

Sur la structure des équations de la dynamique des poutres et une classification des problèmes associés

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Kinematical configuration of a beam-like structure

Material coordinates

$S \in [0, L]$ (ξ_1, ξ_2) local chart in S

Configuration in the ambient space

Placement :

$$\varphi(S, t) = OG$$

$$\text{Rotation} : Q(S, t) \in SO(3)$$

$$d_i(S, t) = Q(S, t)e_i$$

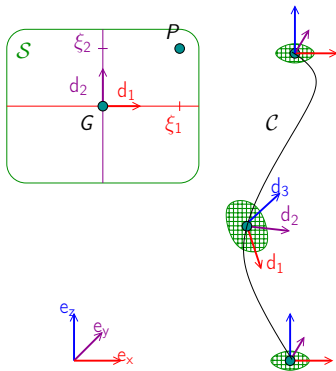
Place of a point P of the beam

$$(\xi_1, \xi_2, S, t) \rightarrow OP = OG + GP$$

$$GP = \xi_1 d_1 + \xi_2 d_2$$

S is **rigidly** transformed

d_3 is **not** tangent to C



E.Cosserat, F.Cosserat, *Théorie des Corps déformables*, 1909
 S.P.Timoshenko, J.C.Gere, *Theory of Elastic Stability*, 1961

G is the center of mass of S

$\{d_i\}$ is an inertial principal basis

Vectors on the tangent space TC Speed v

$$v := \frac{\partial \varphi}{\partial t}$$

Strain ε

$$\varepsilon := \frac{\partial \varphi}{\partial S} - d_3$$

Spin ω

$$\frac{\partial d_i}{\partial t} = \omega \wedge d_i$$

$$\omega = \text{axial}\left(Q^T \frac{\partial Q}{\partial t}\right)$$

$$Q^T \frac{\partial Q}{\partial t} \in so(3)$$

Curvature κ

$$\frac{\partial d_i}{\partial S} = \kappa \wedge d_i$$

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Vectors on the tangent space TC Speed v

$$v := \frac{\partial \varphi}{\partial t}$$

$$v = v_1 d_1 + v_2 d_2 + v_3 d_3$$

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Associated moments and energy

Momentum $\mathbf{p} = \rho A \mathbf{v}$

$$\mathbf{p} := \begin{bmatrix} \rho A & 0 & 0 \\ 0 & \rho A & 0 \\ 0 & 0 & \rho A \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbb{A} \mathbf{v}$$

Force \mathbf{N}

$$\mathbf{N} := \begin{bmatrix} GA & 0 & 0 \\ 0 & GA & 0 \\ 0 & 0 & EA \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \mathbb{G} \boldsymbol{\varepsilon}$$

Moment of momentum $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} := \begin{bmatrix} \rho I_1 & 0 & 0 \\ 0 & \rho I_2 & 0 \\ 0 & 0 & \rho I_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \mathbb{J} \boldsymbol{\omega}$$

Torque \mathbf{M}

$$\mathbf{M} := \begin{bmatrix} EI_1 & 0 & 0 \\ 0 & EI_2 & 0 \\ 0 & 0 & GI_3 \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{bmatrix} = \mathbb{H} \boldsymbol{\kappa}$$

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Kinetic energy density

$$T(\mathbf{v}, \boldsymbol{\omega}) = \frac{1}{2} (\mathbf{v} \mathbb{A} \mathbf{v} + \boldsymbol{\omega} \mathbb{J} \boldsymbol{\omega})$$

Deformation energy density

$$U(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \frac{1}{2} (\boldsymbol{\varepsilon} \mathbb{G} \boldsymbol{\varepsilon} + \boldsymbol{\kappa} \mathbb{H} \boldsymbol{\kappa})$$

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$$\text{Lagrangian} \quad \mathcal{L}(\mathbf{v}, \boldsymbol{\omega}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}) := \int_0^L \frac{1}{2} \mathbf{v} \mathbb{A} \mathbf{v} + \frac{1}{2} \boldsymbol{\omega} \mathbb{J} \boldsymbol{\omega} - \frac{1}{2} \boldsymbol{\varepsilon} \mathbb{G} \boldsymbol{\varepsilon} - \frac{1}{2} \boldsymbol{\kappa} \mathbb{H} \boldsymbol{\kappa} \, dS$$

$$\text{Hamiltonian} \quad \mathcal{H}(\mathbf{v}, \boldsymbol{\omega}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}) := \int_0^L \frac{1}{2} \mathbf{v} \mathbb{A} \mathbf{v} + \frac{1}{2} \boldsymbol{\omega} \mathbb{J} \boldsymbol{\omega} + \frac{1}{2} \boldsymbol{\varepsilon} \mathbb{G} \boldsymbol{\varepsilon} + \frac{1}{2} \boldsymbol{\kappa} \mathbb{H} \boldsymbol{\kappa} \, dS$$

Application for (time) derivation of a quadratic form

$$f = \frac{1}{2} \mathbf{u} \mathbb{X} \mathbf{u}, \quad \mathbb{X}^T = \mathbb{X}$$

Derivation of a vector

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial u_i}{\partial t} \mathbf{d}_i + \boldsymbol{\omega} \wedge \mathbf{u}$$

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{\partial u_i}{\partial t} \mathbb{X}_{ij} u_j \\ &= \left(\frac{\partial \mathbf{u}}{\partial t} - \boldsymbol{\omega} \wedge \mathbf{u} \right) \cdot (\mathbb{X} \mathbf{u}) \end{aligned}$$

$$\frac{\partial f}{\partial t} = \frac{\partial \mathbf{u}}{\partial t} \mathbb{X} \mathbf{u} - (\mathbf{u} \wedge (\mathbb{X} \mathbf{u})) \cdot \boldsymbol{\omega}$$

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \mathbf{u} \mathbb{X} \mathbf{u} \right) \neq \frac{\partial \mathbf{u}}{\partial t} \mathbb{X} \mathbf{u}$$

$\frac{\partial \mathbf{u}}{\partial t} - \boldsymbol{\omega} \wedge \mathbf{u}$ is a corotational derivative

Equilibrium

$$\frac{\partial N}{\partial S} = \frac{\partial \Delta v}{\partial t} \quad (1)$$

$$\frac{\partial M}{\partial S} + \frac{\partial \varphi}{\partial S} \wedge N = \frac{\partial \mathbb{J} \omega}{\partial t} \quad (2)$$

$$N := G \epsilon$$

$$M := H \kappa$$

$$v := \frac{\partial \varphi}{\partial t} \quad \omega := \text{axial}(Q^{-1} \frac{\partial Q}{\partial t}) \quad \kappa := \text{axial}(Q^{-1} \frac{\partial Q}{\partial S}) \quad \epsilon := \frac{\partial \varphi}{\partial S} - d_3$$

Equilibrium

$$\frac{\partial \mathbb{G}\varepsilon}{\partial S} = \frac{\partial \mathbb{A}v}{\partial t} \quad (1)$$

$$\frac{\partial \mathbb{H}\kappa}{\partial S} + (\varepsilon + d_3) \wedge \mathbb{G}\varepsilon = \frac{\partial \mathbb{J}\omega}{\partial t} \quad (2)$$

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$$\mathbb{N} := \mathbb{G}\boldsymbol{\varepsilon}$$

$$\mathbb{M} := \mathbb{H}\boldsymbol{\kappa}$$

$$\mathbf{v} := \frac{\partial \varphi}{\partial t} \quad \boldsymbol{\omega} := \text{axial}(\mathbb{Q}^{-1} \frac{\partial \mathbb{Q}}{\partial t}) \quad \boldsymbol{\kappa} := \text{axial}(\mathbb{Q}^{-1} \frac{\partial \mathbb{Q}}{\partial S}) \quad \boldsymbol{\varepsilon} := \frac{\partial \varphi}{\partial S} - \mathbf{d}_3$$

- $\varphi(S, t)$ and $\mathbb{Q}(S, t)$ are not used as explicit unknowns
- Four vectors are the unknowns \mathbf{v} , $\boldsymbol{\omega}$, $\boldsymbol{\kappa}$, $\boldsymbol{\varepsilon}$.
- Ill-posed problem with 12 unknown components and 6 scalar equations
- first order, non-linear, partial differential equations.

Equilibrium

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- Four vectors are the unknowns $v, \omega, \kappa, \varepsilon$.
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Additional *closure* relations

$$\frac{\partial \kappa}{\partial t} = \frac{\partial \omega}{\partial S} + \omega \wedge \kappa \quad \frac{\partial \varepsilon}{\partial t} = \frac{\partial v}{\partial S} - \omega \wedge d_3$$

obtained from

$$\frac{\partial}{\partial t} \frac{\partial d_i}{\partial S} = \frac{\partial}{\partial S} \frac{\partial d_i}{\partial t} \quad \frac{\partial}{\partial t} \frac{\partial \varphi}{\partial S} = \frac{\partial}{\partial S} \frac{\partial \varphi}{\partial t}$$

Two systems

First system: unknowns $v, \omega, \kappa, \varepsilon$

$$\frac{\partial \mathbb{G}\varepsilon}{\partial S} = \frac{\partial \mathbb{A}v}{\partial t} \quad (3)$$

$$\frac{\partial \mathbb{H}\kappa}{\partial S} + (\varepsilon + d_3) \wedge (\mathbb{G}\varepsilon) = \frac{\partial \mathbb{J}\omega}{\partial t} \quad (4)$$

$$\frac{\partial \omega}{\partial S} + \omega \wedge \kappa = \frac{\partial \kappa}{\partial t} \quad (5)$$

$$\frac{\partial v}{\partial S} - \omega \wedge d_3 = \frac{\partial \varepsilon}{\partial t} \quad (6)$$

Second system: unknowns Q, d_i and φ

$$\omega := \text{axial}(Q^{-1} \frac{\partial Q}{\partial t}) \quad \kappa := \text{axial}(Q^{-1} \frac{\partial Q}{\partial S}) \quad v := \frac{\partial \varphi}{\partial t} \quad \varepsilon := \frac{\partial \varphi}{\partial S} - d_3$$

- One second order PDE \Rightarrow Two first order PDE

!! Initial and boundary conditions

Numerical point of view

$$\frac{\partial \mathbb{G}\boldsymbol{\varepsilon}}{\partial S} = \frac{\partial \mathbb{A}\mathbf{v}}{\partial t} \quad (7)$$

$$\frac{\partial \mathbb{H}\boldsymbol{\kappa}}{\partial S} + (\boldsymbol{\varepsilon} + \mathbf{d}_3) \wedge (\mathbb{G}\boldsymbol{\varepsilon}) = \frac{\partial \mathbb{J}\boldsymbol{\omega}}{\partial t} \quad (8)$$

$$\frac{\partial \boldsymbol{\omega}}{\partial S} + \boldsymbol{\omega} \wedge \boldsymbol{\kappa} = \frac{\partial \boldsymbol{\kappa}}{\partial t} \quad (9)$$

$$\frac{\partial \mathbf{v}}{\partial S} - \boldsymbol{\omega} \wedge \mathbf{d}_3 = \frac{\partial \boldsymbol{\varepsilon}}{\partial t} \quad (10)$$

$$\begin{bmatrix} 0 & 0 & \mathbb{G} & 0 \\ 0 & 0 & 0 & \mathbb{H} \\ \mathbb{G} & 0 & 0 & 0 \\ 0 & \mathbb{H} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}' \\ \boldsymbol{\omega}' \\ \boldsymbol{\varepsilon}' \\ \boldsymbol{\kappa}' \end{bmatrix} + \begin{bmatrix} -\mathbb{W}\mathbb{A} & 0 & \mathbb{K}\mathbb{G} & 0 \\ 0 & -\mathbb{W}\mathbb{J} & \mathbb{E}\mathbb{G} & \mathbb{K}\mathbb{H} \\ \mathbb{G}\mathbb{K} & \mathbb{G}\mathbb{E} & -\mathbb{G}\mathbb{W} & 0 \\ 0 & \mathbb{H}\mathbb{K} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\omega} \\ \boldsymbol{\varepsilon} \\ \boldsymbol{\kappa} \end{bmatrix} = \begin{bmatrix} \mathbb{A} & 0 & 0 & 0 \\ 0 & \mathbb{J} & 0 & 0 \\ 0 & 0 & \mathbb{G} & 0 \\ 0 & 0 & 0 & \mathbb{H} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\boldsymbol{\omega}} \\ \dot{\boldsymbol{\varepsilon}} \\ \dot{\boldsymbol{\kappa}} \end{bmatrix}$$

with non-linearities induced by the skew-matrices:

$$\mathbb{W} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \quad \mathbb{K} = \begin{bmatrix} 0 & -\kappa_3 & \kappa_2 \\ \kappa_3 & 0 & -\kappa_1 \\ -\kappa_2 & \kappa_1 & 0 \end{bmatrix} \quad \mathbb{E} = \begin{bmatrix} 0 & -(\varepsilon_3 + 1) & \varepsilon_2 \\ \varepsilon_3 + 1 & 0 & -\varepsilon_1 \\ -\varepsilon_2 & \varepsilon_1 & 0 \end{bmatrix}$$

Three point of views

$$\frac{\partial \mathbb{G}\varepsilon}{\partial S} = \frac{\partial \mathbb{A}v}{\partial t}$$

Newton's laws

$$\frac{\partial \mathbb{H}\kappa}{\partial S} + (\varepsilon + d_3) \wedge (\mathbb{G}\varepsilon) = \frac{\partial \mathbb{J}\omega}{\partial t}$$

Rotational analogues of Newton's laws

$$\frac{\partial \omega}{\partial S} + \omega \wedge \kappa = \frac{\partial \kappa}{\partial t}$$

compatibility of rotations

$$\frac{\partial v}{\partial S} - \omega \wedge d_3 = \frac{\partial \varepsilon}{\partial t}$$

compatibility of translations

$$\begin{bmatrix} 0 & 0 & \mathbb{G} & 0 \\ 0 & 0 & 0 & \mathbb{H} \\ \mathbb{G} & 0 & 0 & 0 \\ 0 & \mathbb{H} & 0 & 0 \end{bmatrix} \begin{bmatrix} v' \\ \omega' \\ \varepsilon' \\ \kappa' \end{bmatrix} + \begin{bmatrix} -\mathbb{W}\mathbb{A} & 0 & \mathbb{K}\mathbb{G} & 0 \\ 0 & -\mathbb{W}\mathbb{J} & \mathbb{E}\mathbb{G} & \mathbb{K}\mathbb{H} \\ \mathbb{G}\mathbb{K} & \mathbb{G}\mathbb{E} & -\mathbb{G}\mathbb{W} & 0 \\ 0 & \mathbb{H}\mathbb{K} & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \\ \varepsilon \\ \kappa \end{bmatrix} = \begin{bmatrix} \mathbb{A} & 0 & 0 & 0 \\ 0 & \mathbb{J} & 0 & 0 \\ 0 & 0 & \mathbb{G} & 0 \\ 0 & 0 & 0 & \mathbb{H} \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \\ \dot{\varepsilon} \\ \dot{\kappa} \end{bmatrix}$$

$$\mathcal{D} u' + \mathcal{Y}(u) u = \mathcal{M} \dot{u}$$

\mathcal{D} Symetric
Constant

$\mathcal{Y}(u)$ Skew-symetric
Non-linear

\mathcal{M} Diagonal
Constant

Determining the placement

Position $\varphi(S, t)$

One has to solve

$$\text{either} \quad \frac{\partial \varphi}{\partial S} = \varepsilon + d_3 \quad \text{or} \quad \frac{\partial \varphi}{\partial t} = v.$$

As $\{v, \omega, \varepsilon, \kappa\}$ are supposed to be known, these relations write:

$$\dot{\varphi}' + \kappa \wedge \varphi = \varepsilon + d_3 \quad \text{or} \quad \dot{\varphi} + \omega \wedge \varphi = v.$$

The left-side case can be solved if a BC $\varphi(0, t)$ is given, and the right-side case can be solved if an IC $\varphi(S, 0)$ is prescribed.

Orientation $Q(S, t)$ or equivalently $\{d_i\}$

One has to solve

$$\text{either} \quad \frac{\partial Q}{\partial t} - QW = 0 \quad \text{or} \quad \frac{\partial Q}{\partial S} - QK = 0$$

The left-side case can be solved if a BC $\{d_i(0, t)\}$ is given, and the right-side case can be solved if an IC $\{d_i(S, 0)\}$ is prescribed.

Motivation

- *"The thread model does not exist"* P.Seppecher, Ecole d'été d'Oleron.

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- ... but some counterexample
 - Tension/compression of bar : linear, dynamical
$$EAu_3'' = \rho A\ddot{u}_3, \quad u_3 : \text{longitudinal displacement}$$
 - String : \sim linear, dynamical
$$Fu_1'' = \rho A\ddot{u}_1, \quad F = EAu_3', \quad u_1 : \text{transverse displacement}$$
 - Catenary : non-linear, static
 - Rope jumped : non-linear, quasi-static
- ... and interesting phenomena
 - Non-unicity of the solution
 - Non regularity of the solution
 - Zero-Euler's critical load

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- Non-unicity of the solution
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- Boundary condition plays a fundamental role

$$\|\varphi(L) - \varphi(0)\| \sim L$$

$$\|\varphi(L) - \varphi(0)\| < L$$

$$\|\varphi(L) - \varphi(0)\| > L$$

Hypotheses

Circular cross-section $\Leftrightarrow I_1 = I_2$

No shear or bending rigidity \Leftrightarrow Only longitudinal or torsional rigidity

$$GA \rightarrow 0, \quad EA \neq 0, \quad EI_1 \rightarrow 0, \quad EI_2 \rightarrow 0, \quad GI_3 \neq 0$$

$$\mathbb{G} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & EA \end{bmatrix}$$

$$\mathbb{H} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & GI_3 \end{bmatrix}$$

Application to standard structural mechanics

$$\frac{GA}{EA} \ll 1 \quad \frac{EI_1}{GI_3} \ll 1 \quad \text{but} \quad I_3 = I_1 + I_2 \quad \Rightarrow \quad \frac{1}{2} \ll \frac{G}{A} \ll 1$$

Thread model exists, if one consider that

- it's an effective model for a microstructured material
- it's not only a 1D-manifold

Rq a thread is composed of an arrangement of filaments !! **yes !!!**

Degeneracy

$$G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & EA \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Gl_3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & G & 0 \\ 0 & 0 & 0 & H \\ G & 0 & 0 & 0 \\ 0 & H & 0 & 0 \end{bmatrix} \begin{bmatrix} v' \\ \omega' \\ \varepsilon' \\ \kappa' \end{bmatrix} + \begin{bmatrix} -WA & 0 & KG & 0 \\ 0 & -WJ & EG & KH \\ GK & GE & -GW & 0 \\ 0 & HK & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \\ \varepsilon \\ \kappa \end{bmatrix} = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & J & 0 & 0 \\ 0 & 0 & G & 0 \\ 0 & 0 & 0 & H \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \\ \dot{\varepsilon} \\ \dot{\kappa} \end{bmatrix}$$

- Problem

- Lines 7,8,11 and 12 of this algebraic problem are $0 = 0$
- ⇒ No possibility to express some compatibility condition

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- First alternative

- Do not multiply compatibility condition by \mathbb{G} and \mathbb{H}
 \Rightarrow The structure of the problem is broken :
 \mathcal{D} is no more symmetric, $\mathcal{Y}(u)$ is no more skew-symmetric,

Degeneracy

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- Second alternative

- Consider $El_1 = \epsilon$, $El_2 = \epsilon$, $GA = \epsilon$, with $\epsilon \rightarrow 0$
 \Rightarrow Critic for the numerical purpose

$$\lim_{\epsilon \rightarrow 0} u_\epsilon \stackrel{?}{=} u_0$$

Bi-dimensional analysis in the plane $(d_1, d_3) = (e_x, e_z)$, $d_2 = e_y$

$$u = \begin{bmatrix} v_1 \\ v_3 \\ \omega_2 \\ \varepsilon_1 \\ \varepsilon_3 \\ \kappa_2 \end{bmatrix} \quad \hat{D} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & EA & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{D} u' + \hat{Y}(u)u = \hat{M} \dot{u} \quad \hat{Y}(u) = \begin{bmatrix} 0 & -\rho A \omega_2 & 0 & 0 & 0 & EA \kappa_2 & 0 \\ \rho A \omega_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -EA \varepsilon_1 & 0 \\ 0 & \kappa_2 & -1 - \varepsilon_3 & 0 & 0 & 0 & 0 \\ -\kappa_2 & 0 & \varepsilon_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{M} = \begin{bmatrix} \rho A & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho A & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho I_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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Reorganisation of the plane problem

The problem is cast into two subproblem.

- The first preserve a standard transport equation structure

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & EA \\ 0 & 0 & 0 & 0 \\ 0 & EA & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1' \\ v_3' \\ \omega_2' \\ \varepsilon_3' \end{bmatrix} + \begin{bmatrix} 0 & -\rho A \omega_2 & 0 & EA \kappa_2 \\ \rho A \omega_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -EA \varepsilon_1 \\ -EA \kappa_2 & 0 & EA \varepsilon_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_3 \\ \omega_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} \rho A & 0 & 0 & 0 \\ 0 & \rho A & 0 & 0 \\ 0 & 0 & \rho I_3 & 0 \\ 0 & 0 & 0 & EA \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_3 \\ \dot{\omega}_2 \\ \dot{\varepsilon}_3 \end{bmatrix}$$

- The second is composed of the two last compatibility equation

$$\begin{aligned}
 v_1' + \kappa_2 v_3 - (1 + \varepsilon_3) \omega_2 &= \dot{\varepsilon}_1 \\
 \omega_2' &= \dot{\kappa}_2
 \end{aligned}$$

Rq These two problems are coupled

Euler-Bernoulli hypothesis

- No shear $\varepsilon_1 \rightarrow 0$
- The conservation of angular momentum is neglected (like for string)

The problem is cast into two subproblem.

- The first preserve a standard transport equation structure

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & EA \\ 0 & EA & 0 \end{bmatrix} \begin{bmatrix} v_1' \\ v_3' \\ \varepsilon_3' \end{bmatrix} + \begin{bmatrix} 0 & -\rho A \omega_2 & EA \kappa_2 \\ \rho A \omega_2 & 0 & 0 \\ -EA \kappa_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_3 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} \rho A & 0 & 0 \\ 0 & \rho A & 0 \\ 0 & 0 & EA \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_3 \\ \dot{\varepsilon}_3 \end{bmatrix}$$

- The second is composed of the two last compatibility equation

$$\begin{aligned} v_1' + \kappa_2 v_3 - (1 + \varepsilon_3) \omega_2 &= 0 \\ \omega_2' &= \kappa_2 \end{aligned}$$

⇒ The problem persists : additional kinematical hypotheses looks not to be a solution to preserve the structure of the problem.

Paradigm shift

Important remark

The loss of symmetry of matrices \mathcal{Y} and \mathcal{D} is related to

Compatibility conditions:

$$\begin{aligned} \frac{\partial \kappa}{\partial t} &= \frac{\partial \omega}{\partial S} + \omega \wedge \kappa & \frac{\partial \varepsilon}{\partial t} &= \frac{\partial v}{\partial S} - \omega \wedge d_3 \\ \text{obtained from } \frac{\partial}{\partial t} \frac{\partial Q}{\partial S} &= \frac{\partial}{\partial S} \frac{\partial Q}{\partial t} & \frac{\partial}{\partial t} \frac{\partial \varphi}{\partial S} &= \frac{\partial}{\partial S} \frac{\partial \varphi}{\partial t} \end{aligned}$$

If the variables are not (only) vectors $\{v, \omega, \varepsilon, \kappa\} \in TC$ but (also) kinematical quantities $\{\varphi, Q\} \in C$, the compatibility conditions are immediately satisfied!

Paradigm shift

Important remark

The loose of symetry of matrices \mathcal{Y} and \mathcal{D} is related to

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If the variable are not (only) vectors $\{v, \omega, \varepsilon, \kappa\} \in TC$ but (also) kinematical quantities $\{\varphi, Q\} \in C$, the compatibility condition are imediately satisfied!

Ben justement, c'est ce que l'on a fait avec Oscar !!!

Timoshenko beam under finite and dynamic transformations: Lagrangian coordinates and Hamiltonian structures, Communications in Analysis and Mechanics, 17(4), 2025

First formulation

 $\{p, \sigma, \varepsilon, \kappa\}$

$$H(p, \sigma, \varepsilon, \kappa) = \int_0^L \frac{1}{2} p \mathbb{A}^{-1} p + \frac{1}{2} \sigma \mathbb{J}^{-1} \sigma + \frac{1}{2} \varepsilon \mathbb{G} \varepsilon + \frac{1}{2} \kappa \mathbb{H} \kappa \, dS$$

Poisson bracket, for any f and g functions of the variables $\{p, \sigma, \varepsilon, \kappa\}$:

$$\begin{aligned} \{f, g\} = & \int_0^L \left\langle \frac{\partial f}{\partial p}, \frac{\partial}{\partial S} \left(\frac{\partial g}{\partial \varepsilon} \right) \right\rangle - \left\langle \frac{\partial g}{\partial p}, \frac{\partial}{\partial S} \left(\frac{\partial f}{\partial \varepsilon} \right) \right\rangle \\ & + \left\langle \frac{\partial f}{\partial \sigma}, \frac{\partial}{\partial S} \left(\frac{\partial g}{\partial \kappa} \right) \right\rangle - \left\langle \frac{\partial g}{\partial \sigma}, \frac{\partial}{\partial S} \left(\frac{\partial f}{\partial \kappa} \right) \right\rangle \\ & + \left\langle \kappa, \frac{\partial g}{\partial \varepsilon} \wedge \frac{\partial f}{\partial p} - \frac{\partial f}{\partial \varepsilon} \wedge \frac{\partial g}{\partial p} \right\rangle \\ & + \left\langle \sigma, \frac{\partial g}{\partial \sigma} \wedge \frac{\partial f}{\partial \sigma} \right\rangle + \left\langle p, \frac{\partial g}{\partial \sigma} \wedge \frac{\partial f}{\partial p} - \frac{\partial f}{\partial \sigma} \wedge \frac{\partial g}{\partial p} \right\rangle \\ & + \left\langle \varepsilon + d_3, \frac{\partial g}{\partial \varepsilon} \wedge \frac{\partial f}{\partial \sigma} - \frac{\partial f}{\partial \varepsilon} \wedge \frac{\partial g}{\partial \sigma} \right\rangle \\ & + \left\langle \kappa, \frac{\partial g}{\partial \kappa} \wedge \frac{\partial f}{\partial \sigma} - \frac{\partial f}{\partial \kappa} \wedge \frac{\partial g}{\partial \sigma} \right\rangle dS. \end{aligned}$$

Second formulation

 $\{\varphi, \mathbb{Q}, \mathbf{p}, \Sigma\}$

$$\begin{aligned}
 H(\varphi, \mathbb{Q}, \mathbf{p}, \Sigma) = & \frac{1}{2} \int_0^L \langle \mathbf{p}, \mathbb{A}^{-1} \mathbf{p} \rangle + \langle j(\mathbb{Q}^{-1} \Sigma), \mathbb{J}^{-1} j(\mathbb{Q}^{-1} \Sigma) \rangle \\
 & + \langle \frac{\partial \varphi}{\partial \mathcal{S}} - \mathbf{d}_3, \mathbb{G}(\frac{\partial \varphi}{\partial \mathcal{S}} - \mathbf{d}_3) \rangle \\
 & + \langle j(\mathbb{Q}^{-1} \frac{\partial \mathbb{Q}}{\partial \mathcal{S}}), \mathbb{H} j(\mathbb{Q}^{-1} \frac{\partial \mathbb{Q}}{\partial \mathcal{S}}) \rangle \, d\mathcal{S}.
 \end{aligned}$$

Poisson bracket, for any f and g functions of the variables $\{\varphi, \mathbb{Q}, \mathbf{p}, \Sigma\}$:

$$\begin{aligned}
 \{f, g\} = & \int_0^L \langle \frac{\partial f}{\partial \varphi}, \frac{\partial g}{\partial \mathbf{p}} \rangle - \langle \frac{\partial g}{\partial \varphi}, \frac{\partial f}{\partial \mathbf{p}} \rangle \\
 & + \ll \frac{\partial f}{\partial \mathbb{Q}}, \frac{\partial g}{\partial \Sigma} \gg - \ll \frac{\partial g}{\partial \mathbb{Q}}, \frac{\partial f}{\partial \Sigma} \gg \, d\mathcal{S}.
 \end{aligned}$$

where

$$\Sigma = \mathbb{Q} j^{-1} (\mathbb{J} j(\mathbb{Q}^{-1} \delta \mathbb{Q}))$$

Third formulation

 $\{\varphi, p, Q, \sigma\}$

$$\begin{aligned}
 H(\varphi, p, Q, \sigma) = & \frac{1}{2} \int_0^L \langle p, \mathbb{A}^{-1} p \rangle + \langle \sigma, \mathbb{J}^{-1} \sigma \rangle \\
 & + \langle \left(\frac{\partial \varphi}{\partial S} - d_3 \right), \mathbb{G} \left(\frac{\partial \varphi}{\partial S} - d_3 \right) \rangle \\
 & + \langle j(Q^{-1} \frac{\partial Q}{\partial S}), \mathbb{H} j(Q^{-1} \frac{\partial Q}{\partial S}) \rangle dS
 \end{aligned}$$

Poisson bracket, for any f and g functions of the variables $\{\varphi, p, Q, \sigma\}$:

$$\begin{aligned}
 \{f, g\} = & \int_0^L \langle \frac{\partial f}{\partial \varphi}, \frac{\partial g}{\partial p} \rangle - \langle \frac{\partial g}{\partial \varphi}, \frac{\partial f}{\partial p} \rangle \\
 & + \ll \frac{\partial f}{\partial Q}, \mathbb{Q} j^{-1} \left(\frac{\partial g}{\partial \sigma} \right) \gg - \ll \frac{\partial g}{\partial Q}, \mathbb{Q} j^{-1} \left(\frac{\partial f}{\partial \sigma} \right) \gg dS
 \end{aligned}$$

Conclusion

- **Le modèle de fil existe**, il y'en a même deux (avec ou sans cisaillement). Il suffit de le considérer comme une dégénérescence d'une variété matérielle unidimensionnelle particulière : un fibré des repères.
- Un fil est un **milieu microstructuré**
- La dégénérescence induit une **perte de structure** du problème...
- Cette étude n'est **qu'un début**.

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- Un fil est un **milieu microstructuré**
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La vie ne tient qu'à un fil !