

Distance-to-Instability and Distance-to-Singularity Metrics for Continuum Robots

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Instability and singularities in continuum robotics

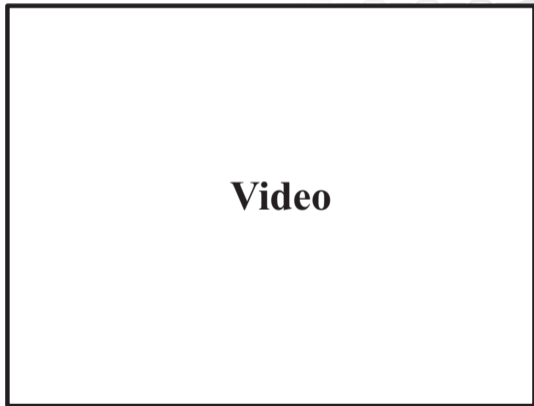
Instabilities

- Sudden dynamic reconfiguration
- Dangerous for the external environment

Singularities

- Loss of physical performances (accuracy, stiffness, payload, etc)
- Some singularities are limits of stability areas^a

^aS. Briot, and A. Goldsztejn, "Singularity Conditions for Continuum Parallel Robots," IEEE Transactions on Robotics, 2022, Vol. 38, No. 1, pp. 507-525.



Instability and singularities in continuum robotics

Checking for singularities and stability

- Criteria exist, to be sure that we are not in singularity or in instability
- Based on checking singular/eigen-values of some matrices or more advanced tools (optimum control theory)

Checking the proximity of singularities and stability

It is another story



Proximity of singularities and stability

Criteria exist (list non exhaustive)

- Generic criteria: Threshold on determinants, condition numbers, singular/eigen-values of some matrices, “distance” to conjugate point
- Dedicated criteria: for concentric tube robots only, slope of the S-curve

Issues with these criteria

- When based on checking matrix properties: sensitive to the change of units, to the discretization type, difficult to decide what is a “good threshold”
- Conjugate points: no straightforward proof that the robot reaches the limit of stability when the conjugate points appear at the abscissa zero
- Slopes of S-curves: possible to concentric tube robots only

Proximity of singularities and stability

A good criterion^a

- It should not add terms with different units,
- It should admit an analytical expression, in order to be able to compute its gradient for optimization purpose,
- It should be bounded in magnitude, at least in one direction of its interval of values, and this bound should be related to an extremum of performance
- It should have a physical meaning in order to ease the comparison between different systems. This does not mean that the index must have a unit, but it should be based on a clear physical interpretation.

^aPatel, S., and Sobh, T., 2015. "Manipulator performance measures-a comprehensive literature survey".

Journal of Intelligent & Robotic Systems, 77, pp. 547-570

Proximity of singularities and stability

Good criterion for characterizing distance to instability

First attempt made in: F. Zaccaria, E. IdÃ , S. Briot, "Directional Critical Load Index: a Distance-to-Instability Metric for Continuum Robots," IEEE Transactions on Robotics, 2024, Vol. 40, No. 1, pp. 3620-3637.54

- DCLI has a well defined unit (Newton)
- Lower-bounded by 0

But

- Directional metrics (depends on the direction of the force applied)
- Not applicable to any type of models

In this work

We try to overpass the previous limitations

Metrics inspired by DCLI but

- No direction limitation
- For any type of modelling approach (continuous or discrete)
- For distances to instability or singularity
- Not only “distances” in force



Generic form of the robot model

Geometrico-static model

- Based on the Cosserat formulation (n bodies)
- Extremum of the total potential energy V_{tot} :

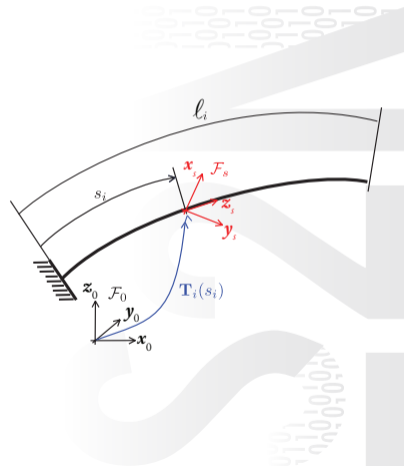
$$V_{tot} = V_{tot}(\mathbf{q}_a, \mathbf{q}_p, \xi_1(s_1), \dots, \xi_n(s_n), \mathbf{f}) \quad (1)$$

Where

- $\mathbf{q}_a, \mathbf{q}_p$: Motor and end-effector variables
- $\xi_i(s_i)$: space-rate of change at abscissa s_i
- \mathbf{f} : external force

Usually, subject to geometric constraints

$$\Phi(\mathbf{q}_a, \mathbf{q}_p, \xi_1(s_1), \dots, \xi_n(s_n)) = \mathbf{0} \in \mathbb{R}^{n_\Phi} \quad (2)$$



Generic form of the robot model

In all cases, for computing the robot configuration

We must find the roots of a residual \mathcal{R}

$$\mathcal{R}(\mathbf{z}, \boldsymbol{\pi}) = \mathbf{0} \quad (3)$$

Where

- in the continuous case: $\mathbf{z} = \mathbf{y} = [\mathbf{q}_a^T \mathbf{q}_p^T \mathbf{q}_u^T]^T$, with \mathbf{q}_u containing the wrenches on the ground, actuator efforts or initial strains,
- in the discrete case: $\mathbf{z} = [\mathbf{y}^T \boldsymbol{\lambda}^T]^T = [\mathbf{q}_a^T \mathbf{q}_p^T \mathbf{q}_u^T \boldsymbol{\lambda}^T]^T$, with \mathbf{q}_u containing variables parameterizing the beams deformations, and $\boldsymbol{\lambda}$ a vector of Lagrange multipliers

Moreover

- $\boldsymbol{\pi}$: $\boldsymbol{\pi} = [\mathbf{f}^T \mathbf{q}_a^{des T}]^T$ for the FGSM,
- $\boldsymbol{\pi} = [\mathbf{f}^T \mathbf{q}_c^{des T}]^T$ for the IGSM

Checking stability

Two cases

- Continuous case: The strong *Legendre-Clebsch condition* + the *Jacobi condition*
- Discrete case: Positive-definiteness of the reduced Hessian of the potential energy

$$\mathbf{H}^r = \mathbf{Z}^T \mathbf{H} \mathbf{Z} \in \mathbb{R}^{(n_p+n_u-n_\phi) \times (n_p+n_u-n_\phi)} \quad (4)$$

Where

$$\mathbf{H} = \left[\frac{\partial^2 \mathcal{L}}{\partial \mathbf{x}^2} \right]; \quad \mathbf{H} \in \mathbb{R}^{(n_p+n_u) \times (n_p+n_u)} \quad (5)$$

- $\mathcal{L} = V_{tot} + \boldsymbol{\lambda}^T \boldsymbol{\Phi}$, $\mathbf{x} = [\mathbf{q}_p^T \ \mathbf{q}_u^T]^T$
- $\mathbf{Z} \in \mathbb{R}^{(n_p+n_u) \times (n_p+n_u-n_\phi)}$ a matrix that spans the nullspace of the matrix $\mathbf{G} = D_{\mathbf{x}} \boldsymbol{\Phi} \in \mathbb{R}^{n_\phi \times (n_p+n_u)}$, i.e. $\mathbf{G} \mathbf{Z} = \mathbf{0}$.

Kinemato-static model

Whatever the type of model

$$\mathbf{A}\Delta\mathbf{q}_a + \mathbf{P}\Delta\mathbf{q}_p + \mathbf{U}\Delta\mathbf{q}_u = \mathbf{0} \quad (6)$$

Main robot singularities

- Singularities of the matrix $\mathbf{N} = [\mathbf{A} \ \mathbf{U}] \in \mathbb{R}^{(n_p+n_u) \times (n_a+n_u)}$, also called Type 1 singularities, correspond to end-effector configuration space boundaries,
- Singularities of the matrix $\mathbf{M} = [\mathbf{P} \ \mathbf{U}] \in \mathbb{R}^{(n_p+n_u) \times (n_p+n_u)}$, also called Type 2 singularities, correspond to active joint configuration space boundaries; it has been proven *in the discrete modeling case* that Type 2 singularities and instabilities issues, which are by essence to distinct types of properties, are indeed related to each other: Type 2 singularities are limits of the *CR* stable configurations domain^a.

^aS. Briot, and A. Goldsztejn, "Singularity Conditions for Continuum Parallel Robots.

Generic criterion for assessing stability

- $\mathbf{M} = [\mathbf{P} \ \mathbf{U}]$ obtained from the kinemato-static model have been shown to be the limits of the stable domain
- As a result, we will use the matrix \mathbf{M} to study the distance to the limits of the stable domain, whatever the type of modeling approach. Starting from a configuration known to be stable and exploring the robot workspace, a generic criterion for characterizing the limits of the stability domain could then be to study the singular values σ_k ($k = 1, \dots, n_p + n_u$) of \mathbf{M} to verify that they do not vanish in the neighborhood of the initial stable configuration:

$$\sigma_{\min} = \min_k \sigma_k > 0 \quad (7)$$

Critical load leading to instability

Problem statement

Let us denote by \mathbf{f}_0 the initial concentrated force applied on the robot in the nominal configuration \mathbf{z}_0 .

Goal: to find the critical force \mathbf{f}_{crit} (magnitude and direction) as close as possible to \mathbf{f}_0 that will cause robot instability, i.e. that leads to $\sigma_{\min} = 0$

Mathematically speaking

For fixed actuator configuration $\mathbf{q}_a^{\text{des}} = \mathbf{q}_{a0}$, given initial force \mathbf{f}_0 :

$$\begin{aligned} (\mathbf{f}_{\text{crit}}, \mathbf{z}_{\text{crit}}) &= \arg \min_{\mathbf{f}, \mathbf{z}} \frac{1}{2} \|\mathbf{f} - \mathbf{f}_0\|_2^2 \\ &\text{subject to } \mathcal{R}(\mathbf{z}, \boldsymbol{\pi}) = \mathbf{0}, \quad \sigma_{\min}(\mathbf{z}, \boldsymbol{\pi}) = 0. \end{aligned} \quad (8)$$

where $\boldsymbol{\pi} = [\mathbf{f}^T \mathbf{q}_{a0}^T]^T$, and $\mathcal{R} = \mathbf{0}$ then represents the FGSM.

Critical load leading to instability

Main issue

- Highly nonlinear problem
- Very dependent of the initial guess (see case studies)
- Time consuming

Proposition

⇒ Take an approximated solution!



Critical load: Approximated computation

First-order Taylor's expansion of σ_k around the initial concentrated force \mathbf{f}_0 in the nominal configuration \mathbf{z}_0 :

$$\tilde{\sigma}_k(\mathbf{f}_k) \simeq \sigma_k(\mathbf{z}_0, \boldsymbol{\pi}_0) + \tilde{\boldsymbol{\Sigma}}_{k,\mathbf{f}}(\mathbf{f}_k - \mathbf{f}_0) \quad (9)$$

where $\boldsymbol{\pi}_0 = [\mathbf{f}_0^T \mathbf{q}_{a0}^T]^T$ and $\tilde{\boldsymbol{\Sigma}}_{k,\mathbf{f}} = [D_{t,f_1} \sigma_k \dots D_{t,f_{n_f}} \sigma_k](\mathbf{z}_0, \boldsymbol{\pi}_0) \in \mathbb{R}^{n_f \times 1}$ and

$$D_{t,f_i} \sigma_k = \mathbf{u}_k^T(\mathbf{z}_0, \boldsymbol{\pi}_0) (D_{t,f_i} \mathbf{M}(\mathbf{z}_0, \boldsymbol{\pi}_0)) \mathbf{v}_k(\mathbf{z}_0, \boldsymbol{\pi}_0) \quad (10)$$

Force \mathbf{f}_k^* for which instability occurs

$$c_{fk} = 0 = \sigma_k(\mathbf{z}_0, \boldsymbol{\pi}_0) + \tilde{\boldsymbol{\Sigma}}_{k,\mathbf{f}}(\mathbf{f}_k - \mathbf{f}_0) \quad (11)$$

Critical load: Approximated computation

New problem to solve

$$\mathbf{f}_k^* = \arg \min_{\mathbf{f}_k} \frac{1}{2} \|\mathbf{f}_k - \mathbf{f}_0\|_2^2 \quad (12)$$

subject to $c_{fk} = 0$.

Analytical solution

$$\begin{bmatrix} \mathbf{f}_k^* \\ \mu_k \end{bmatrix} = \mathbf{C}_{k,f}^{-1} \begin{bmatrix} \mathbf{f}_0 \\ \tilde{\Sigma}_{k,f} \mathbf{f}_0 - \sigma_k \end{bmatrix} \quad \text{with } \mathbf{C}_{k,f} = \begin{bmatrix} \mathbf{1}_{n_f} & \tilde{\Sigma}_{k,f}^T \\ \tilde{\Sigma}_{k,f} & 0 \end{bmatrix} \quad (13)$$

Critical load: Approximated computation

Finally

The *critical load* \mathbf{f}_{crit} is defined as the force among all \mathbf{f}_k^* ($k = 1, \dots, n_p + n_u$) for which the vector $(\mathbf{f}_k^* - \mathbf{f}_0)$ has the smallest magnitude, i.e.

$$\mathbf{f}_{\text{crit}} = \arg \min_{\mathbf{f}_k^*, \forall k \in \mathcal{I}} \{ \|\mathbf{f}_1^* - \mathbf{f}_0\|, \dots, \|\mathbf{f}_{n_p+n_u}^* - \mathbf{f}_0\| \} \quad (14)$$

where $\mathcal{I} = \{1, \dots, n_p + n_u\}$. The critical load index (*CLI*) can then be defined as:

$$CLI = \mathbf{f}_{\text{crit}} - \mathbf{f}_0. \quad (15)$$

Physically speaking, *CLI* has a well-defined unit (Newton)

A *CLI* of [100] N, for instance, would mean that if we push on the robot with a force of 1 N along \mathbf{x} , then the robot risks to fall into instability.

The Critical End-effector Configuration (*CEC*)

Problem statement

Let us denote by \mathbf{q}_{c0} the vector of initial controlled variables (usually a subset of \mathbf{q}_p) when the robot is in its nominal configuration \mathbf{z}_0 .

Goal: to find the new desired end-effector pose $\mathbf{q}_c = \mathbf{q}_c^{des}$ in the parameter vector $\boldsymbol{\pi} = [\mathbf{f}_0^T \mathbf{q}_c^T]^T$ of the IGSM which is a solution to the following optimization problem:

$$\begin{aligned} (\mathbf{q}_{c,cr}, \mathbf{z}_{crit}) &= \arg \min_{\mathbf{q}_c, \mathbf{z}} \frac{1}{2} \|\mathbf{e}_c\|_2^2 & (16) \\ \text{subject to } \mathcal{R}(\mathbf{z}, \boldsymbol{\pi}) &= \mathbf{0}, \\ \sigma_{\min}(\mathbf{z}, \boldsymbol{\pi}) &= 0. \end{aligned}$$

where $\mathcal{R} = \mathbf{0}$ represents here the robot IGSM and

$$\mathbf{e}_c = \mathbf{L}_c (\mathbf{q}_c - \mathbf{q}_{c0}) \quad (17)$$

The Critical Actuator Configuration (CAC)

Problem statement

Let us denote by \mathbf{q}_{a0} the vector of initial motor variables when the robot is in its nominal configuration \mathbf{z}_0 .

Goal: to find the new desired actuator configuration $\mathbf{q}_a = \mathbf{q}_a^{des}$ in the parameter vector $\boldsymbol{\pi} = [\mathbf{f}_0^T \mathbf{q}_a^T]^T$ of the IGSM which is a solution to the following optimization problem:

$$\begin{aligned} (\mathbf{q}_{a,cr}, \mathbf{z}_{crit}) &= \arg \min_{\mathbf{q}_a, \mathbf{z}} \frac{1}{2} \|\mathbf{e}_a\|_2^2 & (18) \\ \text{subject to } \mathcal{R}(\mathbf{z}, \boldsymbol{\pi}) &= \mathbf{0}, \\ \sigma_{\min}(\mathbf{z}, \boldsymbol{\pi}) &= 0. \end{aligned}$$

where $\mathcal{R} = \mathbf{0}$ represents here the robot FGSM and

$$\mathbf{e}_a = \mathbf{L}_a (\mathbf{q}_a - \mathbf{q}_{a0}) \quad (19)$$

For all these indices

Main issue

- Highly nonlinear problem
- Very dependent of the initial guess (see case studies)
- Time consuming

Proposition

⇒ Take an approximated solution, as previously (not detailed)

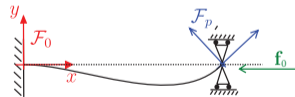
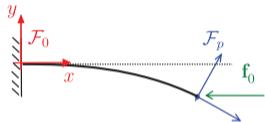
Other indices

Taking into account other matrices (e.g. \mathbf{N} for Type 1 singularities, or any other)

Buckling of beams

Possibility to have analytical expressions

- clamped-free: $f_{EUL} = \frac{EI\pi^2}{(2L)^2}$
- clamped-pinned: $f_{EUL} = \frac{EI\pi^2}{(0.699L)^2}$



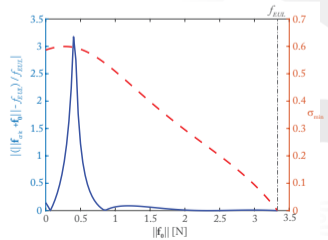
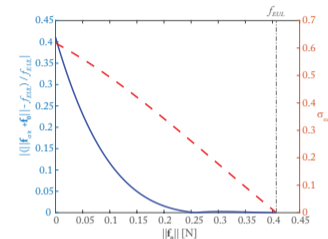
Buckling of beams

Possibility to have analytical expressions

- clamped-free: $f_{EUL} = \frac{EI\pi^2}{(2L)^2}$
- clamped-pinned: $f_{EUL} = \frac{EI\pi^2}{(0.699L)^2}$

Results

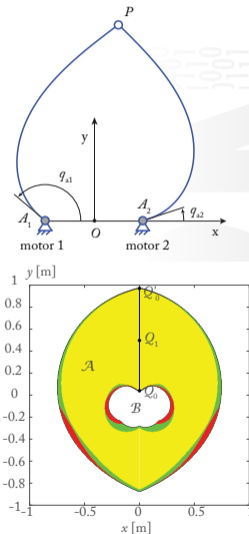
- As we approach the exact critical force f_{EUL} , the value of σ_{\min} approaches 0 (and is exactly 0 at $\|\mathbf{f}_{\text{crit}} + \mathbf{f}_0\| = f_{EUL}$),
- The approximated calculation of the critical force is exact at f_{EUL} , and with less than 5 % error for $\|\mathbf{f}_0\| > 0.15$ N (clamped-free) or for $\|\mathbf{f}_0\| > 1.6$ N (clamped-pinned).



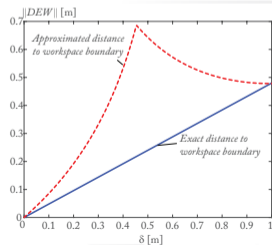
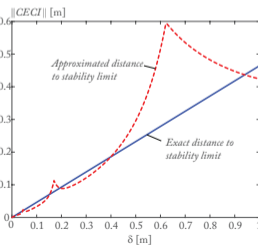
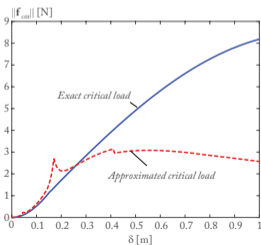
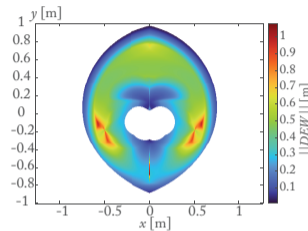
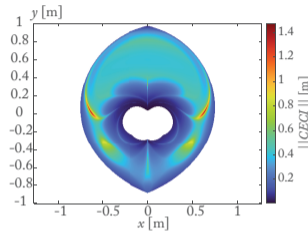
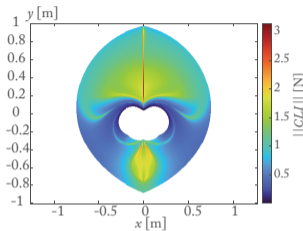
Planar CPR with two controlled DoF

Workspace

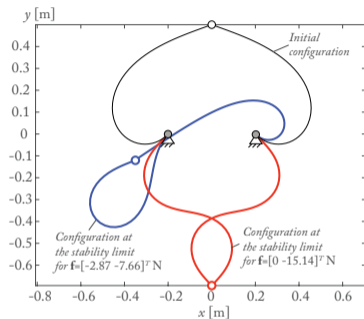
- zone \mathcal{A} : stable domain
- zone \mathcal{B} : unstable domain
- zone in green: close to Type 1 singularities
- zone in red: close to Type 2 singularities



Planar CPR with two controlled DoF



Planar CPR with two controlled DoF



Configurations of the robot for three identical values of the motor angles: when the end-effector is at Q_1 (initial configuration in black), when deformed by critical load exactly computed from problem (8) with the initial (black) configuration as initial guess (red solution), and when deformed by critical load exactly computed from problem (8) with an initial guess obtained by continuation when moving on the path $Q_0 - Q_1$ (blue solution).

A spatial two tubes CTR with four controlled DoF

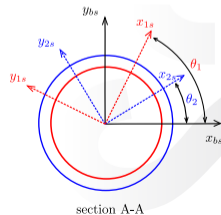
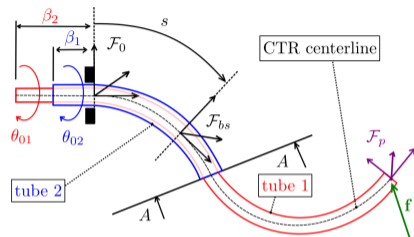
Possibility to have analytical expressions for stability check

The *CTR* equilibrium is globally stable if the robot verifies:

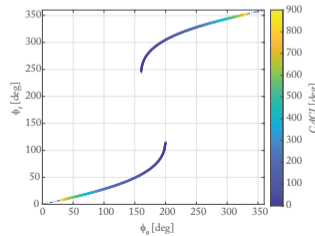
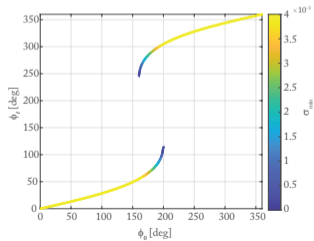
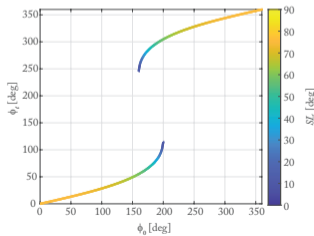
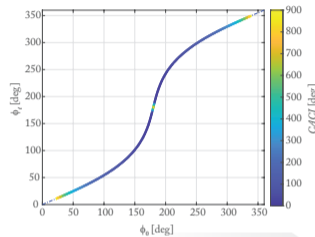
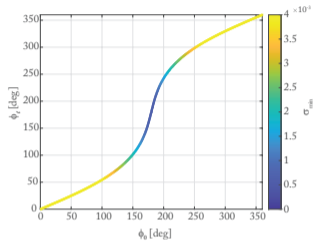
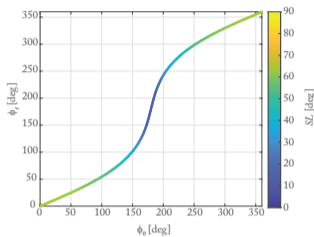
$$\zeta_\gamma = \cot(\gamma)/\sqrt{\gamma} < \zeta_{lim} \quad (20)$$

$$\text{where } \gamma = \ell_2^2 u_{1x}^* u_{2x}^* \frac{k_{1b} k_{2b} (k_{1t} + k_{2t})}{k_{1t} k_{2t} (k_{1b} + k_{2b})}$$

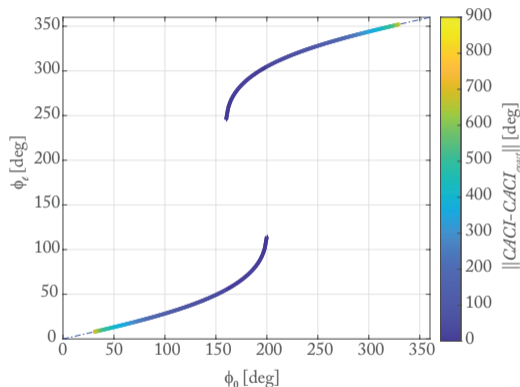
where the terms k_{bi} and k_{ti} are the bending and torsional stiffness of the i th tube, respectively, while the term ζ_{lim} is equal to zero in the case where $\beta_1 = \beta_2 = 0$ m.



A spatial two tubes CTR with four controlled DoF

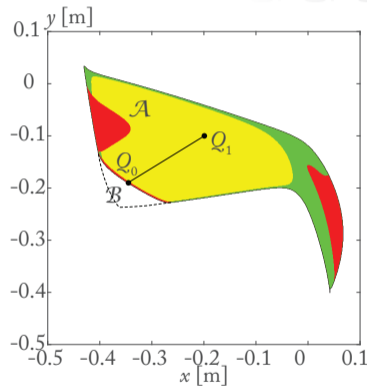
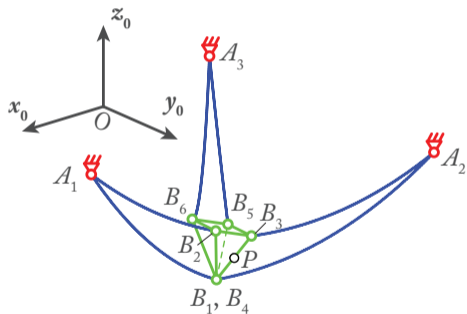


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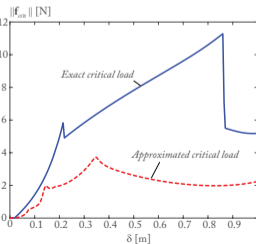
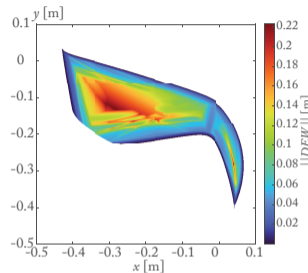
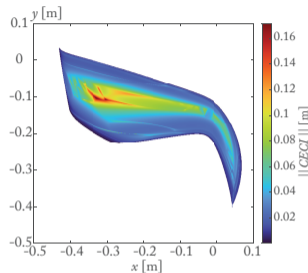
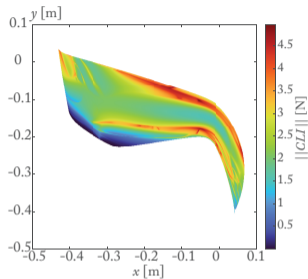


Absolute error of prediction of the $CACI$ w.r.t. the exact computation of the distance of θ_{10} ($CACI_{exact}$) to instability.

A spatial sagging cable-driven parallel robot with six controlled DoF



A spatial sagging cable-driven parallel robot with six controlled DoF



Conclusions

Conclusions

We proposed new metrics for characterizing the distance to instability or to singularities

- No direction limitation
- For any type of modelling approach (continuous or discrete)
- Not only “distances” in force
- Based on linear approximations: Less accurate, but unique and more computationally effective

Conclusions

Future works

- Characterizing the “neighborhood of trust”
- Sensitivity of the indices to model uncertainties and/or approximations
- Taking into account dissipative effects (friction, 3D moments)
- Applications to design and control