Splitting integrators for stochastic Lie–Poisson systems

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Joint work with
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C.-E. Bréhier, D. Cohen, and T. Jahnke, *Splitting integrators for stochastic Lie–Poisson systems*, arxiv preprint 2111.07387.

Plan

Poisson systems

Poisson systems $\leftrightarrow$ ordinary differential equations in $\mathbb{R}^d$:

$$x'(t) = B(x(t))\nabla H(x(t)), \qquad x(0) = x_0$$

where

- $x \in \mathbb{R}^d \mapsto B(x) \in \mathcal{M}_d(\mathbb{R})$ is the structure or Poisson mapping *(assumptions: it is skew–symmetric and satisfies Jacobi identity)*
- $x \in \mathbb{R}^d \mapsto H(x) \in \mathbb{R}$ is the Hamiltonian function.

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A function $x \in \mathbb{R}^d \mapsto C(x) \in \mathbb{R}$ is a Casimir if: $\nabla C(\cdot)^\star B(\cdot) = 0$.

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Main properties:

- preservation of energy H, preservation of Casimir functions C.
- the flow is a Poisson map: $D_y\varphi_t(y)B(y)D_y\varphi_t(y)^\star = B(\varphi_t(y))$ for all $(t, y) \in \mathbb{R} \times \mathbb{R}^d$.

Stochastic differential equations

Itô stochastic integral: $\Delta t = T/N$, $t_n = n\Delta t$,

$$\int_0^T Z(t)dW(t) = \lim_{N\to\infty} \sum_{n=0}^{N-1} Z(t_n)\big(W(t_{n+1})) - W(t_n)\big),$$

where $\big(W(t)\big)_{t\geq 0}$ is a Wiener process/Brownian motion.
Example: $\int_0^T W(t)dW(t) = \frac{W(T)^2 - T}{2}$.

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Itô stochastic differential equation:

$$dX(t) = f(X(t))dt + \sigma(X(t))dW(t), \quad X(0) = x_0,$$

with **integral interpretation**: for all $t \geq 0$

$$X(t) = x_0 + \int_0^t f(X(s))ds + \int_0^t \sigma(X(s))dW(s).$$

Stochastic differential equations

$$dX(t) = f(X(t))dt + \sigma(X(t))dW(t), \quad X(0) = x_0.$$

Itô formula:

$$dh(X(t)) = h'(X(t))\Big(f(X(t))dt + \sigma(X(t))dW(t)\Big) + \frac{1}{2}h''(X(t))\sigma(X(t))^2 dt.$$

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Stratonovich stochastic differential equations:

$$\begin{aligned} dY(t) &= f(Y(t))dt + \sigma(Y(t)) \circ dW(t) \\ &= \big(f(Y(t)) + \frac{1}{2}\sigma'(Y(t))\sigma(Y(t))\big)dt + \sigma(Y(t))dW(t), \end{aligned}$$

the **chain rule** is satisfied

$$dh(Y(t)) = h'(Y(t)) \circ dY(t) = h'(Y(t))f(Y(t))dt + h'(Y(t))\sigma(Y(t)) \circ dW(t).$$

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Stratonovich integral:

$$\int_0^T Z(t) \circ dW(t) = \lim_{N \to \infty} \sum_{n=0}^{N-1} \frac{Z(t_n) + Z(t_{n+1})}{2} \big(W(t_{n+1})) - W(t_n)\big).$$

Examples of stochastic differential equations

Linear stochastic differential equations:

- Itô form: $dX(t) = \sigma X(t) dW(t)$ has solution $X(t) = X(0) e^{\sigma W(t) - \frac{\sigma^2 t}{2}}$
- Stratonovich form: $dX(t) = \sigma X(t) \circ dW(t)$ has solution $X(t) = X(0) e^{\sigma W(t)}$ (geometric Brownian motion).

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Stratonovich equation driven by real-valued noise: the solution of

$$dX(t) = \sigma(X(t)) \circ dW(t)$$

is given by

$$X(t) = \varphi_{W(t)}(X(0))$$

where $\left(\varphi_s\right)_{s \in \mathbb{R}}$ is the flow of the ordinary differential equation

$$\dot{x}_s = \sigma(x_s).$$

Stochastic (Lie–)Poisson systems

Two classes of stochastic Poisson systems:

$$dy(t) = B(y(t))\nabla H(y(t))dt + B(y(t))\nabla \widehat{H}(y(t)) \circ dW(t),$$

and

$$dy(t) = B(y(t))\nabla H(y(t))dt + \sum_{k=1}^{m} B(y(t))\nabla \widehat{H}_k(y(t)) \circ dW_k(t),$$

with

- Hamiltonian function $\widehat{H} = \sum_{k=1}^{m} \widehat{H}_k$
- Wiener process W, independent Wiener processes $W_1, \ldots, W_m$
- the **same structure matrix** B for drift and all diffusion parts
- (deterministic) initial value $y(0) = y_0$.

Properties of stochastic Poisson systems

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Properties of such stochastic Poisson systems:

- Casimir functions C ($\nabla C(\cdot)^\star B(\cdot) = 0$) are preserved,
- the flow is a Poisson map.

A practically useful condition: there exists a **Casimir function** C **with compact level sets**, this ensures **almost sure bounds** on the solutions and **global well-posedness**.

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Stochastic **Lie–Poisson** systems: the mapping $y \mapsto B(y)$ is linear.

Examples

In this talk, **two examples** of stochastic Lie–Poisson systems:

- Maxwell–Bloch system

  M. Puta, *Lie-Trotter formula and Poisson dynamics*, Nonlinearity, bifurcation and chaos: the doors to the future, 1999.

- rigid body system

Examples

In this talk, **two examples** of stochastic Lie–Poisson systems:

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- rigid body system

A **third example** is treated in our paper:

- sine-Euler system.

K. Engø and S. Faltinsen, *Numerical integration of Lie-Poisson systems while preserving coadjoint orbits and energy*, SIAM J. Numer. Anal., 2001.

R. I. McLachlan, *Explicit Lie-Poisson integration and the Euler equations*, Phys. Rev. Lett., 1993.

V. Zeitlin, *Finite-mode analogs of 2D ideal hydrodynamics: coadjoint orbits and local canonical structure*, Phys. D, 1991.

Example 1: stochastic Maxwell–Bloch system

Deterministic system:

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = y_1 y_3 \\ \dot{y}_3 = -y_1 y_2. \end{cases}$$

with

$$B(y) = \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & 0 \\ -y_2 & 0 & 0 \end{pmatrix}, \quad H(y) = \frac{1}{2} y_1^2 + y_3, \quad C(y) = \frac{1}{2}(y_2^2 + y_3^2),$$

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Stochastic Maxwell–Bloch system:

$$dy = B(y)\left(\nabla H(y) dt + \sigma_1 \nabla \widehat{H}_1(y) \circ dW_1(t) + \sigma_3 \nabla \widehat{H}_3(y) \circ dW_3(t)\right),$$

where $\widehat{H}_1(y) = \frac{1}{2} y_1^2$ and $\widehat{H}_3(y) = y_3$, $\sigma_1, \sigma_3 \geq 0$.

Example 2: stochastic rigid body system

Deterministic system:

$$\begin{cases} \dot{y}_1 = (I_3^{-1} - I_2^{-1}) y_3 y_2 \\ \dot{y}_2 = (I_1^{-1} - I_3^{-1}) y_1 y_3 \\ \dot{y}_3 = (I_2^{-1} - I_1^{-1}) y_2 y_1, \end{cases}$$

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Stochastic rigid body system:

$$\begin{aligned} dy &= B(y)\nabla H(y)dt \\ &\quad + B(y)\Big(\nabla \widehat{H}_1(y) \circ dW_1(t) + \nabla \widehat{H}_2(y) \circ dW_2(t) + \nabla \widehat{H}_3(y) \circ dW_3(t)\Big), \end{aligned} \quad (1)$$

where $\widehat{H}_k(y) = \frac{y_k^2}{\widehat{I}_k}$, for $k = 1, 2, 3$, with $\big(\widehat{I}_1, \widehat{I}_2, \widehat{I}_3\big) \neq (I_1, I_2, I_3)$.

Stochastic integrators

Stochastic Poisson systems:

$$dy(t) = B(y(t))\nabla H(y(t))dt + B(y(t))\nabla \widehat{H}(y(t)) \circ dW(t)$$

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Stochastic integrator:

$$y^{[n]} = \Phi_h(y^{[n-1]}, \Delta_n W)$$

$$y^{[n]} = \Phi_h(y^{[n-1]}, \Delta_n W_1, \ldots, \Delta_n W_m),$$

with initial value $y^{[0]} = y_0$, time-step size h and Wiener increments

$$\Delta_n W = W(nh) - W((n-1)h)$$

$$\Delta_n W_k = W_k(nh) - W_k((n-1)h), \qquad k = 1, \ldots, m.$$

Stochastic Poisson integrators

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is a stochastic Poisson integrator if it preserves the Poisson map property and the Casimir functions of the exact system:

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is a stochastic Poisson integrator if it preserves the Poisson map property and the Casimir functions of the exact system:
for all $h > 0$ and all $\Delta w, \Delta w_1, \ldots, \Delta w_m \in \mathbb{R}$,

- the mappings

$$y \mapsto \Phi_h(y, \Delta w), \qquad y \mapsto \Phi_h(y, \Delta w_1, \ldots, \Delta w_m)$$

  are Poisson maps,

- if C is a Casimir of the stochastic Poisson system, for all $y \in \mathbb{R}^d$

$$C\big(\Phi_h(y, \Delta w)\big) = C(y) = C\big(\Phi_h(y, \Delta w_1, \ldots, \Delta w_m)\big).$$

Construction of stochastic Poisson integrators

The Euler–Maruyama scheme

$$y^{[n]} = y^{[n-1]} + hB(y^{[n-1]})\nabla H(y^{[n-1]}) + B(y^{[n-1]})\nabla \widehat{H}(y^{[n-1]})\Delta_n W$$

is not a stochastic Poisson integrator. In addition, it approximates the stochastic Poisson systems with Itô interpretation of the noise

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For a stochastic Poisson integrator, if there exists a Casimir with compact level sets, one has almost sure bounds for the numerical solution $y^{[n]}$.

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For deterministic and stochastic **Lie–Poisson** systems: explicit splitting schemes can be easily constructed in many cases:

R. I. McLachlan, *Explicit Lie-Poisson integration and the Euler equations*, Phys. Rev. Lett., 1993.

Splitting scheme 1

Consider stochastic Poisson systems of class 1:

$$dy(t) = B(y(t))\nabla H(y(t))dt + \sum_{k=1}^{m} B(y(t))\nabla \widehat{H}_k(y(t)) \circ dW_k(t)$$

Splitting of the Hamiltonian function: $H = \sum_{k=1}^{p} H_k$.

Notation: $\big(\exp(tY_{\mathcal{H}})\big)_{t\in\mathbb{R}}$ is the flow of the ODE $\dot{y} = B(y)\nabla\mathcal{H}(y)$.

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Explicit splitting stochastic integrator:

$$\begin{aligned}\Phi_h(\cdot,\Delta W_1,\ldots,\Delta W_m) = &\exp(hY_{H_p}) \circ \exp(hY_{H_{p-1}}) \circ \ldots \circ \exp(hY_{H_1})\\ &\circ \exp(\Delta W_m Y_{\widehat{H}_m}) \circ \exp(\Delta W_{m-1} Y_{\widehat{H}_{m-1}}) \circ \ldots \circ \exp(\Delta W_1 Y_{\widehat{H}_1}).\end{aligned}$$

This defines a (consistent) stochastic Poisson integrator.

Splitting scheme 2

Consider stochastic Poisson systems of class 2:

$$dy(t) = B(y(t))\nabla H(y(t))dt + B(y(t))\nabla \widehat{H}(y(t)) \circ dW(t)$$

Splitting of Hamiltonian functions: $H = \sum_{k=1}^{p} H_k$ and $\widehat{H} = \sum_{k=1}^{m} \widehat{H}_k$.
The splitting scheme

$$\Phi_h(\cdot, \Delta W) = \exp(hY_{H_p}) \circ \ldots \circ \exp(hY_{H_1}) \circ \exp(\Delta W Y_{\widehat{H}_m}) \circ \ldots \circ \exp(\Delta W Y_{\widehat{H}_1})$$

is not consistent.
Explicit splitting stochastic integrator:

$$\begin{aligned}\Phi_h(\cdot, \Delta W) &= \exp(hY_{H_p}) \circ \exp(hY_{H_{p-1}}) \circ \ldots \circ \exp(hY_{H_1}) \\ &\circ \exp(\frac{\Delta W}{2} Y_{\widehat{H}_1}) \circ \ldots \circ \exp(\Delta W Y_{\widehat{H}_m}) \circ \ldots \circ \exp(\frac{\Delta W}{2} Y_{\widehat{H}_1}).\end{aligned}$$

This defines a (consistent) stochastic Poisson integrator.

Other methods for stochastic Poisson systems

- D. Cohen and G. Dujardin, *Energy-preserving integrators for stochastic Poisson systems*, Commun. Math. Sci., 2014.
- D. Cohen and G. Vilmart, *Drift-preserving numerical integrators for stochastic Poisson systems*, Int. J. Comput. Math., 2021.
- J. Hong, J. Ruan, L. Sun, and L. Wang, *Structure-preserving numerical methods for stochastic Poisson systems*, Commun. Comput. Phys., 2021.
- L. Wang, P. Wang, and Y. Cao, *Numerical methods preserving multiple Hamiltonians for stochastic Poisson systems*, Discrete Contin. Dyn. Syst. S, 2021.
- S. J. A. Malham, and A. Wiese, *Stochastic Lie Group Integrators*, SIAM J. Sci. Comput., 2008.
- E. Luesink, S. Ephrati, P. Cifani, and B. Geurts, *Casimir preserving stochastic Lie-Poisson integrators*, arxiv preprint, 2021.

Convergence analysis for SDEs

General stochastic differential equation:

$$dX(t) = f(X(t))dt + \sigma(X(t))dW(t), \quad X(0) = x_0,$$

and stochastic integrator

$$X^{[n]} = \Phi_h(X^{[n-1]}, \Delta_n W)$$

with time-step size $h = T/N$.

Strong/mean-square error estimates:

$$\left(\mathbb{E}[\|X^{[N]} - X(T)\|^2]\right)^{\frac{1}{2}} \leq C_{T,x_0} h^p.$$

Weak error estimates: for smooth φ

$$\left|\mathbb{E}[\varphi(X^{[N]})] - \mathbb{E}[\varphi(X(T))]\right| \leq C_{T,\varphi,x_0} h^q.$$

One has $q \geq p$, often $q > p$.

Strong convergence

Stochastic Poisson system:

$$dy(t) = B(y(t))\nabla H(y(t))dt + \sum_{k=1}^{m} B(y(t))\nabla \widehat{H}_k(y(t)) \circ dW_k(t)$$

Stochastic splitting Poisson integrator:

$$\Phi_h(\cdot, \Delta W_1, \ldots, \Delta W_m) = \exp(hY_{H_p}) \circ \exp(hY_{H_{p-1}}) \circ \ldots \circ \exp(hY_{H_1}) \\ \circ \exp(\Delta W_m Y_{\widehat{H}_m}) \circ \exp(\Delta W_{m-1} Y_{\widehat{H}_{m-1}}) \circ \ldots \circ \exp(\Delta W_1 Y_{\widehat{H}_1}).$$

Strong convergence result[1]:

- strong order of convergence $p = 1/2$ in general,
- one has $p = 1$ if $m = 1$.

[1] assumptions: existence of a Casimir with compact level sets + regularity

Weak convergence

Stochastic Poisson system:

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Stochastic splitting Poisson integrator:

$$\Phi_h(\cdot, \Delta W_1, \ldots, \Delta W_m) = \exp(hY_{H_p}) \circ \exp(hY_{H_{p-1}}) \circ \ldots \circ \exp(hY_{H_1})$$
$$\circ \exp(\Delta W_m Y_{\widehat{H}_m}) \circ \exp(\Delta W_{m-1} Y_{\widehat{H}_{m-1}}) \circ \ldots \circ \exp(\Delta W_1 Y_{\widehat{H}_1}).$$

Weak convergence result[2]:

- weak order of convergence $q = 1$.

[2] assumptions: existence of a Casimir with compact level sets + regularity

A method with weak order 2

Using a Strang splitting strategy, set

$$y^{[n]} = \Phi_{h,\gamma_n}(y^{[n-1]}) = \Phi_{h/2}^{det,S} \circ \Phi_{h,\gamma_n}^{sto}(\cdot, \Delta_n W_1, \ldots, \Delta_n W_m) \circ \Phi_{h/2}^{det,S}(y^{[n-1]}),$$

with i.i.d. γ_n uniformly distributed in $\{\pm 1\}$, a Strang splitting scheme for the drift part

$$\Phi_{h/2}^{det,S} = \exp(\frac{h}{4} Y_{H_1}) \circ \ldots \circ \exp(\frac{h}{4} Y_{H_{p-1}}) \circ \exp(\frac{h}{2} Y_{H_p}) \circ \exp(\frac{h}{4} Y_{H_{p-1}}) \circ \ldots \circ \exp(\frac{h}{4} Y_{H_1})$$

and a Lie–Trotter splitting scheme for the diffusion part

$$\Phi_{h,\gamma_n}^{sto}(\cdot, \Delta_n W_1, \ldots, \Delta_n W_m) = \begin{cases} \exp(\Delta W_m Y_{\widehat{H}_m}) \circ \ldots \circ \exp(\Delta W_1 Y_{\widehat{H}_1}), & \gamma_n = 1 \\ \exp(\Delta W_1 Y_{\widehat{H}_1}) \circ \ldots \circ \exp(\Delta W_m Y_{\widehat{H}_m}), & \gamma_n = -1 \end{cases}$$

Based on

S. Ninomiya and N. Victoir, *Weak approximation of stochastic differential equations and application to derivative pricing*, Appl. Math. Finance, 2008.

Stochastic Maxwell–Bloch system

$$dy = B(y)\left(\nabla H(y)dt + \sigma_1 \nabla \widehat{H}_1(y) \circ dW_1(t) + \sigma_3 \nabla \widehat{H}_3(y) \circ dW_3(t)\right),$$

where $H(y) = H_1(y) + H_3(y)$, $H_1(y) = \widehat{H}_1(y) = \frac{1}{2}y_1^2$ and $H_3(y) = \widehat{H}_3(y) = y_3$.
Deterministic subsystems:

$$\begin{cases} \dot{y}_1 = 0 \\ \dot{y}_2 = y_3 y_1 \\ \dot{y}_3 = -y_2 y_1 \end{cases} \qquad \begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = 0 \\ \dot{y}_3 = 0 \end{cases}$$

give the flows used in the splitting scheme

$$\exp(tY_{H_1})y = \exp(tY_{\widehat{H}_1})y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(y_1 t) & \sin(y_1 t) \\ 0 & -\sin(y_1 t) & \cos(y_1 t) \end{pmatrix} y$$

$$\exp(tY_{H_3})y = \exp(tY_{\widehat{H}_3})y = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} y$$

Preservation of the Casimir (MB)

Stochastic Poisson integrator:

$$\Phi_h = \exp(hY_{H_3}) \circ \exp(hY_{H_1}) \circ \exp(\sigma_3 \Delta W_3 Y_{\widehat{H}_3}) \circ \exp(\sigma_1 \Delta W_1 Y_{\widehat{H}_1})$$

Casimir function: $C(y) = \frac{1}{2}(y_2^2 + y_3^2)$.

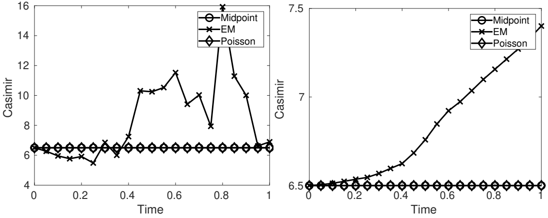

Figure: Stochastic Maxwell–Bloch system: preservation of the Casimir by the explicit stochastic Poisson integrator ($\diamond$). Comparison with the Euler–Maruyama scheme ($\times$) and the midpoint scheme ($\circ$). Left: $(\sigma_1, \sigma_3) = (1, 0)$. Right: $(\sigma_1, \sigma_3) = (0, 1)$.

Strong convergence (MB), $m = 1$

Stochastic Poisson integrator:

$$\Phi_h = \exp(hY_{H_3}) \circ \exp(hY_{H_1}) \circ \exp(\sigma_3 \Delta W_3 Y_{\widehat{H}_3}) \circ \exp(\sigma_1 \Delta W_1 Y_{\widehat{H}_1})$$

With $m = 1$: order $p = 1$

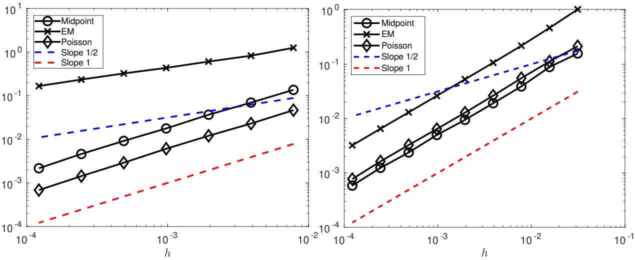

Figure: Stochastic Maxwell–Bloch system with a single Wiener process: Strong errors of the Euler–Maruyama scheme ($\times$), the midpoint scheme ($\circ$), and the explicit stochastic Poisson integrator ($\diamond$). Left: $(\sigma_1, \sigma_3) = (1, 0)$. Right: $(\sigma_1, \sigma_3) = (0, 1)$.

Strong convergence (MB), $m = 2$

Stochastic Poisson integrator:

$$\Phi_h = \exp(hY_{H_3}) \circ \exp(hY_{H_1}) \circ \exp(\sigma_3 \Delta W_3 Y_{\widehat{H}_3}) \circ \exp(\sigma_1 \Delta W_1 Y_{\widehat{H}_1})$$

With $m = 2$: order $p = 1/2$

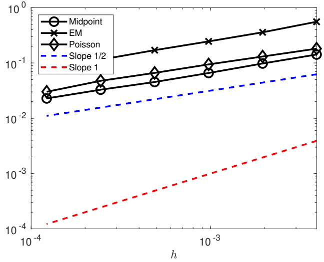

Figure: Stochastic Maxwell–Bloch system with $(\sigma_1, \sigma_3) = (1, 1)$: Strong errors of the Euler–Maruyama scheme ($\times$), the midpoint scheme ($\circ$), and the explicit stochastic Poisson integrator ($\diamond$).

Weak convergence (MB)

Stochastic Poisson integrator:

$$\Phi_h = \exp(hY_{H_3}) \circ \exp(hY_{H_1}) \circ \exp(\sigma_3 \Delta W_3 Y_{\widehat{H}_3}) \circ \exp(\sigma_1 \Delta W_1 Y_{\widehat{H}_1})$$

Order $q = 1$ (left) vs $q = 2$ (right).

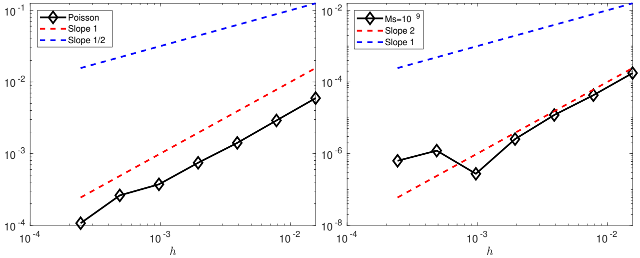

Figure: Stochastic Maxwell–Bloch system. Left: Weak error of the explicit stochastic Poisson integrator. Right: Weak error for the second-order scheme. $\varphi(y) = \sin(2\pi y_1) + \sin(2\pi y_2) + \sin(2\pi y_3)$.

Stochastic rigid body system

$$dy = B(y)\left(\nabla H(y)dt + \sigma_1 \nabla \widehat{H}_1(y) \circ dW_1(t) + \sigma_3 \nabla \widehat{H}_3(y) \circ dW_3(t)\right)$$

with $\widehat{H}_k(y) = \frac{y_k^2}{\widehat{I}_k}$, $H(y) = H_1(y) + H_2(y) + H_3(y)$, $H_k(y) = \frac{y_k^2}{I_k}$. Examples of subsystems $\dot{y} = B(y)\nabla H_1(y)$ and $\dot{y} = B(y)\nabla \widehat{H}_1(y)$

$$\begin{cases} \dot{y}_1 = 0 \\ \dot{y}_2 = y_1 y_3 / I_1 \\ \dot{y}_3 = -y_1 y_2 / I_1. \end{cases} \qquad \begin{cases} \dot{y}_1 = 0 \\ \dot{y}_2 = y_1 y_3 / \widehat{I}_1 \\ \dot{y}_3 = -y_1 y_2 / \widehat{I}_1. \end{cases}$$

gives the flows

$$\exp(tY_{H_1})y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\frac{y_1 t}{I_1}) & \sin(\frac{y_1 t}{I_1}) \\ 0 & -\sin(\frac{y_1 t}{I_1}) & \cos(\frac{y_1 t}{I_1}) \end{pmatrix} y$$

$$\exp(\Delta W_1 Y_{\widehat{H}_1})y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\frac{y_1 \Delta W_1}{\widehat{I}_1}) & \sin(\frac{y_1 \Delta W_1}{\widehat{I}_1}) \\ 0 & -\sin(\frac{y_1 \Delta W_1}{\widehat{I}_1}) & \cos(\frac{y_1 \Delta W_1}{\widehat{I}_1}) \end{pmatrix} y$$

Preservation of the Casimir (RB)

Casimir function $C(y) = y_1^2 + y_2^2 + y_3^2$.

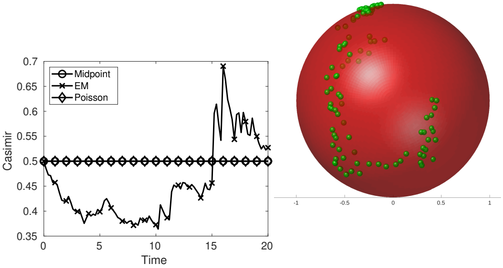

Figure: Stochastic rigid body system: Qualitative behaviour of the Euler–Maruyama scheme ($\times$), the midpoint scheme ($\circ$), and the explicit stochastic Poisson integrator ($\diamond$). Left: preservation of the Casimir. Right: trajectory on the sphere for the splitting scheme.

Strong and weak convergence (RB)

Strong order $p = 1/2$ ($m = 3$)

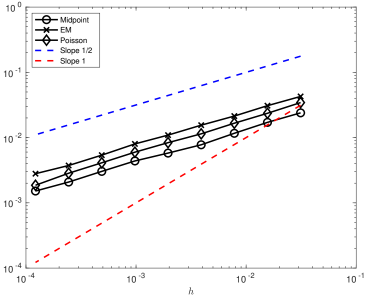

Figure: Stochastic rigid body system: Strong errors for the Euler–Maruyama scheme ($\times$), the midpoint scheme ($\circ$), and the explicit stochastic Poisson integrator ($\diamond$).

Strong and weak convergence (RB)

Weak order $q = 1$ vs $q = 2$

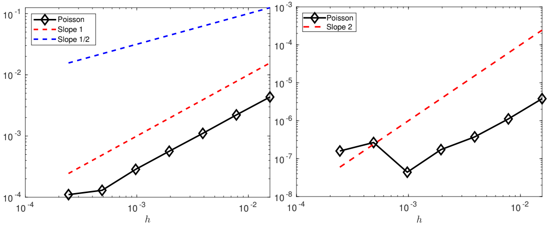

Figure: Stochastic rigid body system. Left: Weak error for the explicit stochastic Poisson integrator. Right: Weak error for the second-order scheme. $\varphi(y) = \sin(2\pi y_1) + \sin(2\pi y_2) + \sin(2\pi y_3)$.

The saturation of the error on the right is due to Monte Carlo statistical error.

Diffusion approximation

The stochastic Poisson system

$$dy(t) = B(y(t))\nabla H(y(t))dt + \sum_{k=1}^{m} B(y(t))\nabla \widehat{H}_k(y(t)) \circ dW_k(t)$$

can be obtained as a limit when $\epsilon \to 0$ of the solution y^ϵ of the multiscale SDE

$$\begin{cases} dy^\epsilon(t) = B(y^\epsilon(t))\nabla H(y^\epsilon(t))dt + \sum_{k=1}^{m} B(y^\epsilon(t))\nabla \widehat{H}_k(y^\epsilon(t))\dfrac{\xi_k^\epsilon(t)}{\epsilon}dt, \\ d\xi_k^\epsilon(t) = -\dfrac{\xi_k^\epsilon(t)}{\epsilon^2}dt + \dfrac{1}{\epsilon}dW_k(t), \quad k = 1, \ldots, m, \end{cases}$$

Preservation of Casimir: $C(y^\epsilon(t)) = C(y^\epsilon(0))$.
Letting $\epsilon \to 0$: gives Stratonovich interpretation of the noise.

Asymptotic preserving schemes

Using a splitting strategy: multiscale stochastic integrator

$$y^{\epsilon,[n]} = \exp(hY_{H_p}) \circ \ldots \circ \exp(hY_{H_1})$$
$$\circ \exp\left(\frac{h\xi_m^{\epsilon,[n]}}{\epsilon} Y_{\widehat{H}_m}\right) \circ \ldots \circ \exp\left(\frac{h\xi_1^{\epsilon,[n]}}{\epsilon} Y_{\widehat{H}_1}\right) (y^{\epsilon,[n-1]}),$$

where for $k = 1, \ldots, m$

$$\xi_k^{\epsilon,[n]} = \xi_k^{\epsilon,[n-1]} - \frac{h}{\epsilon^2}\xi_k^{\epsilon,[n]} + \frac{\Delta_n W_k}{\epsilon}.$$

Asymptotic preserving schemes

Using a splitting strategy: multiscale stochastic integrator

$$y^{\epsilon,[n]} = \exp(hY_{H_p}) \circ \ldots \circ \exp(hY_{H_1})$$
$$\circ \exp\left(\frac{h\xi_m^{\epsilon,[n]}}{\epsilon} Y_{\widehat{H}_m}\right) \circ \ldots \circ \exp\left(\frac{h\xi_1^{\epsilon,[n]}}{\epsilon} Y_{\widehat{H}_1}\right) (y^{\epsilon,[n-1]}),$$

where for $k = 1, \ldots, m$

$$\xi_k^{\epsilon,[n]} = \xi_k^{\epsilon,[n-1]} - \frac{h}{\epsilon^2}\xi_k^{\epsilon,[n]} + \frac{\Delta_n W_k}{\epsilon}.$$

One has $y^{\epsilon,[n]} \underset{\epsilon\to 0}{\to} y^{[n]}$ given by the stochastic Poisson integrator

$$y^{[n]} = \exp(hY_{H_p}) \circ \ldots \circ \exp(hY_{H_1})$$
$$\circ \exp(\Delta_n W_m Y_{\widehat{H}_m}) \circ \ldots \circ \exp(\Delta_n W_1 Y_{\widehat{H}_1})(y^{[n-1]}).$$

Asymptotic preserving schemes

Limits $h, \epsilon \to 0$ can be taken in any order:

$$\begin{array}{ccc} y^{\epsilon,[N]} & \xrightarrow{h\to 0} & y^{\epsilon}(T) \\ \big\downarrow \epsilon\to 0 & & \big\downarrow \epsilon\to 0 \\ y^{[N]} & \xrightarrow{h\to 0} & y(T) \end{array}$$

Such schemes are asymptotic preserving.

C.-E. Bréhier & S. Rakotonirina-Ricquebourg.
On Asymptotic Preserving schemes for a class of Stochastic Differential Equations in averaging and diffusion approximation regimes.
SIAM MMS, 2022.

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Further question: uniform (weak) error estimates?

$$\sup_{\epsilon\in(0,\epsilon_0)} \left|\mathbb{E}[\varphi(y^{\epsilon,[N]})] - \mathbb{E}[\varphi(y^{\epsilon}(T))]\right| \leq C_{T,\varphi,x_0} h^{\alpha}$$

Diffusion approximation for the stochastic rigid body system

Illustration of the convergence when $\epsilon \to 0$ (one realization)

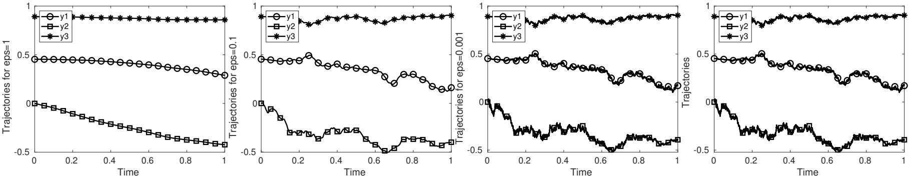

Figure: Stochastic rigid body system: Trajectory of the numerical solutions using the asymptotic preserving scheme ($\epsilon = 1, 0.1, 0.001$) and the stochastic Poisson integrator ($\epsilon = 0$). $h = 10^{-4}$.

Uniform weak error estimates

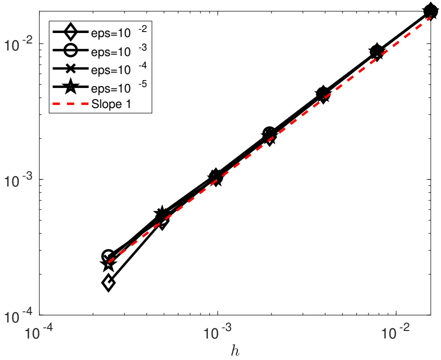

Figure: Stochastic rigid body system: Weak errors of the asymptotic preserving scheme for $\epsilon = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$.

Conclusion

A class of stochastic Poisson integrators, using explicit splitting schemes, for stochastic Lie–Poisson systems has been introduced. One has

- preservation of Casimir,
- strong and weak error estimates,
- asymptotic preserving schemes (diffusion approximation).

Applications: stochastic Maxwell–Bloch, rigid body and sine–Euler systems.

- confirmation of the qualitative and quantitative results,
- superiority compared with other methods.

Reference:

C.-E. Bréhier, D. Cohen, and T. Jahnke, *Splitting integrators for stochastic Lie–Poisson systems*, arxiv preprint 2111.07387.

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Thanks for your attention.