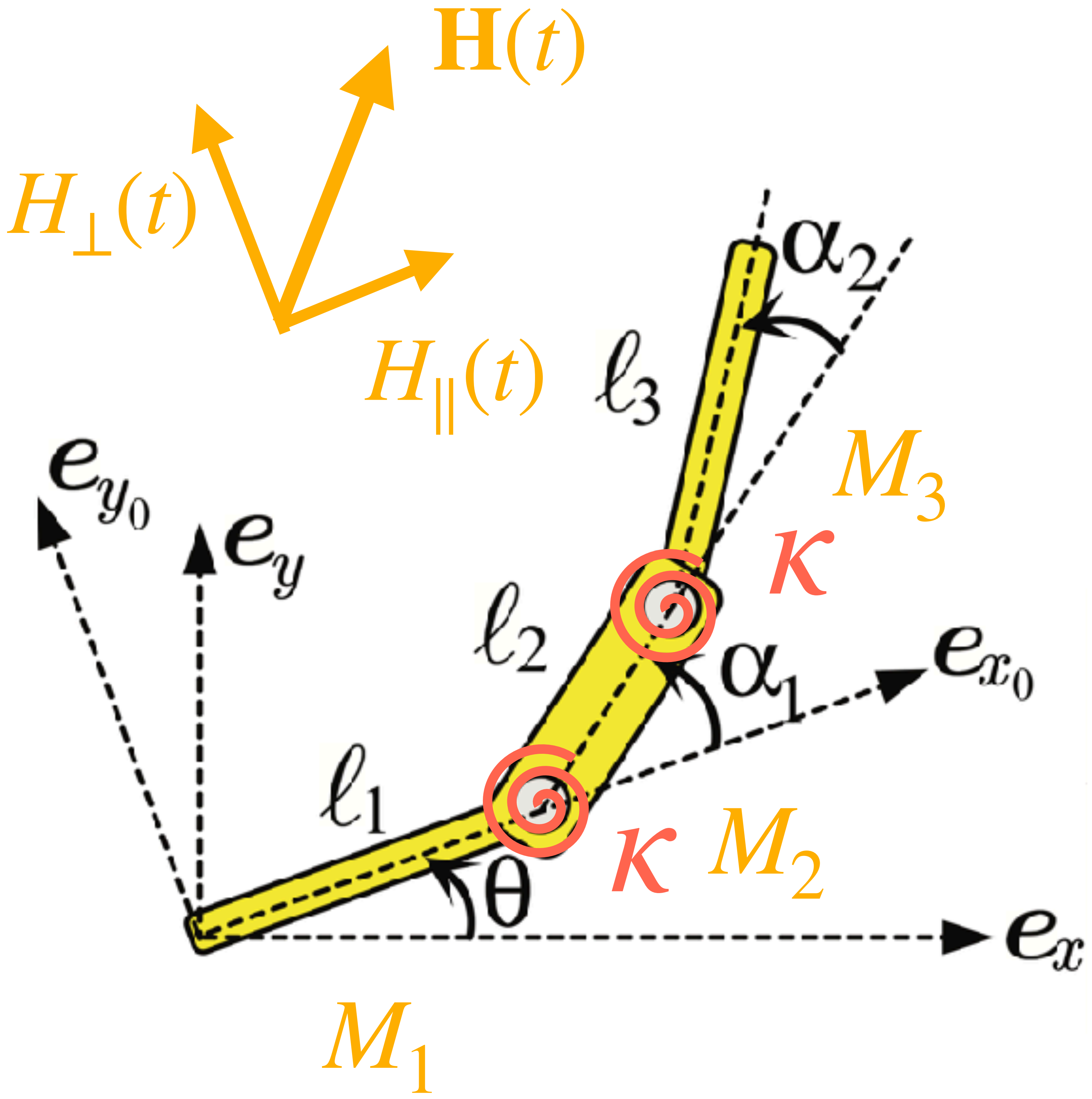


$$\begin{cases} \mathbf{f}_1^h + \mathbf{f}_2^h + \mathbf{f}_3^h = 0 \\ m_1^h + m_2^h + m_3^h = m_1^{mag} + m_2^{mag} + m_3^{mag} \\ m_2^h + m_3^h = m_1^{el} + m_2^{mag} + m_3^{mag} \\ m_3^h = m_2^{el} + m_3^{mag} \end{cases}$$

$$m_i^{\mathrm{mag}} = M_i \mathbf{H} \times \mathbf{e}_{i,\parallel}$$

- State: $\mathbf{z} = (x, y, \theta, \alpha_1, \alpha_2)$
- Controls: $(H_\parallel, H_\perp)$
- Equilibrium points: $\alpha_1 = \alpha_2 = 0, H_\perp = 0, H_\parallel$ **arbitrary**

Drift system: magneto-elastic Purcell swimmer



$$\begin{cases} \mathbf{f}_1^h + \mathbf{f}_2^h + \mathbf{f}_3^h = 0 \\ m_1^h + m_2^h + m_3^h = m_1^{mag} + m_2^{mag} + m_3^{mag} \\ m_2^h + m_3^h = m_1^{el} + m_2^{mag} + m_3^{mag} \\ m_3^h = m_2^{el} + m_3^{mag} \end{cases}$$

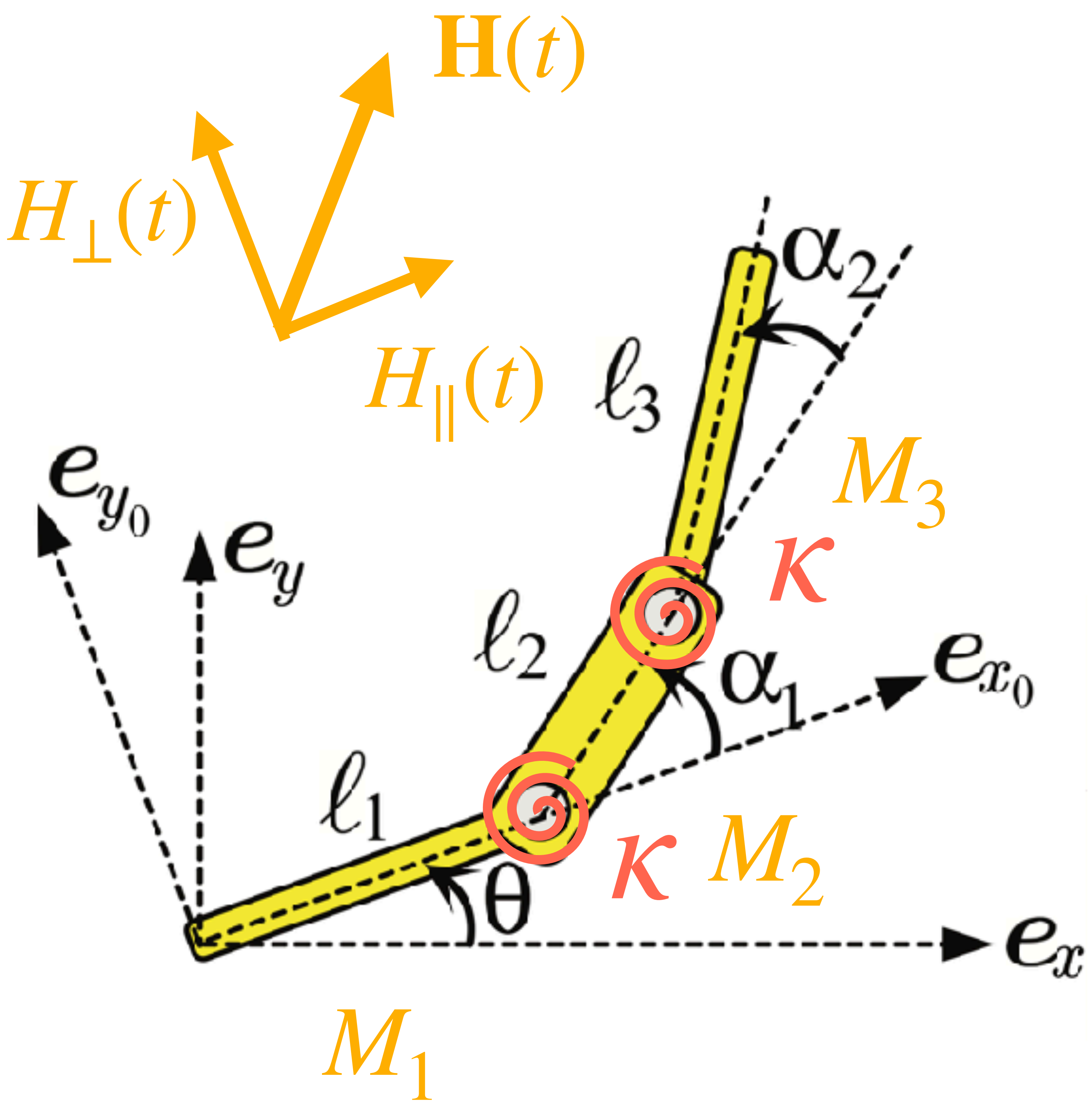
$$m_i^{\mathrm{mag}} = M_i \mathbf{H} \times \mathbf{e}_{i,\parallel}$$

$$\dot{\mathbf{z}} = \mathbf{F}_0(\alpha_1, \alpha_2) + \mathbf{F}_\parallel(\alpha_1, \alpha_2) H_\parallel(t) + \mathbf{F}_\perp(\alpha_1, \alpha_2) H_\perp(t)$$

Control-affine system with a drift

- State: $\mathbf{z} = (x, y, \theta, \alpha_1, \alpha_2)$
- Controls: $(H_\parallel, H_\perp)$
- Equilibrium points: $\alpha_1 = \alpha_2 = 0, H_\perp = 0,$ $H_\parallel$ **arbitrary**

Drift system: magneto-elastic Purcell swimmer



- State: $\mathbf{z} = (x, y, \theta, \alpha_1, \alpha_2)$
- Controls: $(H_\parallel, H_\perp)$
- Equilibrium points: $\alpha_1 = \alpha_2 = 0, H_\perp = 0,$ **$H_\parallel$ arbitrary**

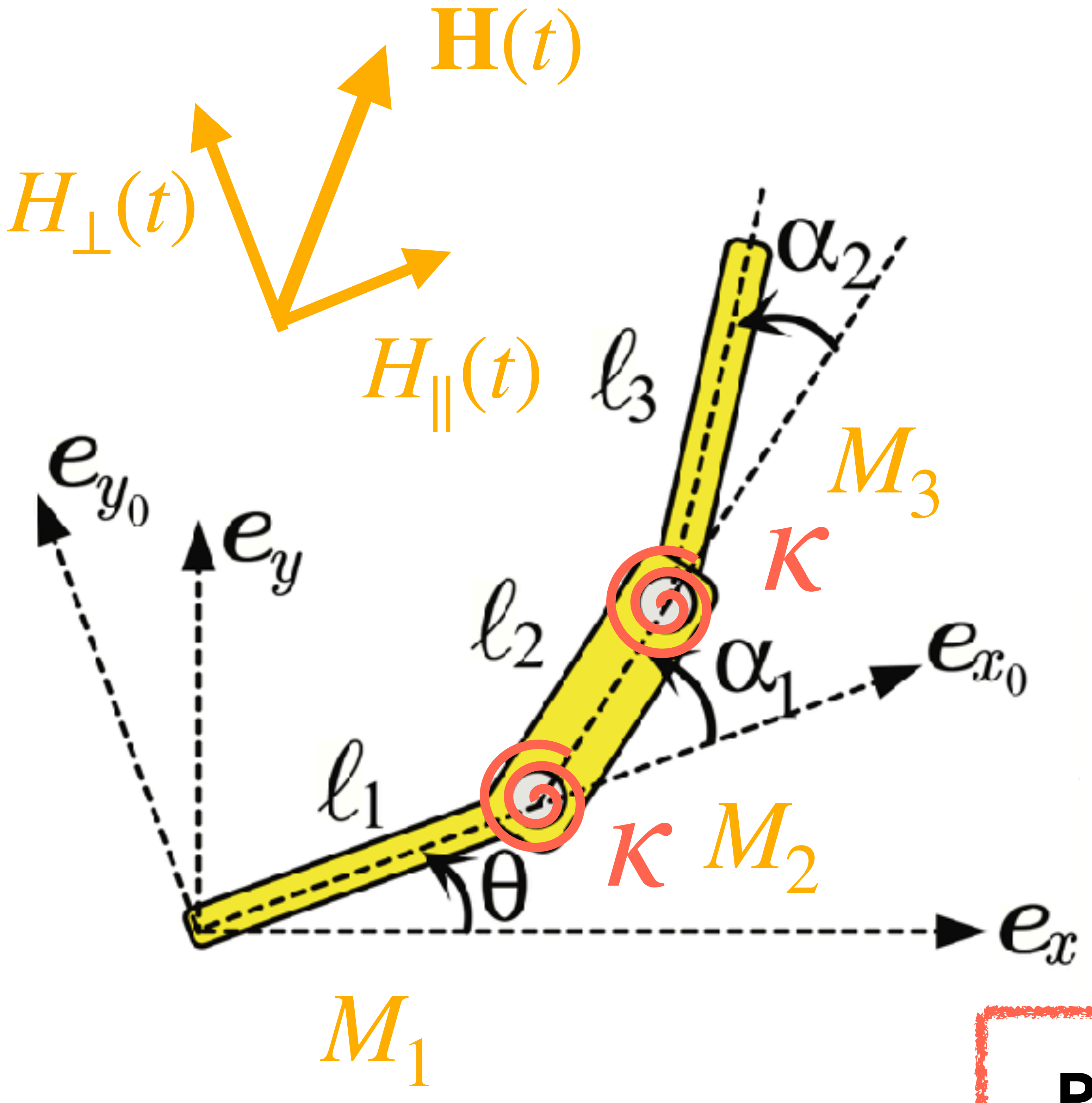
$$\left\{\begin{array}{rcl} \mathbf{f}_1^h + \mathbf{f}_2^h + \mathbf{f}_3^h & = & 0 \\ m_1^h + m_2^h + m_3^h & = & m_1^{mag} + m_2^{mag} + m_3^{mag} \\ m_2^h + m_3^h & = & m_1^{el} + m_2^{mag} + m_3^{mag} \\ m_3^h & = & m_2^{el} + m_3^{mag} \end{array}\right.$$

$$m_i^{\mathrm{mag}} = M_i \mathbf{H} \times \mathbf{e}_{i,\parallel}$$

$$\dot{\mathbf{z}} = \mathbf{F}_0(\alpha_1, \alpha_2) + \mathbf{F}_\parallel(\alpha_1, \alpha_2) H_\parallel(t) + \mathbf{F}_\perp(\alpha_1, \alpha_2) H_\perp(t)$$

Around which points (= **which value of $H_\parallel$**)
do we have local controllabilty?

Drift systems: magneto-elastic Purcell swimmer



$$\left\{\begin{array}{rcl} \mathbf{f}_1^h + \mathbf{f}_2^h + \mathbf{f}_3^h & = & 0 \\ m_1^h + m_2^h + m_3^h & = & m_1^{mag} + m_2^{mag} + m_3^{mag} \\ m_2^h + m_3^h & = & m_1^{el} + m_2^{mag} + m_3^{mag} \\ m_3^h & = & m_2^{el} + m_3^{mag} \end{array}\right.$$

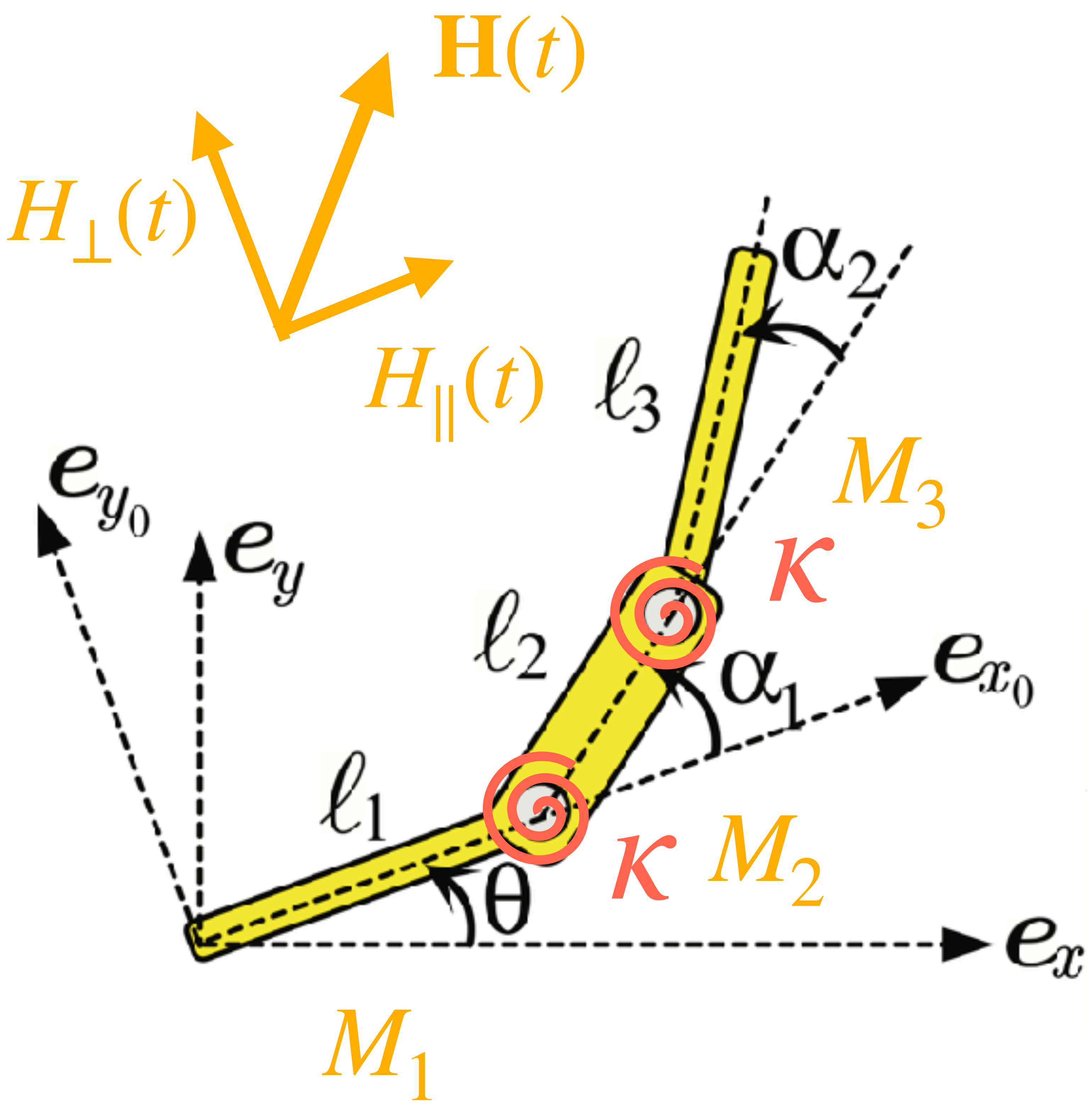
$$\dot{\mathbf{z}} = \mathbf{F}_0(\alpha_1, \alpha_2) + \mathbf{F}_\parallel(\alpha_1, \alpha_2)H_\parallel(t) + \mathbf{F}_\perp(\alpha_1, \alpha_2)H_\perp(t)$$

Proposition: The swimmer is locally controllable if and only if

$$H_\parallel = \frac{[\mathbf{F}_\perp, [\mathbf{F}_0, \mathbf{F}_\perp]](0,0)}{[\mathbf{F}_\perp, [\mathbf{F}_\parallel, \mathbf{F}_\perp]](0,0)} .$$

Moreau, IEEE Control Syst. Lett. 2019

Drift systems: magneto-elastic Purcell swimmer



$$\left\{\begin{array}{rcl} \mathbf{f}_1^h + \mathbf{f}_2^h + \mathbf{f}_3^h & = & 0 \\ m_1^h + m_2^h + m_3^h & = & m_1^{mag} + m_2^{mag} + m_3^{mag} \\ m_2^h + m_3^h & = & m_1^{el} + m_2^{mag} + m_3^{mag} \\ m_3^h & = & m_2^{el} + m_3^{mag} \end{array}\right.$$

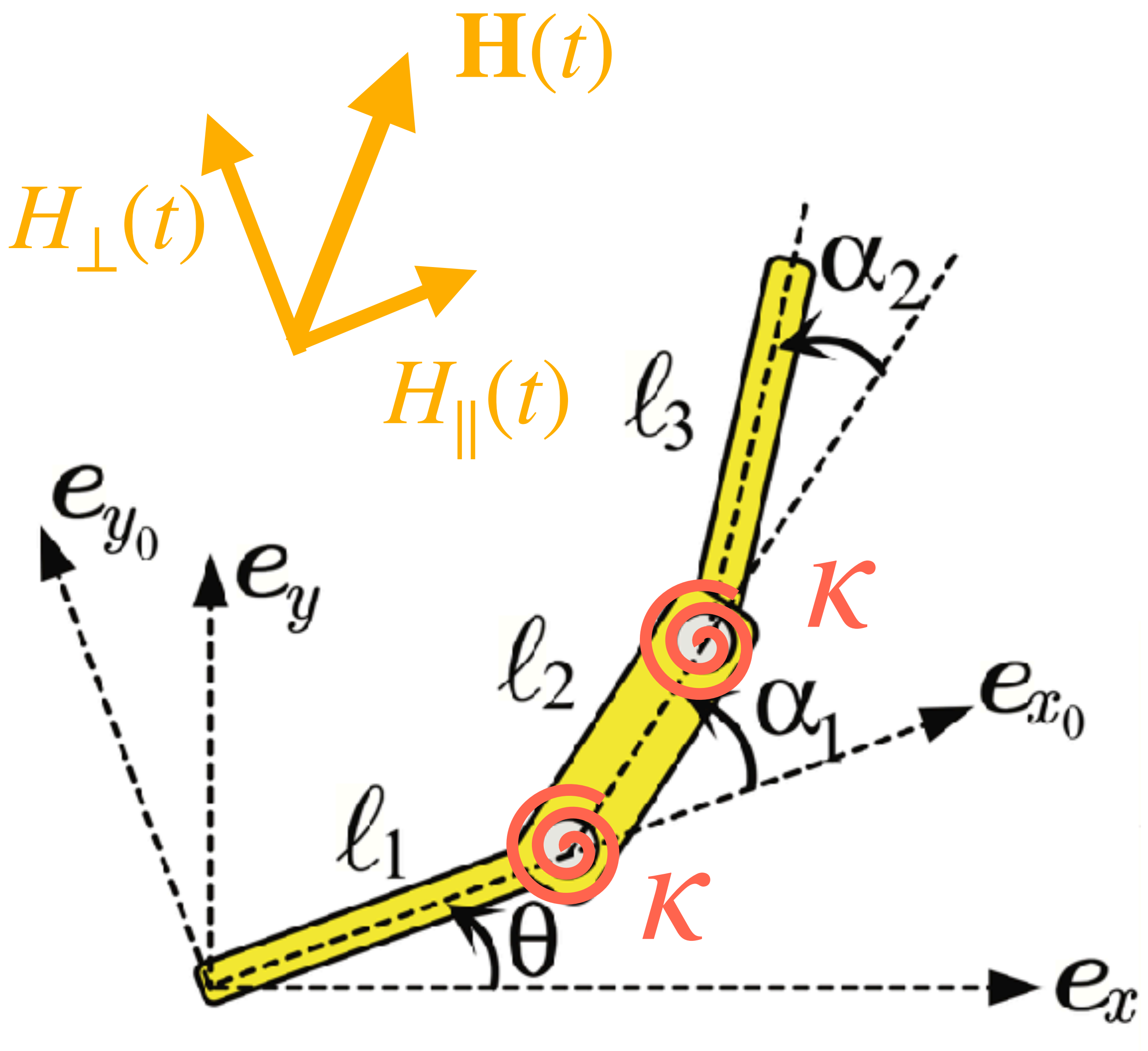
$$\dot{\mathbf{z}} = \mathbf{F}_0(\alpha_1, \alpha_2) + \mathbf{F}_\parallel(\alpha_1, \alpha_2)H_\parallel(t) + \mathbf{F}_\perp(\alpha_1, \alpha_2)H_\perp(t)$$

Proposition: The swimmer is STLC if and only if

$$H_\parallel = \frac{[\mathbf{F}_\perp, [\mathbf{F}_0, \mathbf{F}_\perp]](0,0)}{[\mathbf{F}_\perp, [\mathbf{F}_\parallel, \mathbf{F}_\perp]](0,0)} = \kappa \frac{17(M_1 + M_3) - 16M_2}{-7M_2^2 + 9M_2(M_1 + M_3) + 5M_1M_3}.$$

Moreau, IEEE Control Syst. Lett. 2019

Drift systems: magneto-elastic Purcell swimmer



Proposition: The swimmer is STLC if and only if

$$H_\parallel = \frac{[\mathbf{F}_\perp, [\mathbf{F}_0, \mathbf{F}_\perp]](0,0)}{[\mathbf{F}_\perp, [\mathbf{F}_\parallel, \mathbf{F}_\perp]](0,0)} = \kappa \frac{17(M_1 + M_3) - 16M_2}{-7M_2^2 + 9M_2(M_1 + M_3) + 5M_1M_3}.$$

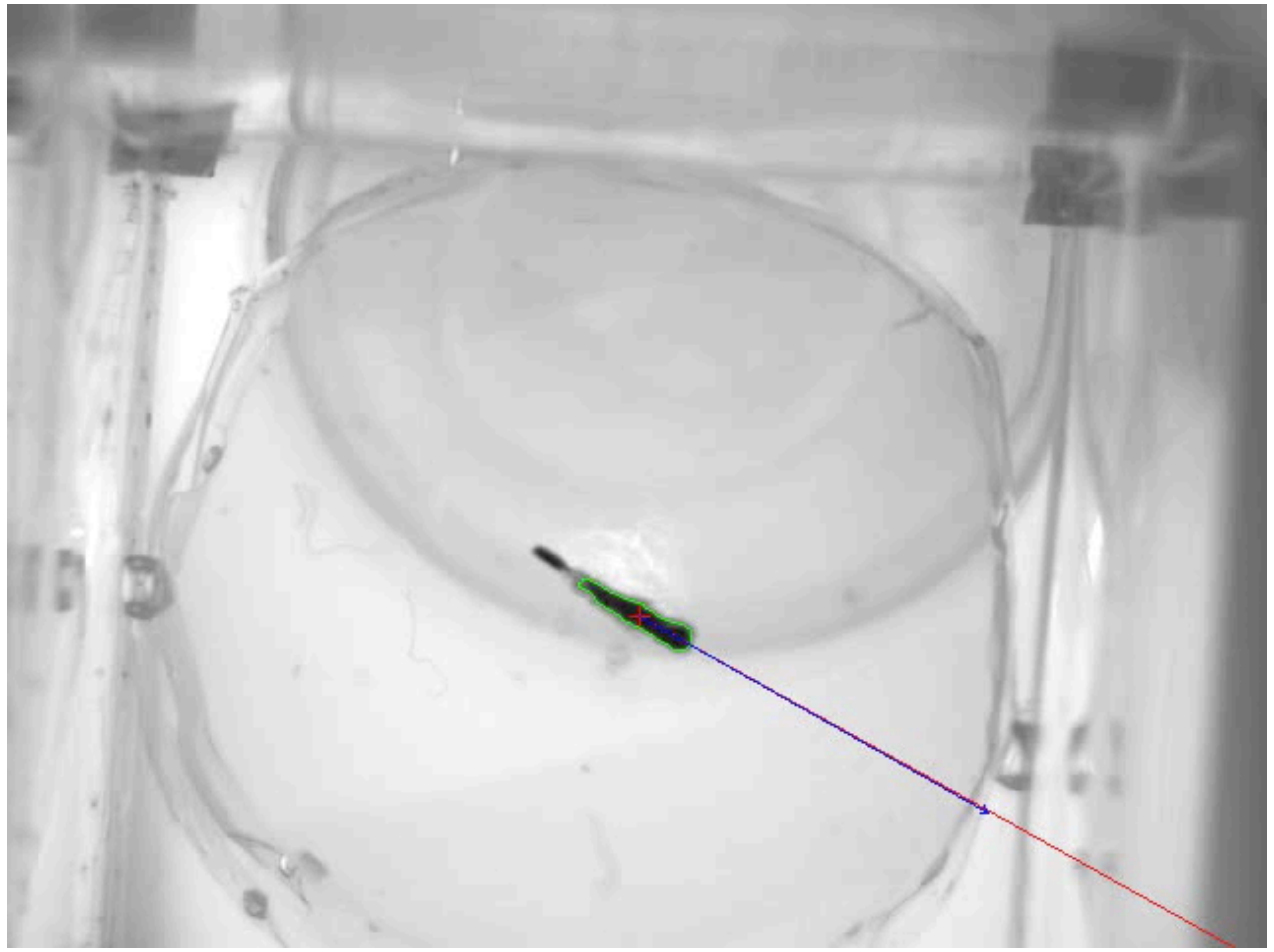
"The control $H_\parallel$ must be close to this value in order to neutralise the drift"

$H_\parallel < \beta$

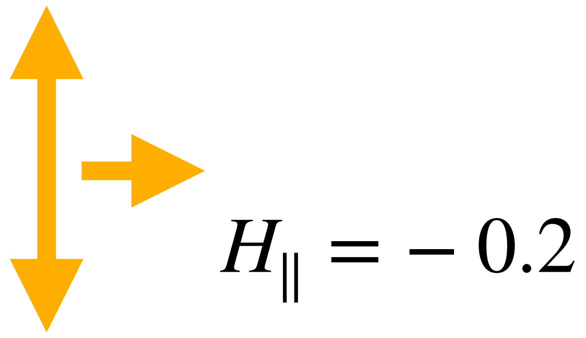
$H_\parallel = \beta$

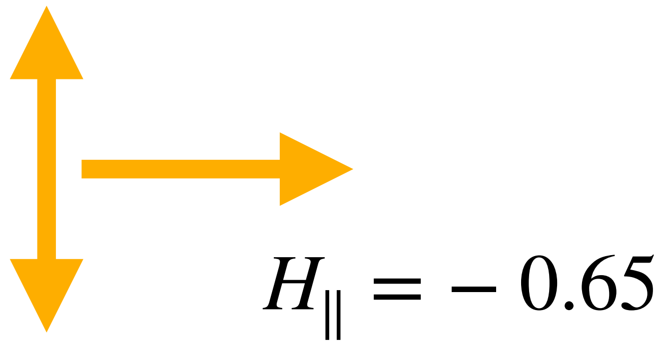
$H_\parallel > \beta$

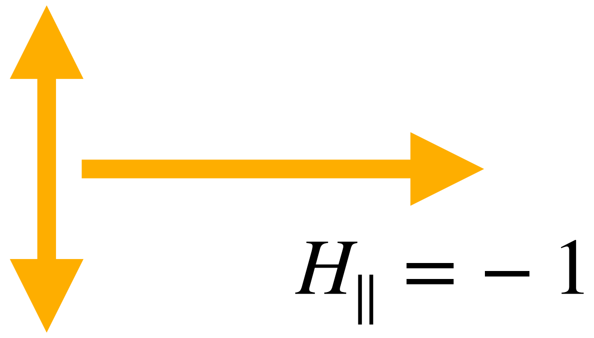
Mystery solved



Magnetic field: vertical oscillating
+ constant horizontal

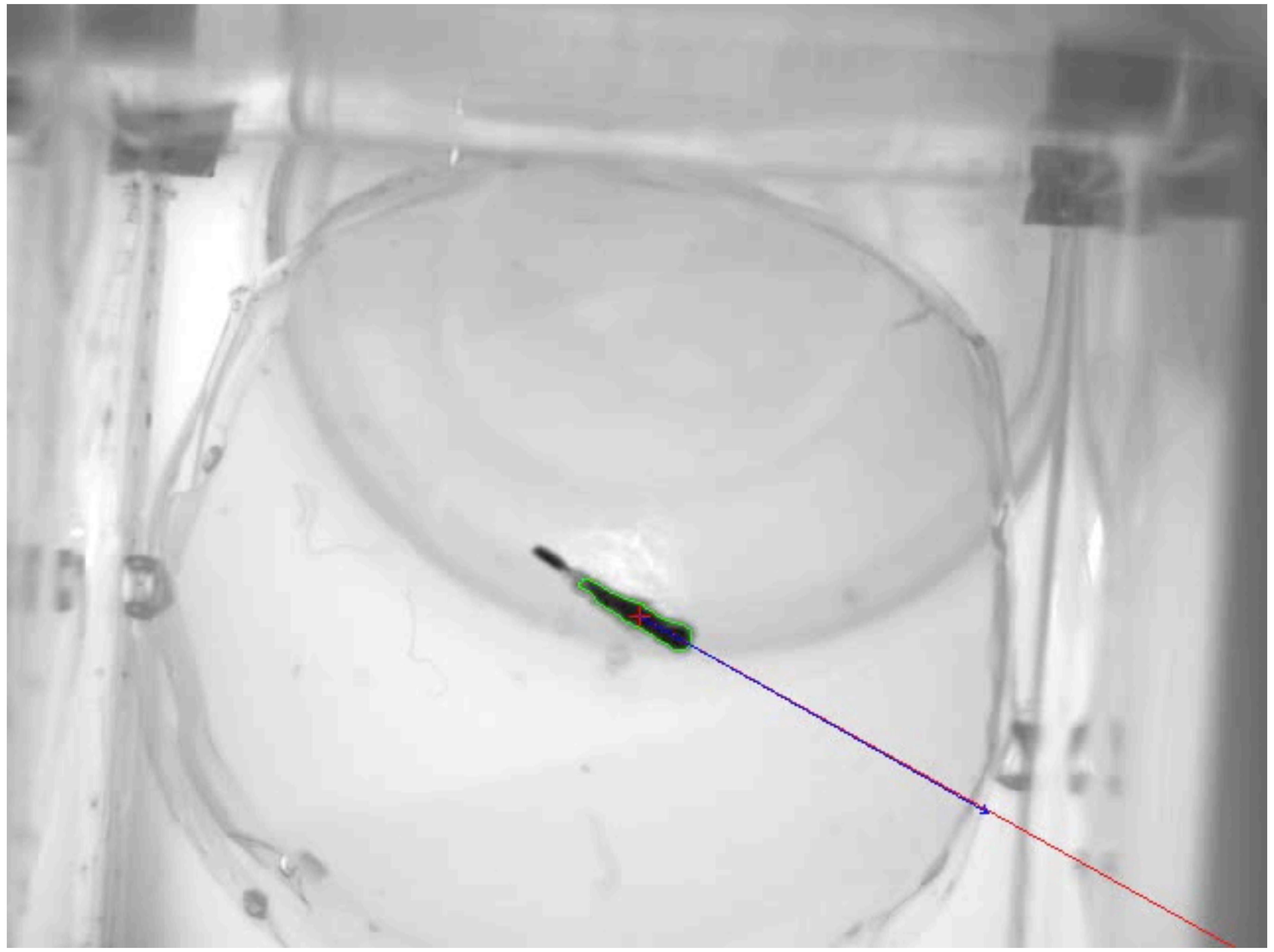




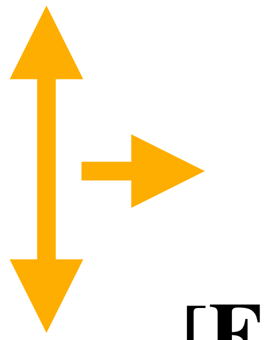


how and why?

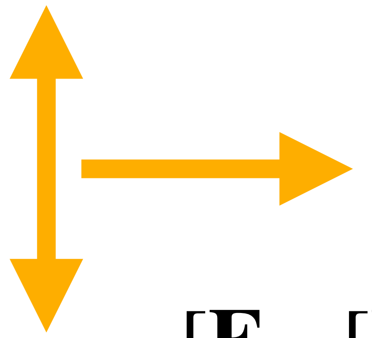
Mystery solved



Magnetic field: vertical oscillating
+ constant horizontal



$$H_\parallel < \frac{[\mathbf{F}_\perp, [\mathbf{F}_0, \mathbf{F}_\perp]](0,0)}{[\mathbf{F}_\perp, [\mathbf{F}_\parallel, \mathbf{F}_\perp]](0,0)}$$



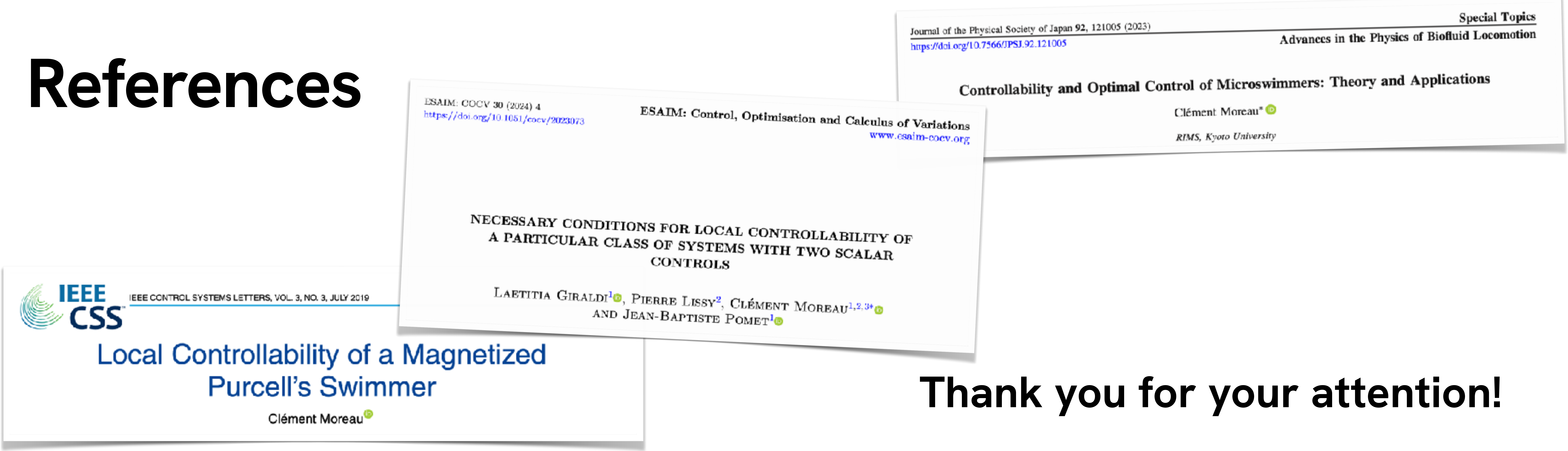
$$H_\parallel = \frac{[\mathbf{F}_\perp, [\mathbf{F}_0, \mathbf{F}_\perp]](0,0)}{[\mathbf{F}_\perp, [\mathbf{F}_\parallel, \mathbf{F}_\perp]](0,0)}$$

$$H_\parallel > \frac{[\mathbf{F}_\perp, [\mathbf{F}_0, \mathbf{F}_\perp]](0,0)}{[\mathbf{F}_\perp, [\mathbf{F}_\parallel, \mathbf{F}_\perp]](0,0)}$$

Summary

- Overview of control theory applied to microswimming
- Future challenges:
    - More controllability results for systems with a drift?
    - Multiple swimmers problem : controllability for more particles
    - “Break more assumptions” from the canonical swimming equation

References



Journal of the Physical Society of Japan **92**, 121005 (2023)
https://doi.org/10.7566/JPSJ.92.121005

Special Topics
Advances in the Physics of Biofluid Locomotion

Controllability and Optimal Control of Microswimmers: Theory and Applications

Clément Moreau*

RIMS, Kyoto University

ESAIM: COCV 30 (2024) 4
https://doi.org/10.1051/cocv/2023073

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

NECESSARY CONDITIONS FOR LOCAL CONTROLLABILITY OF A PARTICULAR CLASS OF SYSTEMS WITH TWO SCALAR CONTROLS

LAETITIA GIRALDI[1], PIERRE LISSY[2], CLÉMENT MOREAU[1,2,3*] AND JEAN-BAPTISTE POMET[1]

IEEE CSS

IEEE CONTROL SYSTEMS LETTERS, VOL. 3, NO. 3, JULY 2019

Local Controllability of a Magnetized Purcell’s Swimmer

Clément Moreau

Thank you for your attention!

Sketch of proof

Chen-Fliess series

$$\exists T > 0,\ \sum_{I} \Big(\int_{I} u \Big) (f_I \Phi)(0) \longrightarrow_{n \to \infty} \Phi(x_u(T))$$

Fliess 1981, Sussmann 1983

Sketch of proof

Real analytic map

$\Phi : \mathbb{R}^n \to \mathbb{R}$

Chen-Fliess series

$$\exists T > 0,\ \sum_I \Big(\int_I u\Big)\,(f_I \Phi)(0) \longrightarrow_{n\to\infty} \Phi(x_u(T))$$

Solution with control u

$x_u(T)$

Fliess 1981, Sussmann 1983

Sketch of proof

Real analytic map

$$\Phi : \mathbb{R}^n \to \mathbb{R}$$

Chen-Fliess series

$$\exists T > 0,\ \sum_I \Big(\int_I u\Big)\,(f_I\,\Phi)(0) \longrightarrow_{n\to\infty} \Phi(x_u(T))$$

Multi-indices

$$(i_1, \dots, i_k), i_j \in \{0,\dots,m\}$$

Iterated integral

$$\int_0^T u_{i_n}(\tau_n)\int_0^{\tau_k} u_{i_{k-1}}(\tau_{k-1})\dots\int_0^{\tau_3} u_{i_2}(\tau_2)\int_0^{\tau_2} u_{i_1}(\tau_1)\,\mathrm{d}\tau_1\mathrm{d}\tau_2\dots\mathrm{d}\tau_{n-1}\mathrm{d}\tau_k$$

$(u_0 = 1)$

Product of differential operators

$$f_{i_1}f_{i_2}\dots f_{i_k}$$

Solution with control u

$$x_u(T)$$

Fliess 1981, Sussmann 1983

Sketch of proof

Chen-Fliess series

$$\Phi(x_u(T)) = \sum_I \left(\int_I u\right) (f_I \Phi)(0)$$

- Objective : show that for a good choice of Φ and for all controls u,

$$\Phi(x_u(T)) > 0$$

- Construction of Φ : $\forall f \in R_1, (f\Phi)(0) = 0$

$$(f_{101}\Phi)(0) = 1$$

- Show that the term associated to f_{101} « dominates » the rest of the series… except when f_{121} comes to the rescue

$\square$

Higher-order brackets, single-input case

$$\dot{x} = f_0(x) + f_1(x)u$$

$$S_2 \not\subset S_1 \,?$$

$$[f_1, [f_0, f_1]](0) \in S_1$$

$$\big[[f_0, f_1], [f_1, [f_0, f_1]]\big]$$

- **may or may not be STLC** ! (Sussmann 1986)
- Refinements of the bracket spaces S: Kawski, Stefani, Krastanov, etc.
- **new point of view** : regularity of controls (Beauchard, Marbach 2018)

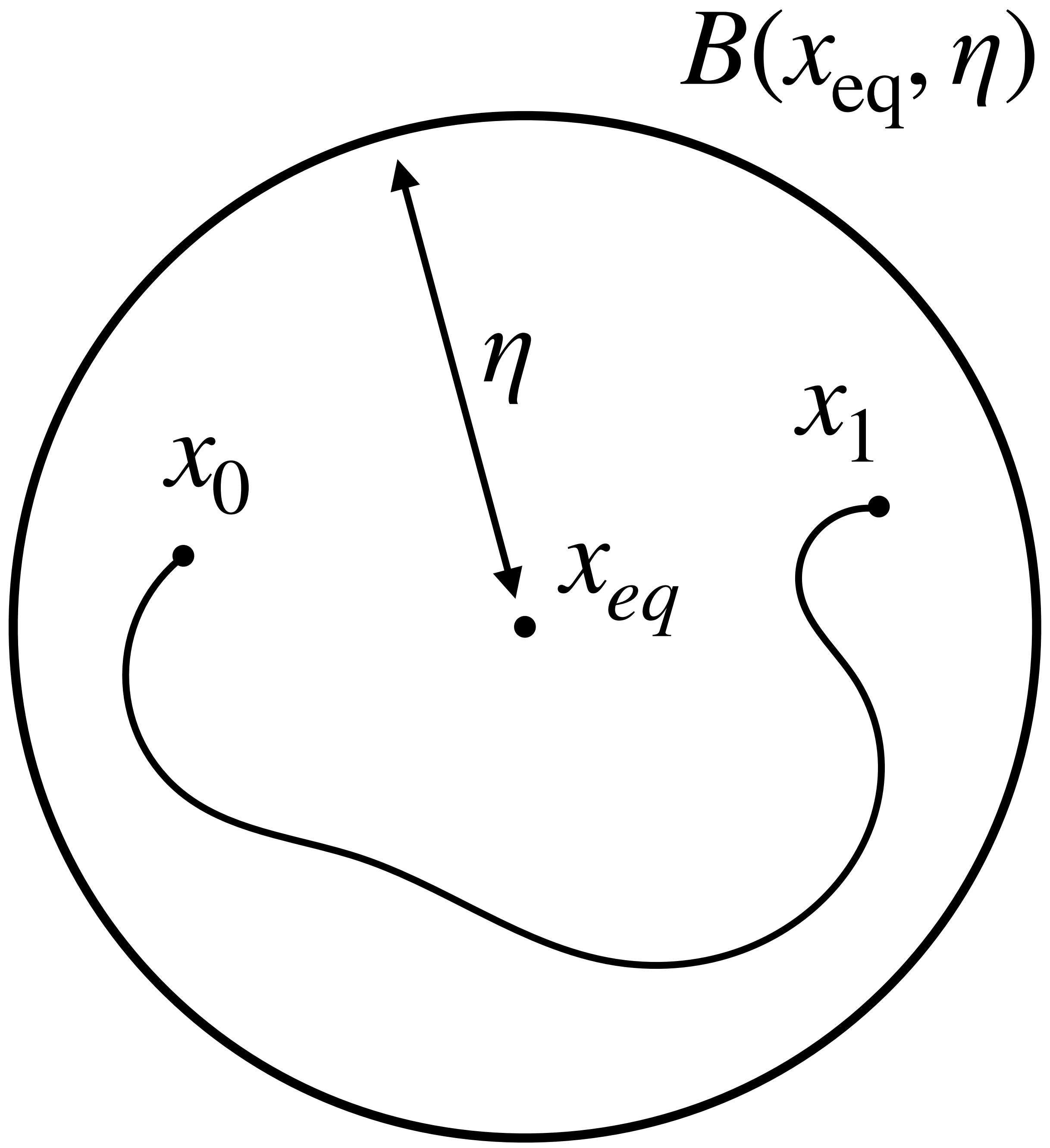
Higher-order brackets, single-input case

$$\dot{x} = f_0(x) + f_1(x)u$$

$$S_2 \not\subset S_1 ?$$

$$[f_1, [f_0, f_1]](0) \in S_1$$

$$\big[[f_0, f_1], [f_0, [f_0, f_1]]\big]$$



STLC: in **small time** and with **small controls**

B-STLC: in **small time** and with **bounded controls**

$W^{k,\infty}$**-STLC**: in **small time** and with **controls small in another norm**

- **may or may not be STLC** ! (Sussmann 1986)
- Refinements of the bracket spaces S: Kawski, Stefani, Krastanov, etc.
- **new point of view** : regularity of controls (Beauchard, Marbach 2018)

Higher-order brackets, multi-input case

$$\dot{x} = f_0(x) + f_1(x)u_1 + f_2(x)u_2$$

$$f_{121}, f_{101} \in R_1$$

Higher-order brackets, multi-input case

$$\dot{x} = f_0(x) + f_1(x)u_1 + f_2(x)u_2$$

$$f_{121}, f_{101} \in R_1$$

8 Lie brackets involved in the higher-order condition…

$$\big[[f_0, f_1], [f_0, [f_0, f_1]]\big] \quad \big[[f_2, f_1], [f_0, [f_2, f_1]]\big] \quad \big[[f_2, f_1], [f_0, [f_0, f_1]]\big] \quad \big[[f_0, f_1], [f_2, [f_2, f_1]]\big]$$

$$\big[[f_0, f_1], [f_2, [f_0, f_1]]\big] \quad \big[[f_2, f_1], [f_2, [f_2, f_1]]\big] \quad \big[[f_2, f_1], [f_2, [f_0, f_1]]\big] \quad \big[[f_0, f_1], [f_0, [f_2, f_1]]\big]$$

Higher-order brackets, multi-input case

$$\dot{x} = f_0(x) + f_1(x)u_1 + f_2(x)u_2$$

$$f_{121}, f_{101} \in R_1$$

8 Lie brackets involved in the higher-order condition...

$$\big[[f_0, f_1], [f_0, [f_0, f_1]]\big] \quad \big[[f_2, f_1], [f_0, [f_2, f_1]]\big] \quad \big[[f_2, f_1], [f_0, [f_0, f_1]]\big] \quad \big[[f_0, f_1], [f_2, [f_2, f_1]]\big]$$

$$\big[[f_0, f_1], [f_2, [f_0, f_1]]\big] \quad \big[[f_2, f_1], [f_2, [f_2, f_1]]\big] \quad \big[[f_2, f_1], [f_2, [f_0, f_1]]\big] \quad \big[[f_0, f_1], [f_0, [f_2, f_1]]\big]$$

A key quadratic form

$$D(\lambda_1, \lambda_2) = \lambda_1^2(f_{01,001} - u_2^{\mathrm{eq}} f_{01,201}) + \lambda_2^2(f_{21,021} - u_2^{\mathrm{eq}} f_{21,221})$$
$$-\lambda_1\lambda_2(f_{21,001} + f_{01,021} - u_2^{\mathrm{eq}}(f_{21,201} + f_{01,221}))$$

Higher-order brackets, multi-input case

$$\dot{x} = f_0(x) + f_1(x)u_1 + f_2(x)u_2$$

$$f_{121}, f_{101} \in R_1$$

A key quadratic form

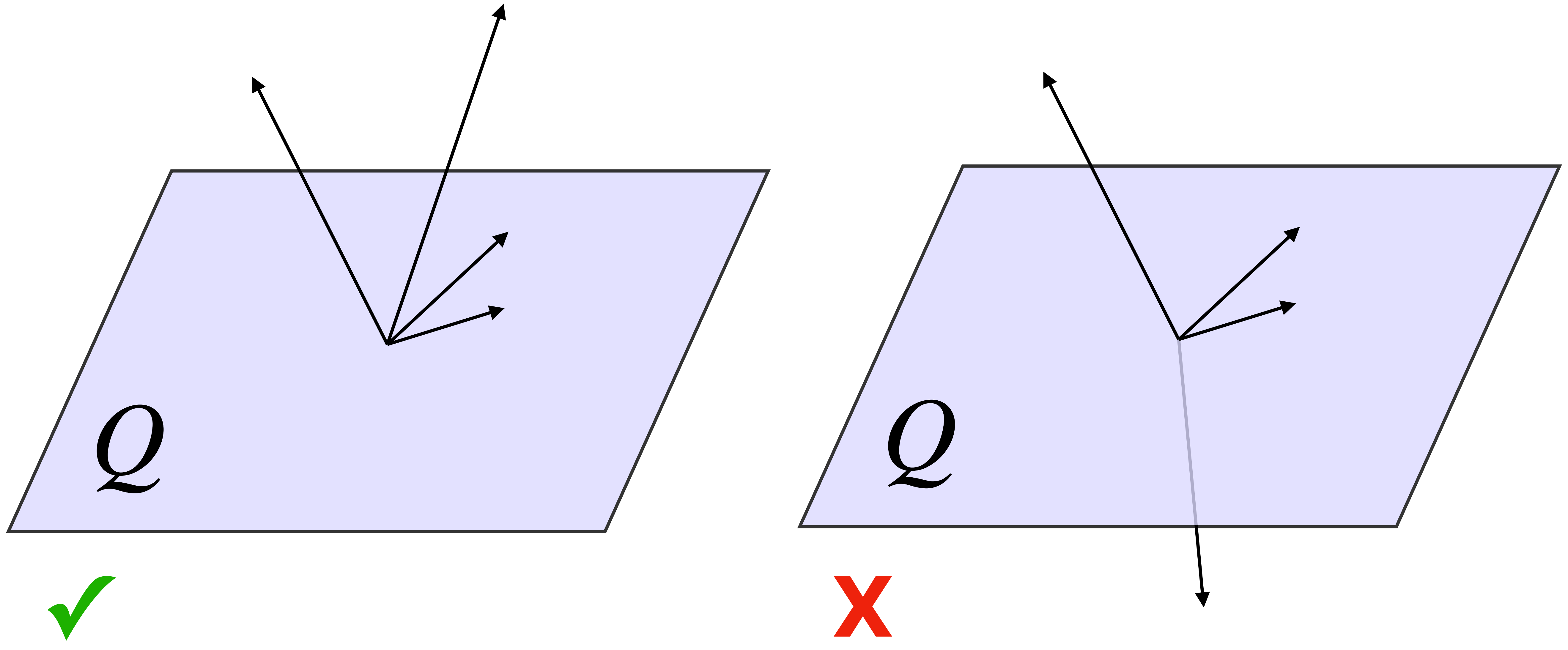
$$D(\lambda_1, \lambda_2) = \lambda_1^2(f_{01,001} - u_2^{\mathrm{eq}} f_{01,201}) + \lambda_2^2(f_{21,021} - u_2^{\mathrm{eq}} f_{21,221}) - \lambda_1\lambda_2(f_{21,001} + f_{01,021} - u_2^{\mathrm{eq}}(f_{21,201} + f_{01,221}))$$

Geometric interpretation:

« $C(Q)$ condition »

$$\exists \varphi \in L(\mathbb{R}^n, \mathbb{R}) \ \text{ s.t. } Q \in \ker \varphi$$

$$\text{and } (\lambda_1, \lambda_2) \mapsto \langle \varphi, D(\lambda_1, \lambda_2) \rangle > 0$$



Higher-order brackets, multi-input case

$$\dot{x} = f_0(x) + f_1(x)u_1 + f_2(x)u_2$$

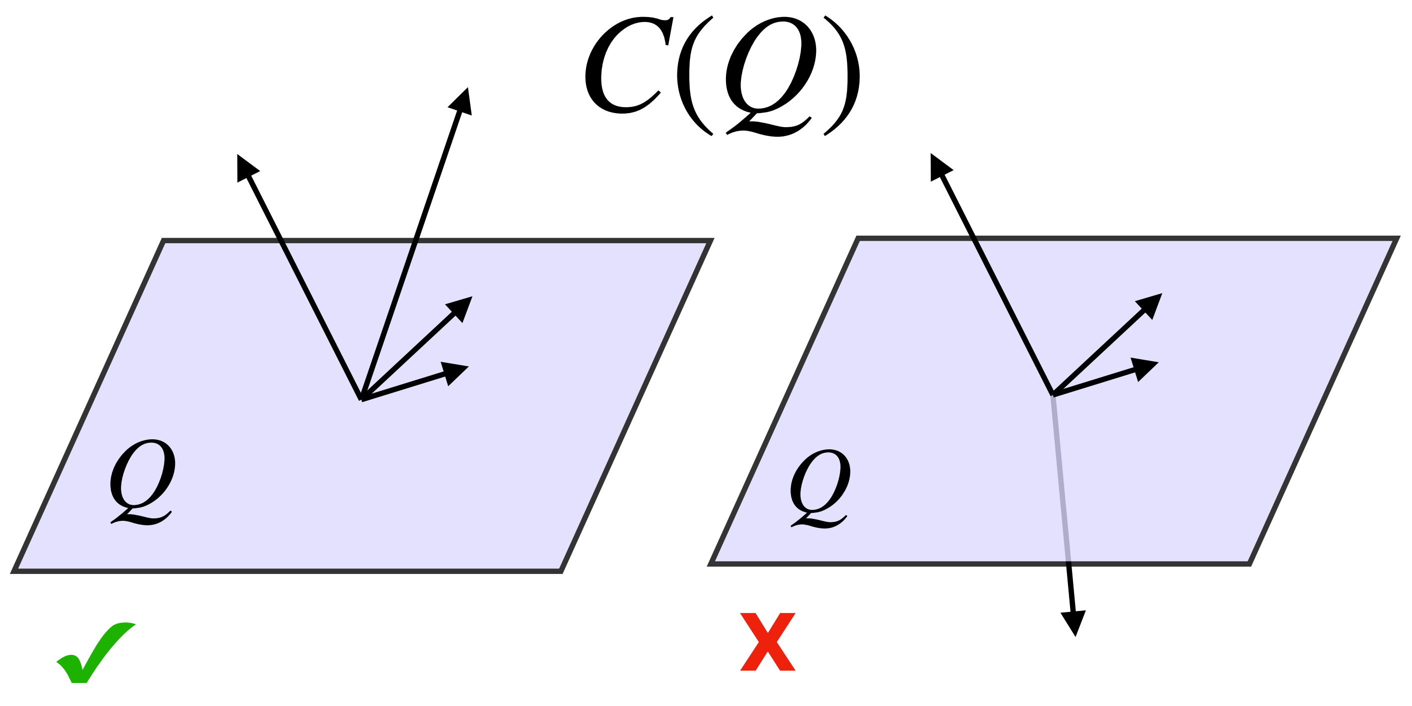
$$f_{121}, f_{101}, f_{21,01} \in R_1$$

$$R' = \mathrm{Span}(f_{01,201}, f_{21,221}, f_{21,201}, f_{01,221})$$

R'' : for Kawski-style condition

Theorem (Giraldi, Lissy, M., Pomet, ESAIM:COCV 2024)

1. If $C(R_1)$ holds : not $(W^{1,\infty}, L^\infty)$-STLC.
2. If $C(R_1 + R')$ holds : not $(W^{1,\infty}, B)$-STLC.
3. If $C(R_1 + R'')$ holds : not (L^∞, L^∞)-STLC.
4. If $C(R_1 + R' + R'')$ holds : not (L^∞, B)-STLC.



1. « The 8 brackets cannot compensate each other if they all drift on the same side »

2. « If the nicer ones have no effect, it's worse »