Quelques intégrateurs géométriques

Dina Razafindralandy, Aziz Hamdouni

LaSIE, La Rochelle Université – UMR CNRS 7356

GDR Géométrie Différentielle et Mécanique
ENS Paris Saclay, 23-25 Novembre 2022

Geometric integrator

Discretization scheme which respects *exactly* some geometric properties

- Symplectic/Poisson integrator
  - Preserves the symplectic form/Poisson bivector

    $\Phi_t^* \ \omega = \omega \quad \longrightarrow \quad \phi_{t^n}^* \ \omega(u^n) = \omega(u^0)$
  - Stability, small error on first integrals
- Variationnal integrator
  - Comes from a calculus of variation

    $$\begin{array}{ccc} \delta\mathcal{L}(t,x,u,u_{(1)},\dots) = 0 & \longrightarrow & \delta\mathcal{L}_h(t^n,x^i,u^{n,i},u^{n,i}_{(1)},\dots) = 0 \\ \downarrow & & \downarrow \\ EL(t,x,u) = 0 & \not\rightarrow & EL(t^n,x^i,u^{n,i}) = 0 \end{array}$$
  - Stability, small error on first integrals and Energy preserving
- Port Hamiltonian, Dirac, . . .
  - Coupled/constrained system
  - Preserves power balance/constraints
- Invariant integrator: Preserves the symmetry group

  $\mathbb{G}_{E(t,x,u)=0} \quad \subset \quad \mathbb{G}_{E_h(t^n,x^i,u^{n,i})=0}$
- Discrete exterior calculus: div curl $= 0$, curl grad $= 0$

} Outline (last two items)

Invariant integrator

Symmetry group

- Differential equation: $E\left(\mathbf{x}, \mathbf{u}, \mathbf{u}_{(1)}, \ldots\right) = 0$
- Transformation $(\mathbf{x}, \mathbf{u}) \longmapsto (\widehat{x}, \widehat{u})$ is a symmetry if
  $E\left(\mathbf{x}, \mathbf{u}, \mathbf{u}_{(1)}, \ldots\right) = 0 \quad \Longrightarrow \quad E\left(\widehat{\mathbf{x}}, \widehat{\mathbf{u}}, \widehat{\mathbf{u}_{(1)}}, \ldots\right) = 0$
- Example
  - ★ $\begin{cases} \operatorname{div} \mathbf{u} = 0 \\ \mathbf{u}_t + (\nabla \mathbf{u})\mathbf{u} + \frac{1}{\rho}\nabla p - \nu \Delta \mathbf{u} + \beta g \theta \mathbf{e}_2 = 0 \\ \theta_t + \nabla\theta \cdot \mathbf{u} - \kappa \Delta\theta = 0 \end{cases}$
  - ★ Symmetries: $(t, x, u) \longmapsto$
    - $(t + \epsilon, \mathbf{x}, \mathbf{u})$
    - $(t, \mathbf{x} + \alpha(t), \mathbf{u} + \dot{\alpha}, p - \rho\,(\mathbf{x} + \alpha)\,.\,\ddot{\alpha}, \theta)$
    - $(t, \mathbf{x}, \mathbf{u}, p + \zeta(t), \theta)$
    - $(t, \mathbf{x}, \mathbf{u}, p + \epsilon\beta g \mathbf{x} \cdot \mathbf{e}_2, \theta + \theta + \frac{\epsilon}{\rho})$
    - $(t, \mathrm{R}\mathbf{x}, \mathrm{R}\mathbf{u}, p, \theta)$ R rotation horizontale
    - $(\mathrm{e}^{2\epsilon}\, t, \mathrm{e}^{\epsilon}\, \mathbf{x}, \mathrm{e}^{-\epsilon}\, \mathbf{u}, \mathrm{e}^{-2\epsilon}\, p, \mathrm{e}^{-3\epsilon}\, \theta)$

$\mathbb{G}$=(4-dim Lie symmetry group) $\vee$ (4 ∞-dim symmetry groups)

Importance of symmetry groups

- Traduce fundamental physical principles/properties
  (Galilean invariance, homogeneity, . . .)
- Conservation laws through Nœther's theorem
- Self-similar solutions
- Existence, stability, . . . theorems, bifurcation
- Modeling (scaling laws, turbulence modeling, . . .)

Invariant scheme and invariantization process

$$E\left(\mathbf{x}, \mathbf{u}, \mathbf{u}_{(1)}, \dots\right) = 0$$

$\underline{\mathbf{x}} = (\mathbf{x}^1, \mathbf{x}^2, \dots)$, $\underline{\mathbf{u}} = (\mathbf{u}^1, \mathbf{u}^2, \dots)$, $\quad \mathbf{u}^i \simeq \mathbf{u}(\mathbf{x}^i)$

$$\mathrm{Schm}(\underline{\mathbf{x}}, \underline{\mathbf{u}}) = 0 \longleftarrow \begin{cases} \mathrm{Grid}(\underline{\mathbf{x}}, \underline{\mathbf{u}}) = 0 \\ E_h(\underline{\mathbf{x}}, \underline{\mathbf{u}}) = 0 \end{cases}$$

Invariant scheme and invariantization process

$$\underline{\mathbf{x}} = (\mathbf{x}^1, \mathbf{x}^2, \dots),\ \underline{\mathbf{u}} = (\mathbf{u}^1, \mathbf{u}^2, \dots), \qquad \mathbf{u}^i \simeq \mathbf{u}(\mathbf{x}^i)$$

$$E\left(\mathbf{x}, \mathbf{u}, \mathbf{u}_{(1)}, \dots\right) = 0 \quad \Big| \quad \mathrm{Schm}(\underline{\mathbf{x}}, \underline{\mathbf{u}}) = 0 \longleftarrow \begin{cases} \mathrm{Grid}(\underline{\mathbf{x}}, \underline{\mathbf{u}}) = 0 \\ E_h(\underline{\mathbf{x}}, \underline{\mathbf{u}}) = 0 \end{cases}$$

$$\underset{\mathbf{m}}{T : (\mathbf{x}, \mathbf{u})} \longmapsto \underset{\widehat{\mathbf{m}}}{(\widehat{\mathbf{x}}, \widehat{\mathbf{u}})} \quad \Big| \quad \underset{\underline{\mathbf{m}}}{T_h : (\underline{\mathbf{x}}, \underline{\mathbf{u}})} \longmapsto \underset{\widehat{\underline{\mathbf{m}}}}{(\widehat{\underline{\mathbf{x}}}, \widehat{\underline{\mathbf{u}}})}$$

Invariant scheme and invariantization process

$E\left(\mathbf{x},\mathbf{u},\mathbf{u}_{(1)},\dots\right)=0$

$T:\underbrace{(\mathbf{x},\mathbf{u})}_{\mathbf{m}}\longmapsto\underbrace{(\widehat{\mathbf{x}},\widehat{\mathbf{u}})}_{\widehat{\mathbf{m}}}$

$\underline{\mathbf{x}}=(\mathbf{x}^1,\mathbf{x}^2,\dots)$, $\underline{\mathbf{u}}=(\mathbf{u}^1,\mathbf{u}^2,\dots)$, $\quad \mathbf{u}^i\simeq\mathbf{u}(\mathbf{x}^i)$

$$\mathrm{Schm}(\underline{\mathbf{x}},\underline{\mathbf{u}})=0 \longleftarrow \begin{cases}\mathrm{Grid}(\underline{\mathbf{x}},\underline{\mathbf{u}})=0\\ E_h(\underline{\mathbf{x}},\underline{\mathbf{u}})=0\end{cases}$$

$T_h:\underbrace{(\underline{\mathbf{x}},\underline{\mathbf{u}})}_{\underline{\mathbf{m}}}\longmapsto\underbrace{(\underline{\widehat{\mathbf{x}}},\underline{\widehat{\mathbf{u}}})}_{\underline{\widehat{\mathbf{m}}}}$

Definition: scheme Schm is invariant if

$$\mathrm{Schm}(\underline{\mathbf{m}})=0 \quad\Longrightarrow\quad \mathrm{Schm}(\underline{\widehat{\mathbf{m}}})=0, \qquad \forall T\in\mathbb{G}_E$$

Sufficient condition

$$\mathrm{Schm}(\underline{\widehat{\mathbf{m}}})=\mathrm{Schm}(\underline{\mathbf{m}}) \qquad \forall T\in\mathbb{G}_E$$

Invariant scheme and invariantization process

$$E\left(\mathbf{x},\mathbf{u},\mathbf{u}_{(1)},\dots\right)=0$$

$$T:\underbrace{(\mathbf{x},\mathbf{u})}_{\mathbf{m}}\longmapsto\underbrace{(\widehat{\mathbf{x}},\widehat{\mathbf{u}})}_{\widehat{\mathbf{m}}}$$

$\underline{\mathbf{x}}=(\mathbf{x}^1,\mathbf{x}^2,\dots)$, $\underline{\mathbf{u}}=(\mathbf{u}^1,\mathbf{u}^2,\dots)$, $\quad \mathbf{u}^i\simeq\mathbf{u}(\mathbf{x}^i)$

$$\mathrm{Schm}(\underline{\mathbf{x}},\underline{\mathbf{u}})=0 \longleftarrow \begin{cases}\mathrm{Grid}(\underline{\mathbf{x}},\underline{\mathbf{u}})=0\\ E_h(\underline{\mathbf{x}},\underline{\mathbf{u}})=0\end{cases}$$

$$T_h:\underbrace{(\underline{\mathbf{x}},\underline{\mathbf{u}})}_{\underline{\mathbf{m}}}\longmapsto\underbrace{(\widehat{\underline{\mathbf{x}}},\widehat{\underline{\mathbf{u}}})}_{\widehat{\underline{\mathbf{m}}}}$$

Definition: scheme Schm is invariant if

$$\mathrm{Schm}(\underline{\mathbf{m}})=0 \quad\Longrightarrow\quad \mathrm{Schm}(\widehat{\underline{\mathbf{m}}})=0,\qquad \forall T\in\mathbb{G}_E$$

Sufficient condition

$$\mathrm{Schm}(\widehat{\underline{\mathbf{m}}})=\mathrm{Schm}(\underline{\mathbf{m}})\qquad \forall T\in\mathbb{G}_E$$

Invariatization process: Schm any scheme

$$\widetilde{\mathrm{Schm}}(\underline{\mathbf{m}})=\mathrm{Schm}(\tau[\underline{\mathbf{m}}]\cdot\underline{\mathbf{m}})$$ is an invariant scheme

(Right) Mobile frame: $\tau:\mathbf{m}\in M\longmapsto\tau[\mathbf{m}]\in\mathbb{G}$

$$\tau[T(\mathbf{m})]=\tau[\mathbf{m}]T^{-1}\qquad \forall T\in\mathbb{G}$$

Application

- Burgers equation: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0$
- 5-dim Lie symmetry group
  - Space translation: $(t+\epsilon_1, x, u)$
  - Time translation: $(t, x+\epsilon_2, u)$
  - Projection: $\left(\frac{t}{1-\epsilon_3 t}, \frac{x}{1-\epsilon_3 t} + \epsilon_3, u(1-\epsilon_3 t) + \epsilon_3 x\right)$
  - Scale transformation: $(\mathrm{e}^{2\epsilon_4}\, t, \mathrm{e}^{\epsilon_4}\, x, \mathrm{e}^{-\epsilon_4}\, u)$
  - Galilean boost: $(t, x+\epsilon_5 t, u+\epsilon_5)$

FTCS, Regular grid, $O(\Delta t, \Delta x^2)$

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} + u_i^n\left[\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}\right] - \nu\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{\Delta x^2} = 0$$

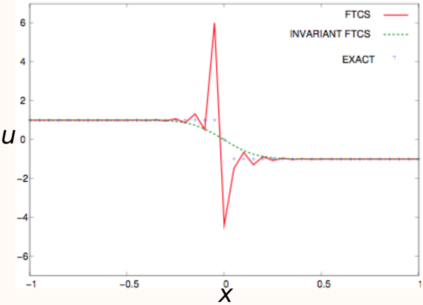
Invariantized FTCS, Regular grid, $O(\Delta t, \Delta x^2)$

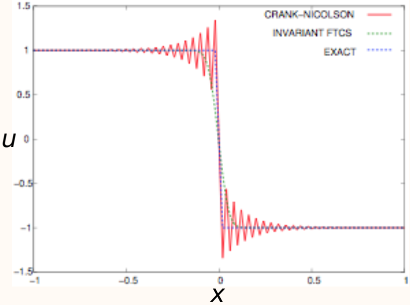
$$\frac{u_i^{n+1}(1-\epsilon_3\Delta t)-u_i^n}{\Delta t}(1-\epsilon_3\Delta t) + [u_i^n+\epsilon_5]\left[\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}+\epsilon_3\right] - \nu\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{\Delta x^2} = 0$$

$$\epsilon_3 = \frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}, \quad \epsilon_5 = fu_{i+1}^n + eu_i^n + fu_{j-1}^n, \quad e+2f=1$$

Numerical results

Pseudo-choc : $$u(t,x) = \frac{\sinh\left(\frac{x}{2\nu}\right)}{\cosh\left(\frac{x}{2\nu}\right)+\exp\left(\frac{-t}{4\nu}\right)}$$





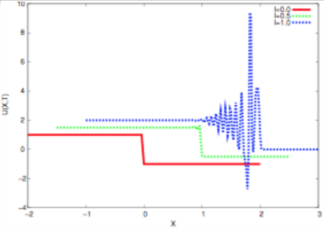
Left: $\nu = 8 \cdot 10^{-3}$. Right: $\nu = 7.5 \cdot 10^{-4}$

Self-similar solution under projection

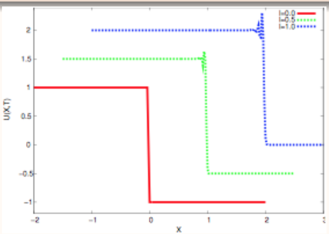
Numerical results

Compatibility with Galilean invariance

$$(t, x, u) \longmapsto (t, x + \epsilon t, u + \epsilon)$$

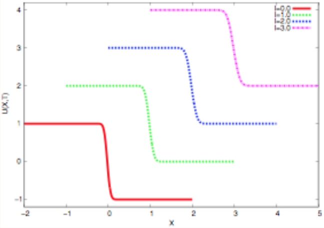


FTCS

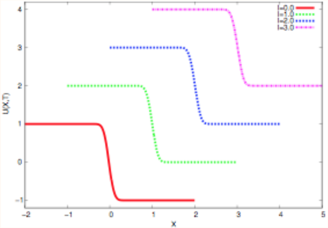


Crank-Nicholson

$\nu = 5 \cdot 10^{-3}$



Invariant FTCS



Invariant Crank-Nicholson

- Invariant integrator: conclusion
  - Compatibility with fundamental invariance properties
  - Compatibility with self-similar solutions
  - May have sensibly higher numerical cost
    But more interesting for coarse time/space grids
  - Should not destroy conservation laws and symmetry based models
  - To be done for Navier-Stokes equations

- Problems for which other properties are more important ?
  - Many Navier-Stokes solvers use : curl grad $= 0$, div curl $= 0$

    Eg: vorticity-stream function formulation

    Not numerically verified $\Longrightarrow$ spurious mass, portance, circulation, ...
  - In exterior calculus: $\mathrm{d}^2 = 0$
    $\mathrm{d} : \Lambda^k \longrightarrow \Lambda^{k+1}$: exterior derivative operator
    acting on differential forms
- Discrete Exterior Calculus: Discrete version of Exterior Calculus theory

Discrete Exterior Calculus (DEC)

Exterior and differential forms

- Exterior k-form = skew-symmetric k-linear form

  Differential k-form $\omega \in \Lambda^k(M)$: Smooth field of exterior k-forms

  $$\omega_{|x} : \; T_{\mathbf{x}}M \times \ldots \times T_{\mathbf{x}}M \longrightarrow \mathbb{R}$$

  Locally: $\omega = \omega_{i_1 \ldots i_k}(\mathbf{x}) \, \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}$

- Often derivated: $\Lambda^0 \xrightarrow{\mathrm{d}} \Lambda^1 \xrightarrow{\mathrm{d}} \Lambda^2 \xrightarrow{\mathrm{d}} \ldots$ with $\mathrm{d}^2 = 0$

  Locally: $\mathrm{d}\omega = \dfrac{\partial \omega_{i_1 \ldots i_k}}{\partial x^j} \, \mathrm{d}x^j \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}$

  In $\mathbb{R}^3$:
  - $\mathrm{d} : \Lambda^0 \longrightarrow \Lambda^1$ ⟷ grad : $\mathcal{F}(R^3) \longrightarrow \mathfrak{X}(R^3)$
  - $\mathrm{d} : \Lambda^1 \longrightarrow \Lambda^2$ ⟷ rot : $\mathfrak{X}(R^3) \longrightarrow \mathfrak{X}(R^3)$
  - $\mathrm{d} : \Lambda^2 \longrightarrow \Lambda^3$ ⟷ div : $\mathfrak{X}(R^3) \longrightarrow \mathcal{F}(R^3)$
  - $\mathrm{d}^2 = 0$ ⟷ rot grad $= 0$, div rot $= 0$

- Sometimes integrated: $\omega : \; \sigma \longmapsto \int_\sigma \omega$

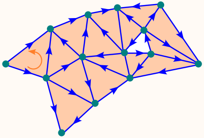
  M orientable manifold, $\omega \in \Lambda^k(M)$, σ k-dim submanifold of M

  Stokes: $$\int_\sigma \mathrm{d}\omega = \int_{\partial\sigma} \iota^*_{\partial\sigma} \omega$$

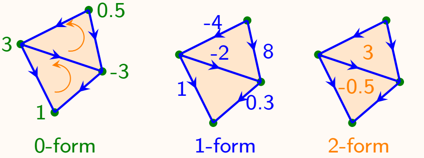
Discretization

- $M \xrightarrow{\text{discr.}}$ Oriented simplicial complex discretization

  $K = \underbrace{K_0}_{\text{vertices}} \cup \underbrace{K_1}_{\text{edges}} \cup \underbrace{K_2}_{\text{triangles}} \cup \underbrace{K_3}_{\text{tetrahedra}} \cup \ldots \cup \underbrace{K_{\dim M}}_{n\text{-simplices}}$



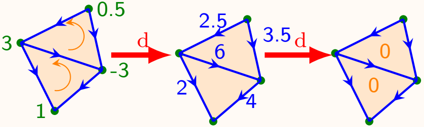
- $\omega \in \Lambda^k \xrightarrow{\text{discr.}} \left(\omega_i = \int_{\sigma_i} \omega\right)_{\sigma_i \in K_k}$



- Discrete exterior derivative

  Stokes: $$\int_\sigma \mathrm{d}\omega = \int_{\partial\sigma} \omega$$

  $$\langle\, \mathrm{d}\theta; \sigma \rangle = \langle \theta; \partial\sigma \rangle$$

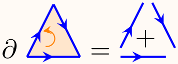


- $\mathrm{d}^2 = 0$ exactly

  because $\partial\partial = \emptyset$

More formally : combinatorial geometry

- **Mesh**
  - ★ k-simplex $[\mathbf{v}_0\mathbf{v}_1\ldots\mathbf{v}_k] = \left\{ \sum_{i=0}^{k}\lambda_i\mathbf{v}_i,\ \lambda_i \geq 0, \sum_{i=1}^{k}\lambda_i = 1 \right\} \in K_k$
  - ★ Simplicial complex: Collection $K = \cup_{k=0}^{n} K_k$ of simplices such that
    - $\sigma \in K$ and τ face of $\sigma \implies \tau \in K$
    - $+$ regularity conditions
  - ★ Orientation
- **Chain** $=$ Element of $\Lambda_k(K)$ — $|K_k|$-dim array
  - ★ $\Lambda_k(K) = \operatorname{span}_{\mathbb{Z}} K_k = \left\{ \sum_{\sigma\in K_k} z_\sigma\sigma,\ z_\sigma \in \mathbb{Z} \right\}$



  - ★ Boundary operator

$$\partial : \begin{array}{ccc} K_k & \longrightarrow & \Lambda_{k-1}(K) \\ \sigma = [\mathbf{v}_0\mathbf{v}_1\ldots\mathbf{v}_k] & \longmapsto & \sum_{i=0}^{k}(-1)^i[\mathbf{v}_0\ldots\mathbf{v}_{i-1},\mathbf{v}_{i+1},\ldots,\mathbf{v}_k] \end{array}$$

    Extends by linearity into $\partial : \Lambda_k(K) \longrightarrow \Lambda_{k-1}(K)$
- **Discrete k-form $=$ k-cochain** $=$ Element of $\Lambda^k(K)$ — $|K_k|$-dim array
  - ★ $\Lambda^k(K) = \mathbb{R}\text{-dual of } \Lambda_k(K) \simeq \left\{ \sum_{\sigma\in K_k} \omega_\sigma\sigma,\ \omega_\sigma \in \mathbb{R} \right\}$
  - ★ Discrete $\mathrm{d} = \partial^\top : \Lambda_k(K) \longrightarrow \Lambda_{k+1}(K)$ — $(|K_{k+1}| \times |K_k|)$-dim array

Hodge $\star$ operator

M: n-dim manifold M with metric g and volume form vol

Continuous Hodge: isomorphism

$\star : \Lambda^k(M) \longrightarrow \Lambda^{n-k}(M)$ such that $\theta \wedge \star\omega = g(\theta, \omega)\,\mathrm{vol}$

- Complement to vol: $\omega \wedge \star\omega = \mathrm{vol}$ if $g(\omega, \omega) = 1$

  Orthogonality: $g(\star\omega, \omega) = 0$

- In $\mathbb{R}^2$ with Euclidean metric and $\mathrm{vol} = \mathrm{d}x \wedge \mathrm{d}y$

  $\star 1 = \mathrm{d}x \wedge \mathrm{d}y, \quad \star\mathrm{d}x = \mathrm{d}y, \quad \star\mathrm{d}y = -\mathrm{d}x, \quad \star\mathrm{d}x \wedge \mathrm{d}y = 1$

- In $\mathbb{R}^3$ with Euclidean metric and $\mathrm{vol} = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$

  $\star 1 = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z, \quad \star\mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}z, \quad \star(\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}z, \quad \ldots$

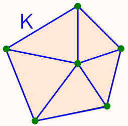
- $\dim \Lambda^k(M) = \binom{n}{k} = \binom{n}{n-k} = \dim \Lambda^{n-k}(M)$

- Usefull to formulate constitutive laws

Dual mesh

Discrete Hodge: isomorphism ?

$\star : \Lambda^k(K) \longrightarrow \Lambda^{n-k}(K)$ Generally impossible



$$\dim \Lambda^k(K) = |K_k| \neq |K_{n-k}| = \dim \Lambda^{n-k}(K)$$

Eg when $n = 2$: nb vertices $\neq$ nb triangles

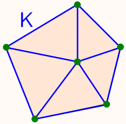
Dual mesh

Discrete Hodge: isomorphism ?

$\star : \Lambda^k(K) \longrightarrow \Lambda^{n-k}(K)$ Generally impossible

$$\dim \Lambda^k(K) = |K_k| \neq |K_{n-k}| = \dim \Lambda^{n-k}(K)$$

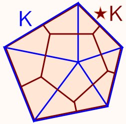
Eg when $n = 2$: nb vertices $\neq$ nb triangles



Discrete Hodge: isomorphism

$\star : \Lambda^k(K) \longrightarrow \Lambda^{n-k}(*K)$ Possible

- Dual mesh $*K = \bigcup_{k=0}^{n} (*K)_k$
  - Bijection $* : K_k \longrightarrow (*K)_{n-k}$
  - Orientation on K ⟶ Orientation on $*K$
  - Dual cochain: element of $\Lambda^k(*K)$
  - ∂ on K ⟶ ∂ on $*K$ ⟶ d on $*K$



Circumcentric dual

- $\dim \Lambda^k(K) = |K_k| = |(*K)_{n-k}| = \dim \Lambda^{n-k}(*K)$

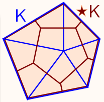
Discrete Hodge operator

- How to define $\star : \Lambda^k(K) \longmapsto \Lambda^{n-k}(*K)$?
- Continuous case : If ω is locally constant and $\sigma \perp *\sigma$ (circumcentric dual)

$$\frac{1}{|*\sigma|}\int_{*\sigma} \star\omega = \frac{1}{|\sigma|}\int_{\sigma}\omega, \qquad |\sigma| = \begin{cases} k\text{-volume de } \sigma, & \dim\sigma \geq 1 \\ 1 & \dim\sigma = 0 \end{cases}$$

Circoncentric Hodge

$$\frac{\langle \star\omega, *\sigma\rangle}{|*\sigma|} = \frac{\langle \omega, \sigma\rangle}{|\sigma|}$$



Circoncentric dual

Diagonal matrix $H_k = \text{diag}\left(\frac{|*\sigma|}{|\sigma|}, \quad \sigma \in K_k\right)$

Navier-Stokes + passive scalar (polluant)

Vector/tensor formulation

$$\frac{\partial \mathbf{u}}{\partial t} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) + \frac{1}{\rho}\operatorname{grad} p - \nu \Delta \mathbf{u} = 0$$

$$\operatorname{div} \mathbf{u} = 0$$

$$\frac{\partial \theta}{\partial t} + \operatorname{div}(\mathbf{u}\theta) - \kappa \Delta \theta = 0$$

Exterior calculus formulation $(\omega = u^\flat)$

$$\frac{\partial \omega}{\partial t} + u \lrcorner \, \mathrm{d}\omega + \frac{1}{\rho}\,\mathrm{d}\left(p + \tfrac{1}{2}\rho||\mathbf{u}||^2\right) - \nu \delta \,\mathrm{d}\omega = 0$$

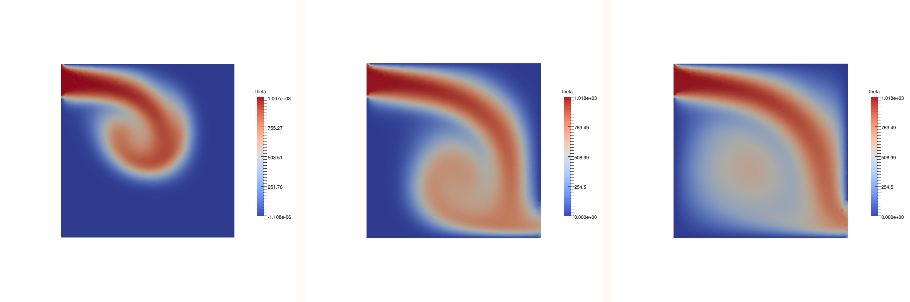
$$\delta\omega = 0$$

$$\frac{\partial \theta}{\partial t} + \delta(\omega \wedge \theta) - \kappa \delta \,\mathrm{d}\theta = 0$$

$$\delta = \pm \star \mathrm{d} \star$$

Resolution: Stream function $\omega = - \star \mathrm{d}\psi$

Consequence: $\operatorname{div} \mathbf{u} = 0$ at machine precision

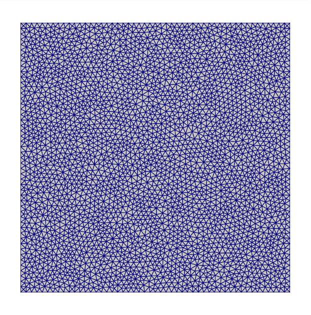


Concentration of polluant at different times

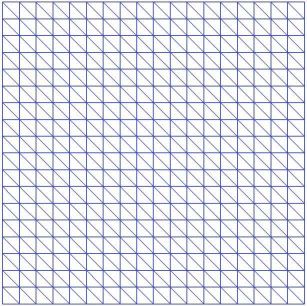
[Razafindralandy, Hamdouni, Chhay. AMSES 2018]

Necessity of an alternative discrete Hodge

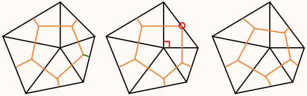
- Circumcentric Hodge needs a "well-centered" primal mesh



OK



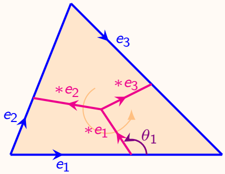
$H_1 = \mathrm{diag}\left(\frac{|e_i|}{|\star e_i|}\right)$ non-invertible



- Change simplex centers
  - ★ barycenter, incenter, ...
  - ★ error minimisation
  - ★ physical consideration

"Analytical" discrete Hodge

- Any dual mesh (circumcenter, barycenter, incenter, ...)



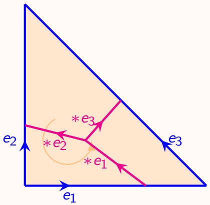
- Exact if ω locally constant

$$H_1 = \begin{pmatrix} \frac{|*e_1|}{|e_1|} & 0 & 0 \\ 0 & \frac{|*e_2|}{|e_2|} & 0 \\ 0 & 0 & \frac{|*e_2|}{|e_2|} \end{pmatrix} \begin{pmatrix} \sin\theta_1 & a_1^2\cos\theta_1 & a_1^3\cos\theta_1 \\ a_2^1\cos\theta_2 & \sin\theta_2 & a_2^3\cos\theta_2 \\ a_3^1\cos\theta_3 & a_3^2\cos\theta_3 & \sin\theta_3 \end{pmatrix}$$

- $\theta_i = (\widehat{\vec{e}_i, \overrightarrow{*e_i}})$, $\qquad -R_{\pi/2}\vec{e}_1 = a_1^2\vec{e}_2 + a_1^3\vec{e}_3, \ \dots$

[Ayoub, Hamdouni, Razafindralandy. AMC 2021]

Example in an unitary triangle



$$H_1^{barycenter} = \frac{1}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad H_1^{incenter} = \frac{1}{4+2\sqrt{2}}\begin{bmatrix} 2 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

- Invertible
- Assembling
- Elementwise inversion

Poisson Equation

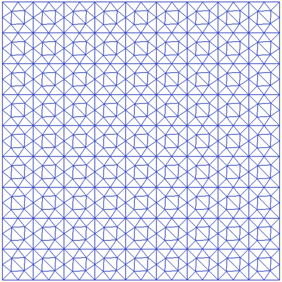
- $\Delta\omega = \overline{f}$

Exterior calculus formulation: $\star \ \mathrm{d} \ \star \ \mathrm{d} \ \omega = f$

-

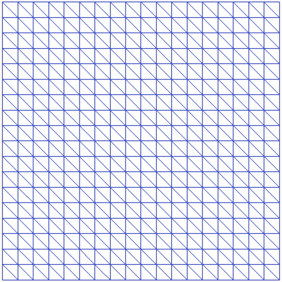
Well-centered mesh

- ★ Circumcentric dual
- ★ Barycentric dual
- ★ Incentric dual



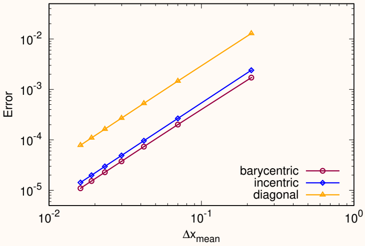
Acute mesh

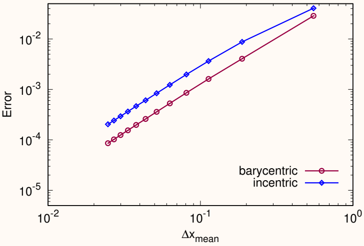
- ★ Barycentric dual
- ★ Incentric dual



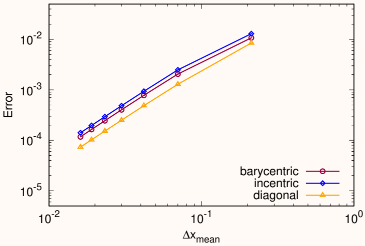
Δx_{mean} =mean edge length

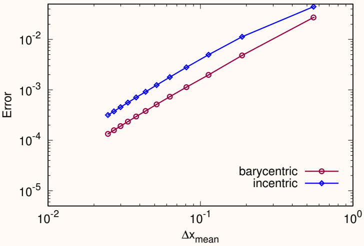
Poisson equation





$$\omega_{exact} = x^2 + y^2$$



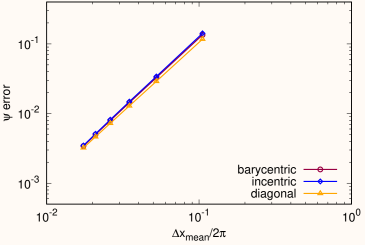


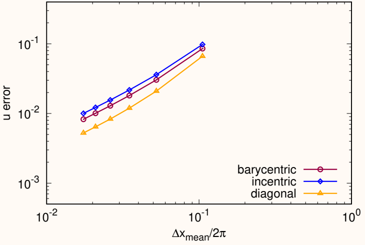
$$\omega_{exact} = \sin(\pi x)\sinh(\pi y)$$

Navier-Stokes: Taylor-Green Vortex

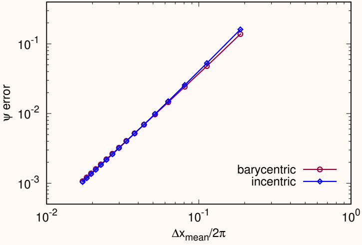
$$u = -\cos x \sin y \, e_x \quad + \quad \sin x \cos y \, e_y$$

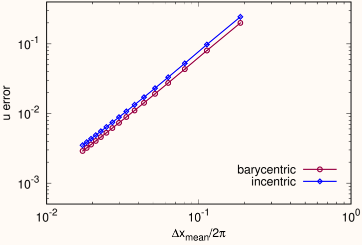
Well-centered mesh





Acute mesh



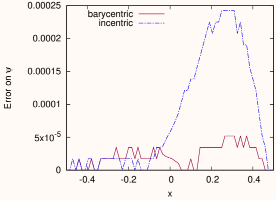


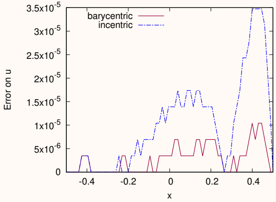
Anisothermal traveling wave

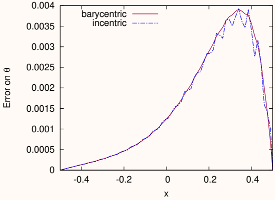
$$\xi = ax + by + ct, \qquad u = (\alpha_1 \, e^{\lambda\xi/\nu} + \alpha_2 \, e^{\lambda\xi/\tau}) \, e_x + \alpha_3 \, e_y$$
$$p = \alpha_p \, e^{\lambda\xi/\tau}, \qquad \theta = \alpha_\theta \, e^{\lambda\xi/\tau}$$

The $\alpha_.$ are non-independent constants

Right mesh, $\Delta x_{mean} = 1.89 \cdot 10^{-2}$, Final time $|\lambda c T|/\tau = 1$





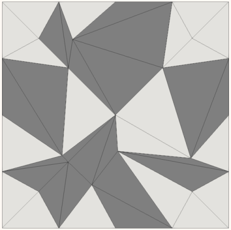


Dual mesh	Stream function	Velocity	Temperature
Barycentric	$2.651 \cdot 10^{-5}$	$7.270 \cdot 10^{-5}$	$5.529 \cdot 10^{-3}$
Incentric	$8.875 \cdot 10^{-5}$	$2.132 \cdot 10^{-4}$	$5.589 \cdot 10^{-3}$

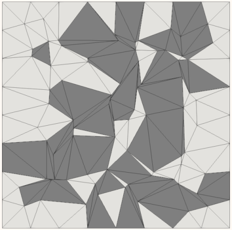
Mean relative error

In both cases, $\operatorname{div} u = 0$ at machine precision

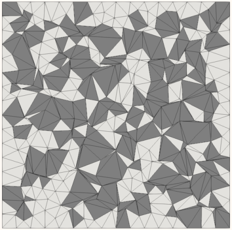
"Weird" meshes



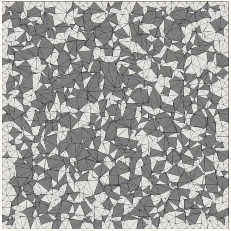
(a) $\Delta x_{mean} = 0.2682$



(b) $\Delta x_{mean} = 0.1265$



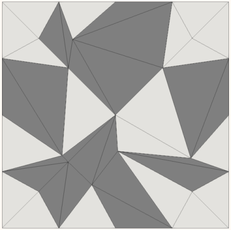
(c) $\Delta x_{mean} = 0.0650$



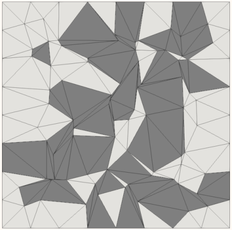
(d) $\Delta x_{mean} = 0.0328$

Ex: 50% of non-Delaunay triangles

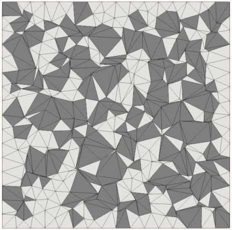
“Weird” meshes



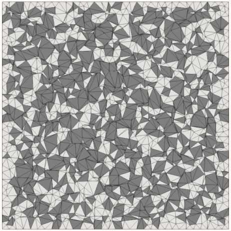
(a) $\Delta x_{mean} = 0.2682$



(b) $\Delta x_{mean} = 0.1265$



(c) $\Delta x_{mean} = 0.0650$



(d) $\Delta x_{mean} = 0.0328$

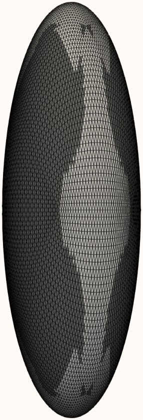
Ex: 50% of non-Delaunay triangles

	ψ	θ
15% non-Delaunay	1.9005	1.5159
25% non-Delaunay	1.6729	1.2154
50% non-Delaunay	1.6591	0.8660

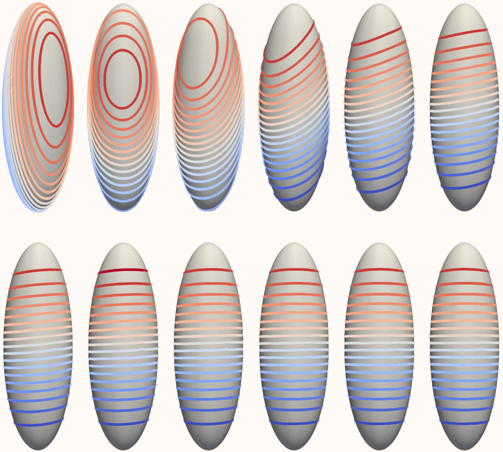
Convergence rate

On a surface

Initial flow: $\psi(t=0) = y + 0.1z$,



40% of well-centered triangles



Streamlines for t from 0 to 11

Conclusion

- Done
  - ★ "Velocity"–pressure formulation + Prediction-correction scheme
    Lid driven cavity (Ghia), Re=100, Re=1000
  - ★ Neumann boundary condition
  - ★ 3D with Whitney Hodge

- Advantages
  - ★ $\mathrm{d}^2 = 0$
  - ★ Coordinate independent
  - ★ Exterior calculus formulation brings clarity in some problems
  - ★ Conservation laws : Circulation preservation for non-viscous fluid
    Needs carefully designed time integrator

- To do
  - ★ Further exploitatiion of $\mathrm{d}^2 = 0$
    Complex geometry
  - ★ Space-time DEC
  - ★ Invariantized space-time DEC ?