Approche variationnelle de la dynamique de Dirac

Vladimir Salnikov

GDR Groupement de recherche
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Géométrie Différentielle et Mécanique

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LaSIE Université La Rochelle

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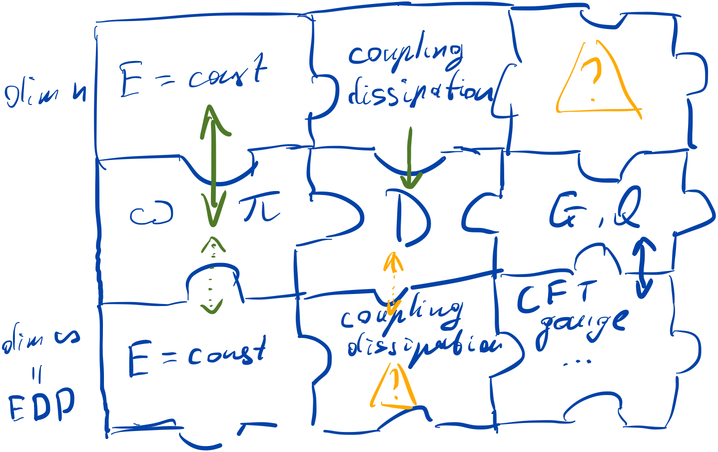
In the previous episodes...

Instead of conclusion – big puzzle and questions

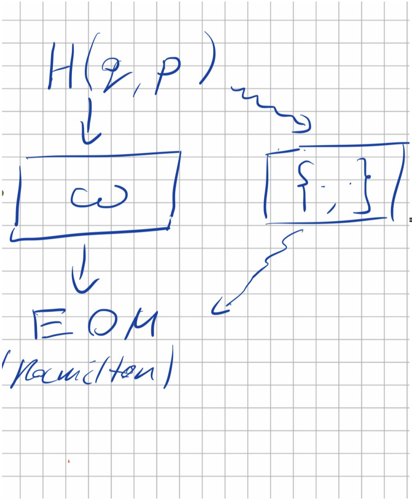


And what precisely about mechanics? What phenomena?

La Rochelle, France, 08/07/2021, 11:40

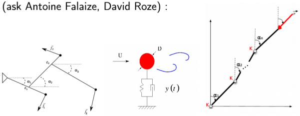
skip recap

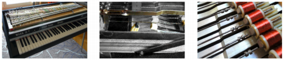
What is the conceptual difference?



Beyond: port-Hamiltonian systems; constraints

A lot of examples
(ask Antoine Falaize, David Roze) :



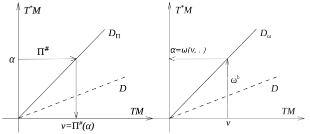


Conjecture (VS): Everything is port-Hamiltonian.

Geometry behind: Courant algebroids, Dirac structures

On $E = TM \oplus T^*M$ (or more generally $F \oplus F^*$)
Symmetric pairing: $< v \oplus \eta, v' \oplus \eta' >= \eta(v') + \eta'(v)$,
Dorfman bracket: $[v \oplus \eta, v' \oplus \eta']_D = [v, v']_{\text{Lie}} \oplus (\mathcal{L}_v \eta' - \mathrm{d}\eta(v'))$.

A *Dirac structure* $\mathcal{D}$ is a maximally isotropic (Lagrangian) subbundle of E closed w.r.t. $[\cdot,\cdot]_D$



$\mathcal{D}_\Pi = graph(\Pi^\sharp) = \{(\Pi^\sharp\alpha, \alpha)\}$ $\quad \mathcal{D}_\omega = graph(\omega^\flat) = \{(v, \iota_v\omega)\}$

Same place, 20 minutes earlier

skip recap

Season 2, Episode 1: What is Dirac dynamics? Main ingredients.

Geometry behind: Courant algebroids, Dirac structures

On $\mathbb{T}M = TM \oplus T^*M$ (or more generally $E \oplus E^*$)
Symmetric pairing: $< v \oplus \eta, v' \oplus \eta' >= \eta(v') + \eta'(v)$,
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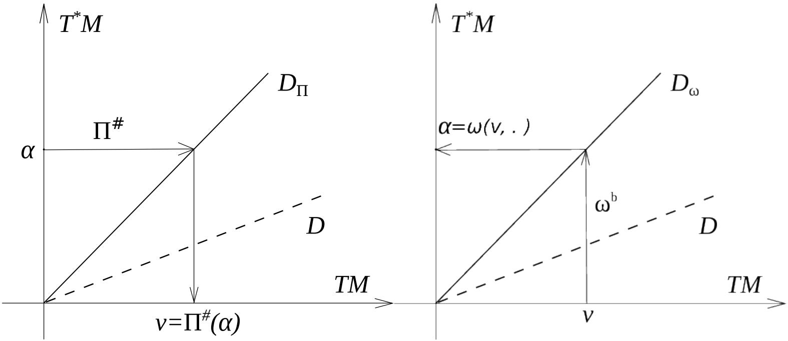
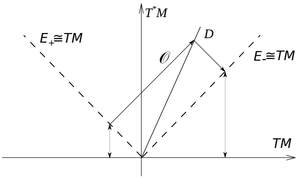


$\mathcal{D}_\Pi = graph(\Pi^\sharp) = \{(\Pi^\sharp \alpha, \alpha)\}$ $\quad$ $\mathcal{D}_\omega = graph(\omega^\flat) = \{(v, \iota_v \omega)\}$

Dirac structures: general

Choose a metric on $M \Rightarrow TM \oplus T^*M \cong TM \oplus TM$,
Introduce the eigenvalue subbundles $E_\pm = \{v \oplus \pm v\}$
of the involution $(v, \alpha) \mapsto (\alpha, v)$. Clearly, $E_+ \cong E_- \cong TM$.



(Almost) Dirac structure – graph of an orthogonal operator $\mathcal{O} \in \Gamma(\mathrm{End}(TM))$:
$(v, \alpha) = ((\mathrm{id} - \mathcal{O})w, g((\mathrm{id} + \mathcal{O})w, \cdot))$
Dirac structure = almost Dirac + (Jacobi-type) integrability condition:

$$g\left(\mathcal{O}^{-1}\nabla_{(\mathrm{id}-\mathcal{O})\xi_1}(\mathcal{O})\xi_2, \xi_3\right) + cycl(1,2,3) = 0$$

Remark. If the operator $(\mathrm{id} + \mathcal{O})$ is invertible, one recovers D_Π with $\Pi = \frac{\mathrm{id} - \mathcal{O}}{\mathrm{id} + \mathcal{O}}$ (Cayley transform), integrability $\Leftrightarrow [\Pi, \Pi]_{SN} = 0$.

Remark. Lie algebroid structure:
$\rho = (\mathrm{id} - \mathcal{O})$, $C_{ij}^k = (\mathrm{id} - \mathcal{O})_i^m \Gamma_{mj}^k - (i \leftrightarrow j) + \mathcal{O}_j^{m;k}\mathcal{O}_{mi}$

Remark. The same, using degree 1 DG-manifolds (Q-structures)

Reminder about Lie algebroids

Definition

Let M be a smooth manifold. A Lie algebroid $(A, \rho, [\cdot, \cdot])$ is given by a finite-dimensional vector bundle A, a vector bundle morphism $\rho : A \to TM$, called anchor and a ($\mathbb{R}$-bilinear) Lie bracket on the sections of A

$$[\cdot, \cdot] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$$

satisfying for all $f \in C^\infty(M)$, $s, s' \in \Gamma(A)$:

$$[s, fs'] = f[s, s'] + \rho(s)(f) \cdot s'.$$

(Recall R. Louis, O. Cossezat)

Fun facts about Lie algebroids

• Lie algebroids can be alternatively defined as differential graded manifolds of degree 1. In particular, there is a degree 1 (Lichnerowicz) differential $d_A : \Gamma(\Lambda^\bullet A^*) \to \Gamma(\Lambda^\bullet A^*)$:

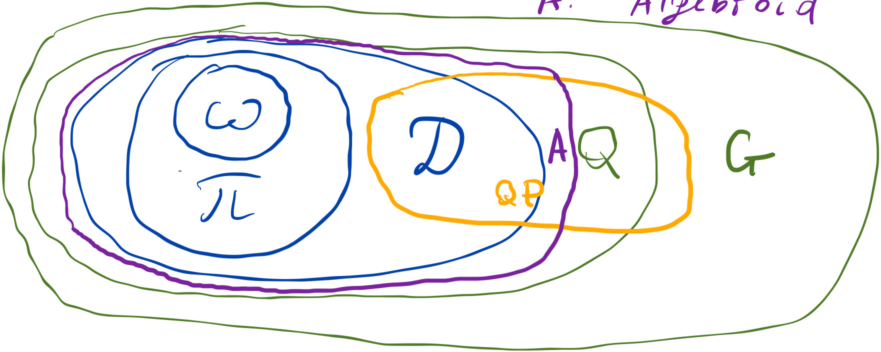
$$(d_A\eta)(\xi_1, ..., \xi_{n+1}) = \sum_i (-1)^{i+1}\rho(\xi_i)(\eta(\xi_1, ..., \hat{\xi}_i, .., \xi_{n+1})) + \sum_{i<j}(-1)^{i+j}\eta([\xi_i, \xi_j], \xi_1, ..., \hat{\xi}_i, ..., \hat{\xi}_j, .., \xi_{n+1})$$

It satisfies $d_A^2 = 0$ and induces the so-called Lie algebroid cohomology and $H^\bullet(A)$.
The anchor ρ induces a morphism $H^\bullet_{dR}(M) \to H^\bullet(A)$.
• A Lie algebroid always induces a singular foliation on M: $\rho(A) \subset TM$ is involutive, hence $M = \bigsqcup_\alpha N_\alpha$ such that $TN_\alpha = \rho(A)|_{N_\alpha}$ for all N_α (immersed connected submanifolds). Moreover, the bracket on A restricts to well-defined brackets on $A|_{N_\alpha}$, turning $A|_{N_\alpha} \to N_\alpha$ into Lie algebroids.

For graded good

G := Graded
Q := Differential Graded
QP := DG symplectic
A := Algebroid

– "Folkloric" applications in geometry



Flashback, La Rochelle 2021

For graded good

– "Folkloric" applications in geometry

G := Graded
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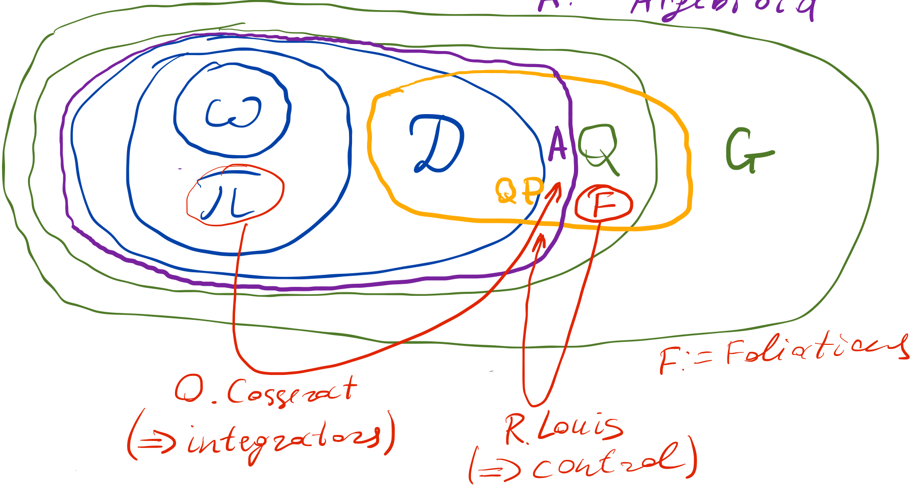


Season 2, Episode 2: Variational approach

O. Cosserat, C. Laurent-Gengoux, A. Kotov, L. Ryvkin, V. Salnikov,
On Dirac structures admitting a variational approach,
Preprint: arXiv:2109.00313

Horizontal cohomology of Lie algebroids

Definition Let $A \stackrel{\rho}{\to} TM$ be a Lie algebroid over the smooth manifold M. We define:

- The subspace of ρ-horizontal forms at $m \in M$ as:

$$(\Lambda^\bullet A_m^*)^{hor} := \{\alpha \in \Lambda^\bullet A_m^* \mid \iota_v \alpha = 0 \ \forall v \in \ker(\rho_m : A_m \to T_m M)\}$$

- The subspaces of ρ-horizontal forms:

$$\Gamma(\Lambda^\bullet A^*)^{hor} = \{\alpha \in \Gamma(\Lambda^\bullet A^*) \mid \alpha_m \text{ and } (d_A \alpha)_m \text{ are horizontal for all } m\}$$

- The horizontal cohomology of A as the quotient

$$\mathcal{H}^\bullet_{hor}(A) = \frac{\ker(d_A : \Gamma(\Lambda^\bullet A^*)^{hor} \to \Gamma(\Lambda^\bullet A^*)^{hor})}{\mathrm{Image}(d_A : \Gamma(\Lambda^\bullet A^*)^{hor} \to \Gamma(\Lambda^\bullet A^*)^{hor})}$$

Horizontal forms – fun facts

Lemma

Let A be an algebroid, $N \subset M$ a leaf of A and $\eta \in \Gamma((\Lambda^k A^*)^{hor})$ a ρ-horizontal form.

- $\eta|_N$ is a horizontal k-form on the restricted Lie algebroid $A|_N \to N$, i.e. it induces a unique k-form $\eta_N \in \Omega^k(N)$.
- η is completely determined by the collection $\{\eta_N \mid N \text{ leaf of } A\}$.
- When η is horizontal, we have $(d_A\eta)_N = d\eta_N$.
- Let $[\eta] = 0 \in \mathcal{H}^k_{hor}(A)$, then $[\eta_N] = 0 \in H^k_{dR}(N)$ for all leaves N of the algebroid A.

The natural horizontal two-cocycle of a Dirac structure

Let $D \subset \mathbb{T}M$ be a Dirac structure. We define $\omega_D \in \Gamma(\Lambda^2 D^*)$ by

$$\omega_D((v,\alpha),(w,\beta)) = \alpha(w) - \beta(v).$$

Since D is isotropic, $\omega_D((v,\alpha),(w,\beta)) = 2\alpha(w) = -2\beta(v)$, i.e. ω_D is horizontal.

Since D is involutive ω_D is closed in Dirac cohomology, i.e. $d_D\omega_D = 0$, and hence ω_D is horizontal. It thus yields a natural class in $\mathcal{H}^2_{hor}(D)$. Hence, in view of Lemma on horizontal forms:

Lemma Let $D \subset \mathbb{T}M$ be a Dirac structure.

- There is a naturally induced horizontals cocycle $\omega_D \in \Gamma(\Lambda^2 D^*)^{hor}$ associated to any Dirac structure D.
- If $[\omega_D] = 0 \in \mathcal{H}^2_{hor}(D)$, then for any leaf N of D, $[(\omega_D)_N] = 0 \in H^2_{dR}(N)$.

Dirac paths

Theorem 1. Let $D \subset \mathbb{T}M$ be a Dirac structure over M, $H \in C^\infty(M)$ be a Hamiltonian function and γ a path on M.

Assume that the basic 2-class $[\omega_D]$ vanishes, and let $\theta \in \Gamma(D^*)^{hor}$ be such that $d_D\theta = \omega_D$, then the following statements are equivalent:

(i) The path γ is a Hamiltonian curve, i.e. $(\dot\gamma(t), dH_{\gamma(t)}) \in D$ for all t.

(ii) All Dirac paths $\zeta : I \to D$ over γ (i.e. $\rho(\zeta) = \dot\gamma$) are critical points among the Dirac paths with the same end points of the following functional:

$$\zeta \mapsto \int_I \big(\theta_{\gamma(t)}(\zeta(t)) + H(\gamma(t))\big)\, dt \tag{1}$$

Inspiration from Tulczyjew's business

Definition. Let $L\colon TQ \to \mathbb{R}$ a (possibly degenerate) Lagrangian.

a) Tulczyjew's differential – map $u \mapsto \mathcal{D}_u L := \kappa(d_u L)$, where $\kappa : T^*TQ \to T^*T^*Q$ is the Tulczyjew isomorphism. Its image is a submanifold of T^*T^*Q.

b) Legendre – map from TQ to T^*Q: $\mathbb{F}L(v)$ for every $v \in T_qQ$:

$$\frac{\partial}{\partial t}\Big|_{t=0} L(v+tw) = \langle \mathbb{F}L(v), w\rangle$$

c) We denote by $\mathfrak{Leg} = \mathbb{F}L(TQ) \subset T^*Q$ the image of $\mathbb{F}L$.

d) We call partial vector fields on $\mathfrak{Leg}$ sections[1] of $\Gamma(TT^*Q)|_{\mathfrak{Leg}}$.

c) An integral curve of a partial vector field X on $\mathfrak{Leg}$ is a path $t \mapsto u_t \in TQ$ such that $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{F}L(u_t) = X_{\mathbb{F}L(u_t)}$.

d) An implicit Lagrangian system for an almost Dirac structure $\mathbb{D} \subset \mathbb{T}T^*Q$ is a pair (X, L), with X a partially defined vector field on $\mathfrak{Leg}$, such that $(X(\mathbb{F}L(u)), \mathcal{D}_u L) \in \mathbb{D}$ for all u in TQ.

[1] For E a vector bundle over a manifold X and $Y \subset X$ an arbitrary subset (not necessarily a manifold), we denote by $\Gamma(E)|_Y$ restrictions to Y of smooth sections of E in a neighborhood of Y in X.

Operations with Dirac structures

Definition Let $D \subset \mathbb{T}M$ be a subbundle.

- For all $\phi\colon M' \to M$, we denote by $\phi^! D$ the set

$$\phi^! D_{m'} := \left\{(X, \phi^*\beta) \text{ with } X \in T_{m'}M', \beta \in T^*_{\phi(m')}M | (\phi_*(X), \beta) \in D_{\phi(m')}\right\}$$

When D is an (almost-)Dirac structure call $\phi^! D$ the pullback of D.

- Let ω be a 2-form $\omega \in \Omega^2(M)$, we denote by $e^\omega D$ the set

$$e^\omega D = \{(v, \beta + \iota_v \omega) \mid (v, \beta) \in D\}$$

and call it the gauge transform of D.

Lemma Let $D \subset \mathbb{T}M$ be a Dirac structure and M' be a manifold.
- For any smooth map $\phi\colon M' \to M$, $\phi^! D$ is a Dirac structure on M'.
- For any closed 2-form $\omega \in \Omega^2(M)$, $e^\omega D$ is a Dirac structure on M.

Implicit Lagrangian systems with magnetic terms

Given $D \subset \mathbb{T}Q$ a Dirac structure on Q, consider
(i) its pull back $\pi^! D$ on T^*Q through the canonical base map $\pi \colon T^*Q \to Q$, then
(ii) the gauge transformation $e^{\Omega}\pi^! D$ of this pull-back with respect to the canonical symplectic 2-form Ω.

Definition Let $D \subset \mathbb{T}Q$ be a Dirac structure on Q. We call constrained magnetic Lagrangian system an implicit Lagrangian system for the Dirac structure $\mathbb{D} = e^{\Omega}\pi^! D \subset \mathbb{T}T^*Q$ as above.

Implicit Lagrangian systems with magnetic terms

Theorem 2. Let $D \subset \mathbb{T}Q$ be a Dirac structure and $L : TQ \to \mathbb{R}$ a Lagrangian. Assume that the 2-form $\omega_D \in \Gamma(\Lambda^2 D^*)^{hor}$ admits a basic primitive $\theta \in \Gamma(D^*)^{hor}$. Then for $q : I \to Q$ the following are equivalent:

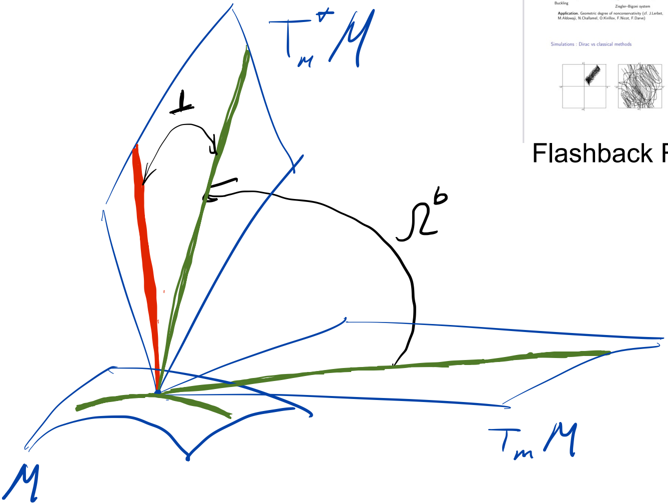
a) There exists a Dirac path $\zeta : I \to D$ such that $\rho(\zeta) = \dot{q}$ which is the critical point among Dirac paths with the same end points of

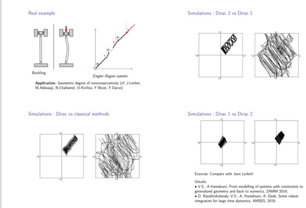
$$\int_I (L(\rho(\zeta(t))) + \theta(\zeta(t)))dt. \tag{2}$$

b) For all $t \in I$, the following condition holds.

$$\left(\frac{\partial}{\partial t}\mathbb{F}L(\dot{q}(t)), \mathcal{D}_{\dot{q}(t)}L\right) \in \mathbb{D} = e^{\Omega}\pi^! D. \tag{3}$$

Example: constraints





Flashback Rennes 2018

$$e^{\Omega}\pi^{!}D = \{(w,\alpha) \in TT^*Q \oplus T^*T^*Q \mid \pi_*(w) \in F, \alpha - \Omega^\flat w \in \pi^{-1}(F)^\circ\}$$

Instead of conclusion: work in progress

(Vladimir's modest plans)

Poisson.
Let $D \subset \mathbb{T}M$ be the graph of a Poisson structure π. Then the Lie algebroid D is isomorphic to T^*M and $H^\bullet(D) \cong H^\bullet_\pi(M)$ is known as the Poisson cohomology. The class of ω_D in $H^\bullet(D)$ corresponds to the class of π in $H^2_\pi(M)$. The class of ω in the finer cohomology $\mathcal{H}^\bullet_{bas}(D)$ is zero if and only if $\pi \in \mathfrak{X}^2(M)$ admits a primitive $E \in \mathfrak{X}(M)$ (a vector field E satisfying $L_E\pi = \pi$), which is tangent to the Poisson structure, i.e. is a section of $\rho(D) \subset TM$.
Question: reformulate the Theorem 2. (cf. Oscar)

Other Dirac structure. Pick your favourite one.

Generalizations.
– Interpret obstructions
– Almost Dirac? $\to$ "almost classes" (cf. Ruben ?)
– ...

Discretization and numerics – very long story

Dirac structures in infinite dimension?..

(Vladimir's not modest plans)

On $E = TM \oplus T^*M$ (or more generally $F \oplus F^*$)
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Remark. The same, using degree 1 DG-manifolds (Q-structures)

Methods from HEP

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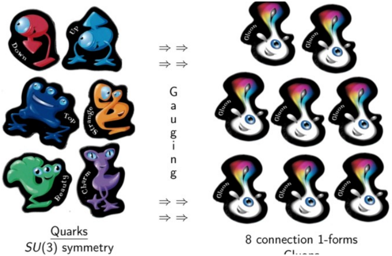


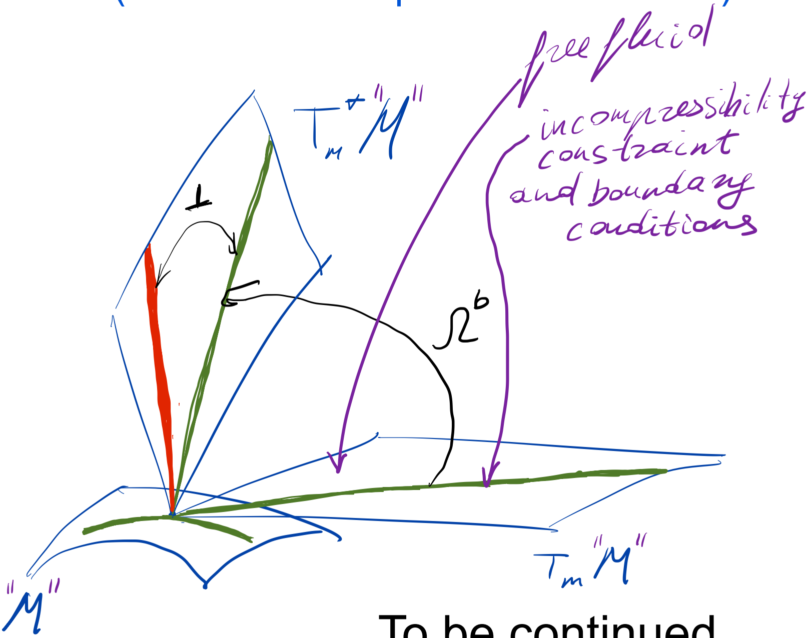
Important "tool": graded geometry, differential graded manifolds.

Equivariant cohomology and gauging in a nutshell

"equations of motion"	↔	Q-morphisms
"symmetries"	↔	Q-homotopies
"gauge invariant"	↔	"equivariantly Q-closed"

Flashback to Vladimir's childhood

Applications (Aziz's serious questions / dreams)



To be continued…

Merci pour votre attention!

Merci pour votre attention!

- Merci à tous les participants
- Merci aux organisateurs locaux!

La Rochelle - été 2023
Paris - automne 2023

- Suivez les annonces et répondez aux mails ;-)

- Pari de bière

score:

12 : 5
(-1)

L'abus d'alcool est dangereux pour votre santé, à consommer avec un modérateur.