Formulation relativiste de l'hyper-élasticité d'après J.-M. Souriau

Rencontre thématique du GDR-GDM 2043 à Saclay
23–25 novembre 2022

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Jeudi 24 Novembre 2022

B. Kolev and R. Desmorat (2022). Relativistic general covariant formulation of hyperelasticity after J.M. Souriau, hal-03861444.
https://hal.archives-ouvertes.fr/hal-03861444

The source of this talk is a paper (1958) by Souriau

Alger, Mathématiques.
Tome V, nº 2, 1958.

Jean-Marie SOURIAU

LA RELATIVITÉ VARIATIONNELLE

Sommaire. — Nous proposons une théorie unitaire, compatible avec la Relativité Générale, fondée sur un principe variationnel qui généralise celui d'Hamilton.

Après des préliminaires mathématiques (§ 1), l'énoncé des principes de la théorie (§ 2), et la démonstration des « théorèmes généraux » (§ 3), nous étudions divers phénomènes : la gravitation (§ 4), la matière parfaite, qui généralise la théorie des fluides parfaits relativistes de LICHNEROWICZ (§ 5), la lumière et la matière électrisée (§ 6).

Diverses applications sont données en cours de route : interprétation du tenseur d'impulsion-énergie, solide relativiste, antimatière et dématérialisation, équations non linéaires de la lumière et des particules de spin 1, notion d'interaction, etc.

La valeur explicative de la théorie est discutée dans le § 7.

- After not easy to read mathematical preliminaries, he proposes a complete and detailed formulation of Hyperelasticity (matière parfaite) in General Relativity.
- His mindset is clearly Gauge theory, which is the common mathematical framework of High Energy Physics and Quantum Mechanics.
- Most of this work has since been rediscovered (with or without some gaps) by many authors (Carter–Quintana, 1972, Kijowski–Magli, 1992, 1997, Beig–Schmidt, 2003), but Souriau is almost never cited.

OUTLINE

1. Geometric frameworks for Continuum Mechanics
2. Lagrangian formulation of Relativistic Hyperelasticity
3. Spacetime structures
4. Relativistic Hyperelasticity in Schwarzschild spacetime
5. 4D Newton-Cartan Galilean limit of Relativistic Hyperelasticity
6. Conclusion

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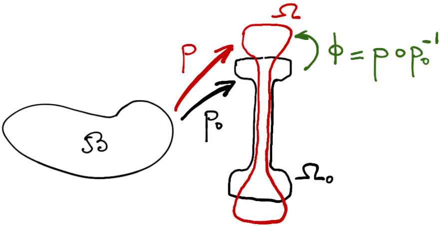
3D formalism for Continuum Mechanics

Truesdell and Noll (1965)

- A configuration is an embedding $p : (\mathscr{B}, \mu) \to (\mathscr{E}, q)$, where
  - $\mathscr{B}$ (the body) is a 3D compact oriented manifold (with boundary),
  - μ is a volume form (the mass measure),
  - $\mathscr{E}$ is the ambiant 3-space,
  - q is the Euclidean metric.
- By pull-back, we get a Riemannian metric $\gamma = p^*q$ on $\mathscr{B}$
- By push-forward, we get a mass measure $p_*\mu$ on $\Omega_p = p(\mathscr{B})$. The mass density ρ is defined by $p_*\mu = \rho\,\mathrm{vol}_q$.



3D HYPERELASTICITY

NOLL (1972, 1978), ROUGÉE (1991, 2006)

Hyperelasticity is then formulated by the definition of an elastic energy

$$W(\gamma) := \int_{\mathscr{B}} \psi(\gamma)\mu, \qquad \boldsymbol{\theta} = 2\frac{\partial \psi}{\partial \gamma},$$

where $\boldsymbol{\theta} = \phi^*(\boldsymbol{\sigma}/\rho)$ and $\boldsymbol{\sigma}$ is the Cauchy stress tensor.

Hyperelasticity on a reference configuration

In Mechanics, one usually prefers to work on a reference configuration $\Omega_0 = p_0(\mathscr{B})$, rather than on the abstract body $\mathscr{B}$ itself, and using $\phi := p \circ p_0^{-1}$ rather than p itself. Then,

- $\gamma \sim \mathbf{C} = \phi^* q$ (Right Cauchy–Green tensor)
- $\boldsymbol{\theta} \sim \mathbf{S} = \phi^*(\boldsymbol{\sigma}/\rho)$ (Second Piola–Kirchhoff tensor)
- $\boldsymbol{\theta} = 2\dfrac{\partial \psi}{\partial \gamma} \sim \mathbf{S} = 2\dfrac{\partial \psi}{\partial \mathbf{C}}$.

4D formalism for Continuum Mechanics

Souriau (1958–1964)

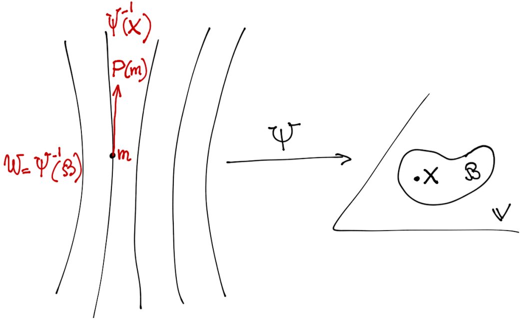
- The modeling of perfect matter adopted by Souriau is inspired by Gauge theory, where matter fields are described by sections of some vector bundle (here a trivial vector bundle).
- A perfect matter field is a smooth vector valued function

$$\Psi : \mathscr{M} \to V = \mathbb{R}^3,$$

  where $\mathscr{M}$ is the Universe, a four dimensional manifold, endowed with a Lorentzian metric g.
- The notation Ψ for the matter field is on purpose: Ψ is the wave function in Quantum Mechanics.

The body, the mass measure and the World tube

- A continuous medium is then described by a 3D compact orientable manifold with boundary $\mathscr{B} \subset \mathbb{R}^3$, the body, which labels the particles and is endowed with a volume form μ, the mass measure.
- It is further assumed that $T_m\Psi$ is of rank 3 at each point m of $\Psi^{-1}(\mathscr{B})$. Thus, $\mathscr{W} := \Psi^{-1}(\mathscr{B})$ is fibered by the particles World lines $\Psi^{-1}(\mathbf{X})$, $\mathbf{X} \in \mathscr{B}$, and is called for this reason the body's World tube.



A point of view reverse to 3D formalism

- In 3D formalism, a configuration is an embedding

$$p\colon \mathscr{B} \to \mathscr{E}$$

from the 3D body $\mathscr{B}$ to the 3D space $\mathscr{E}$.

- In the present 4D formalism, the main concept is a mapping

$$\Psi\colon \mathscr{M} \to \mathscr{B}$$

from the 4D Universe $\mathscr{M}$ to the 3D body $\mathscr{B}$.

A key difference is that, in Classical Continuum Mechanics, the embedding p and its tangent map

$$\mathbf{F} = Tp\colon T\mathscr{B} \to T\mathscr{E}$$

are invertible, whereas here, the matter field Ψ and its tangent map $T\Psi$ are not.

Current of matter

- The pullback by Ψ of the mass measure μ on the body $\mathscr{B}$, $\Psi^*\mu$, is a 3-form. Since $T_m\Psi$ is assumed to be of rank 3 at each point of $\mathscr{W}$, there exists a nowhere vanishing (quadri-)vector field $\mathbf{P}$ on $\mathscr{W}$, such that

$$\Psi^*\mu = i_{\mathbf{P}}\mathrm{vol}_g,$$

  where $i_{\mathbf{P}}$ means the contraction of $\mathbf{P}$ with vol_g. This vector field $\mathbf{P}$ is the current of matter.

- To describe perfect matter, Souriau assumes furthermore that $\mathbf{P}$ is timelike,

$$\|\mathbf{P}\|_g^2 = g(\mathbf{P}, \mathbf{P}) < 0 \quad \text{on the World tube } \mathscr{W}$$

Remark

At each point $m \in \mathscr{W}$, the tangent vector $\mathbf{P}(m)$ spans the one-dimensional subspace $\ker T_m\Psi$:

$$T_m\Psi.\mathbf{P}(m) = 0,$$

Relativistic mass conservation

Lemma (Souriau, 1958)

$$\operatorname{div}^g \mathbf{P} = 0.$$

- We can write
$$\mathbf{P} = \rho_r \mathbf{U}, \quad \text{with} \quad \|\mathbf{U}\|_g^2 = -1.$$
- This defines, on the World tube $\mathscr{W}$, the rest mass density
$$\rho_r := \sqrt{-\|\mathbf{P}\|_g^2},$$

The equation
$$\operatorname{div}^g \mathbf{P} = \operatorname{div}^g(\rho_r \mathbf{U}) = 0$$
is then considered as the relativistic mass conservation (more precisely, conservation of number density of idealized particles of the medium).

Conformation

The conformation $\mathbf{H}$ has been introduced by Souriau in 1958.
It is the cornerstone of the formulation of Relativistic Hyperelasticity at large scale, such as in the modeling of neutron stars (with a solid crust).

Definition

The conformation is the vector-valued function

$$\mathbf{H} : \mathscr{M} \to \mathbb{S}^2(V), \qquad m \mapsto \mathbf{H}(m) := (T_m\Psi)\, g_m^{-1} (T_m\Psi)^\star .$$

- For each $m \in \mathscr{W}$, $\mathbf{H}(m)$ is a positive definite quadratic form on $V^\star$ (consequence of $\mathbf{U}$ timelike).
- Since the mapping $\Psi : \mathscr{W} \to \mathscr{B}$ is not invertible,
  $\mathbf{H}$ is not the pushforward of g^{-1}.

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Variational Relativity

- Souriau has proposed a clear and detailed formulation of Hyperelasticity in the framework of General Relativity, which is part of his Variational Relativity.
- This formulation is inspired by Gauge Theory formulation of General Relativity (Palatini's Method).
- His approach consists in adding Lagrangians *i.e.*, functionals depending on tensorial fields:
  - the metric g,
  - Gauge potentials $\mathbf{A}$, Γ,...
  - matter fields $\Psi, \dots,$

  each of them describing a physical phenomena, and looking for critical points of the total Lagrangian (Principle of Least Action).

EINSTEIN'S EQUATION IN THE VACUUM

- The starting point is the Hilbert-Einstein functional

$$\mathscr{H}(g) = \int (aR_g + b)\,\mathrm{vol}_g, \qquad \begin{cases} \kappa = \frac{8\pi G}{c^4} = \frac{1}{2a}, \\ \Lambda = -\frac{b}{2a}. \end{cases}$$

- The L^2-gradient of $\mathscr{H}$ is the symmetric second order covariant tensor field

$$\mathrm{grad}\,\mathscr{H} = a\,\mathbf{Ric}_g - \frac{1}{2}(a\,R_g + b)g = \frac{1}{2\kappa}\left(\mathbf{G}_g + \Lambda g\right),$$

where $\mathbf{G}_g := \mathbf{Ric}_g - \frac{1}{2}R_g\,g$ is the Einstein tensor.

- The critical points of $\mathscr{H}$ are the solutions of Einstein's equation in the vacuum (with cosmological constant)

$$\mathbf{G}_g + \Lambda g = 0.$$

Einstein's equation in the matter

- To introduce the presence of matter in this framework, a functional $\mathscr{L}^m(g, \Psi)$ (depending of the matter field Ψ), is added to $\mathscr{H}$ to build a new Lagrangian

$$\mathscr{L}(g, \Psi) = \mathscr{H}(g) + \mathscr{L}^m(g, \Psi).$$

- Following Souriau (1958, 1964), for Relativistic continua, one assumes that the Lagrangian for perfect matter $\mathscr{L}^m(g, \Psi)$ depends only on the 0-jet of the metric g and of the 1-jet of the matter field Ψ:

$$\mathscr{L}^m(g, \Psi) = \int L^m \left(g_{\mu\nu}, \Psi^I, \frac{\partial \Psi^I}{\partial x^\mu} \right) \mathrm{vol}_g.$$

- The function L^m is called the Lagrangian density of the functional $\mathscr{L}^m$.

THE STRESS-ENERGY TENSOR

EINSTEIN–HILBERT, 1915

- The Euler-Lagrange stationary equation $\delta \mathscr{L} = 0$ leads to

$$\frac{\delta \mathscr{H}}{\delta g} + \frac{\delta \mathscr{L}^m}{\delta g} = 0,$$

- It recasts as the Einstein field equation

$$\mathbf{G}_g^\sharp + \Lambda g^{-1} = \kappa \mathbf{T},$$

where $\mathbf{G}_g^\sharp = g^{-1} \mathbf{G}_g g^{-1}$, and

$$\mathbf{T} := -2 \frac{\delta \mathscr{L}^m}{\delta g},$$

is the stress-energy tensor (the source term in Einstein's equation).

- $\mathbf{T}$ is a symmetric contravariant second-order tensor field.
It can be considered as a 4D generalization of the stress tensor of 3D Classical Continuum Mechanics.

General Covariance

The main postulate of General Relativity is precisely that
Physical laws must be independent of the choice of coordinates.

- This means that the Lagrangian $\mathscr{L}$ must be invariant by a (local) diffeomorphism φ, *i.e.*,

$$\mathscr{L}(\varphi^* g, \varphi^* \Psi) = \mathscr{L}(g, \Psi),$$

  where

$$\varphi^* g = (T\varphi)^\star (g \circ \varphi)(T\varphi), \quad \text{and} \quad \varphi^* \Psi = \Psi \circ \varphi.$$

- There is also an equivalent infinitesimal formulation of general covariance (Noether, 1918), which is given by

$$\frac{\delta \mathscr{L}}{\delta g} . \mathrm{L}_X \, g + \frac{\delta \mathscr{L}}{\delta \Psi} . \mathrm{L}_X \, \Psi = 0.$$

General covariance of Hilbert-Einstein functional

- Hilbert-Einstein's functional $\mathscr{H}$ is general covariant

$$\mathscr{H}(\varphi^* g) = \int (aR_{\varphi^* g} + b) \operatorname{vol}_{\varphi^* g} = \int \varphi^*[(aR_g + b) \operatorname{vol}_g] = \mathscr{H}(g).$$

- A direct consequence of this invariance is the fundamental property that

$$\operatorname{div}^g (\mathbf{G}_g + \Lambda g) = \operatorname{div}^g \mathbf{G}_g = 0,$$

and thus that the stress-energy tensor $\mathbf{T}$ (verifying Einstein equation $\mathbf{G}_g^\sharp + \Lambda g^{-1} = \kappa \mathbf{T}$) satisfies the conservation law

$$\operatorname{div}^g \mathbf{T} = 0.$$

- As observed by Einstein himself: "$\operatorname{div} \mathbf{T} = 0$, that's mechanics".

Theorem (Souriau (1958))

Suppose that the Lagrangian

$$\mathscr{L}^m(g, \Psi) = \int L_0(g_m, \Psi(m), T_m\Psi)\, \mathrm{vol}_g$$

is general covariant. Then, its Lagrangian density can be written as

$$L^m(g, \Psi, T\Psi) = L(\Psi, \mathbf{H}),$$

for some function L, where $\mathbf{H} = (T\Psi)\, g^{-1} (T\Psi)^\star$ *is the conformation.*

In Classical Continuum Mechanics, the energy density depends on the deformation ϕ only through the right Cauchy–Green tensor $\mathbf{C} = \phi^* q$. Here, $\mathbf{H}$ plays the role of $\mathbf{C}^{-1}$.

A stress-energy tensor for perfect matter

The following splitting has been introduced by Souriau (1958, 1964) and De Witt (1962),

$$L(\Psi, \mathbf{H}) = \rho_r c^2 + E(\Psi, \mathbf{H}) = \rho_r c^2 + \rho_r e(\Psi, \mathbf{H}),$$

where ρ_r is the rest mass density, and E is the internal energy density. Then,

$$\mathbf{T} = -2\frac{\delta \mathscr{L}^m}{\delta g} = \rho_r c^2 \mathbf{U} \otimes \mathbf{U} - \mathbf{S} = L\, \mathbf{U} \otimes \mathbf{U} - \mathbf{\Sigma},$$

where

$$\mathbf{S} := E\, g^{-1} - 2g^{-1}(T\Psi)^\star \frac{\partial E}{\partial \mathbf{H}}(T\Psi)g^{-1}, \qquad \mathbf{S}.\mathbf{U}^\flat = E\,\mathbf{U},$$

$$\mathbf{\Sigma} := -2\rho_r\, g^{-1}(T\Psi)^\star \frac{\partial e}{\partial \mathbf{H}}(T\Psi)g^{-1}, \qquad \mathbf{\Sigma}.\mathbf{U}^\flat = 0,$$

correspond to two choices of a (4D) relativistic stress tensor.

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Time function and spacelike hypersurfaces

Introduction of an observer (Arnowitt et al, 1962, York, 1979, Gourgoulhon, 2012)

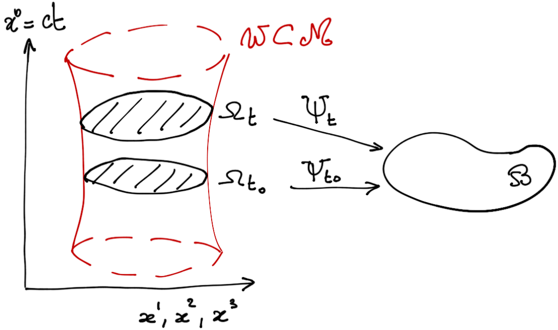
- There is no Mechanics without the proper definition of time and space.
- In General Relativity, one introduces a time function $\hat{t}$ on the Universe $\mathscr{M}$ with a timelike gradient everywhere.
- This defined a foliation (a spacetime structure) of $\mathscr{M}$ by spacelike hypersurfaces

$$\Omega_t := \left\{ m \in \mathscr{M};\ \hat{t}(m) = t \right\},$$

- When a spacetime is defined by one chart and coordinates (x^μ), the time function is chosen as $\hat{t}(m) = x^0(m)/c$, and we set

$$\Omega_t := \mathscr{W} \cap \left\{ x^0(m) = ct \right\}.$$

Foliation of $\mathscr{W}$ by spacelike hypersurfaces Ω_t



The 3D submanifolds Ω_t play the role of the configuration manifolds $\Omega = \Omega_{p(t)}$ of Classical Continuum Mechanics.
The restriction $\Psi_t : \Omega_t \to \mathscr{B}$ is not necessarily a diffeomorphism but its differential $T\Psi_t$ is always a linear isomorphism.

ORTHOGONAL DECOMPOSITION OF A SPACETIME

- The unit normal (future-oriented) to the hypersurfaces Ω_t is

$$\mathbf{N} := -\frac{\operatorname{grad} \hat{t}}{\sqrt{-\left\|\operatorname{grad} \hat{t}\right\|^2}}.$$

- Given a spacetime structure on $\mathscr{M}$, we have thus, at each point $m \in \mathscr{M}$, the orthogonal decomposition

$$T_m\mathscr{M} = \langle \mathbf{N}(m) \rangle \oplus T_m\Omega_t,$$

For mechanicians, it is worth to notice that this orthogonal decomposition is very similar to the one used in Thick Shell theory.

ORTHOGONAL DECOMPOSITION OF $\mathbf{U}$

Consider that

- perfect matter is present and represented by a matter field Ψ, defining a unit timelike quadrivector $\mathbf{U}$ (with $\mathbf{P} = \rho_r \mathbf{U}$),
- a spacetime structure is defined on the Universe $\mathscr{M}$, with unit normal $\mathbf{N}$, which we can assume defines the same time orientation as $\mathbf{U}$, i.e.

$$\langle \mathbf{U}, \mathbf{N} \rangle_g < 0.$$

Then, the orthogonal decomposition of $\mathbf{U}$ is written as

$$\mathbf{U} = \mathbf{U}^N + \mathbf{U}^\top, \qquad \mathbf{U}^N = -\langle \mathbf{U}, \mathbf{N} \rangle_g \mathbf{N},$$

- $\mathbf{U}^N$: normal component of $\mathbf{U}$,
- $\mathbf{U}^\top \in T\Omega_t$: tangential component of $\mathbf{U}$.

Generalized Lorentz factor and spatial velocity

- Introducing

$$\gamma := -\langle \mathbf{U}, \mathbf{N} \rangle_g$$

and setting $\mathbf{U}^\top = \gamma \boldsymbol{u}/c$, we can write

$$\mathbf{U} = \gamma \left(\mathbf{N} + \frac{\boldsymbol{u}}{c} \right),$$

- $\gamma := -\langle \mathbf{U}, \mathbf{N} \rangle_g$ is called the generalized Lorentz factor, since

$$\gamma = 1 \Big/ \sqrt{1 - \frac{\|\boldsymbol{u}\|_g^2}{c^2}}, \quad \text{because} \quad \|\mathbf{U}\|_g^2 = -1.$$

- The spatial velocity $\boldsymbol{u}$ can be expressed on Ω_t as

$$\boldsymbol{u} = -c\, \mathbf{F}\, T\Psi.\mathbf{N} \quad \text{on} \quad \Omega_t,$$

where $\mathbf{F}$ is the inverse of $\mathbf{F}^{-1} = T\Psi_t$, the restriction of $T\Psi$ on Ω_t.

Orthogonal decomposition of $\mathbf{T}$

Stress-energy tensor $\mathbf{T}$

$$\mathbf{T} = E_{\text{tot}}\, \mathbf{N} \otimes \mathbf{N} + \frac{1}{c}\left(\mathbf{N} \otimes \boldsymbol{p} + \boldsymbol{p} \otimes \mathbf{N}\right) + \mathbf{s}.$$

- E_{tot} is the total energy density,
- $\boldsymbol{p}$ is the momentum density vector field,
- and $\mathbf{s}$ is the spatial part of $\mathbf{T}$ (related to the stress field).

Following Eckart (1940) and Souriau (1958), there exists 2 ways to define a spatial stress $\boldsymbol{\sigma}$, which generalizes the 3D stress, using either $\mathbf{S}$ or $\boldsymbol{\Sigma}$. They provide two alternative Relativistic Hyperelasticity laws.

First modeling choice

The 3D stress tensor $\boldsymbol{\sigma}$ is defined as the spatial part of

$$\mathbf{S} = -2\rho_r\, g^{-1}(T\Psi)^\star \frac{\partial e}{\partial \mathbf{H}}(T\Psi)g^{-1} - E\,\mathbf{U}\otimes\mathbf{U} \qquad \big(\text{such as } \mathbf{S}.\mathbf{U}^\flat = E\,\mathbf{U}\big),$$

and is given by

$$\boldsymbol{\sigma} := -\frac{2}{\gamma}\rho\,(g^{3D})^\sharp \mathbf{F}^{-\star}\frac{\partial e}{\partial \mathbf{H}}\mathbf{F}^{-1}(g^{3D})^\sharp - \frac{\gamma^2 E}{c^2}\,\boldsymbol{u}\otimes\boldsymbol{u},$$

Associated (3+1)-decomposition of $\mathbf{T} = \rho_r c^2\mathbf{U}\otimes\mathbf{U} - \mathbf{S}$

$$\begin{cases} E_{\text{tot}} = \gamma\rho c^2 + E\Big(1 + \frac{1}{c^2}\left\|\boldsymbol{u}\right\|^2\Big) - \frac{1}{c^2}\boldsymbol{u}^\flat\cdot\boldsymbol{\sigma}\cdot\boldsymbol{u}^\flat, \\ \boldsymbol{p} = (\gamma\rho c^2 + E)\boldsymbol{u} - \boldsymbol{\sigma}\cdot\boldsymbol{u}^\flat, \\ \mathbf{s} = \gamma\rho\,\boldsymbol{u}\otimes\boldsymbol{u} - \boldsymbol{\sigma}, \end{cases}$$

where $\rho = \gamma\rho_r$ is the relativistic mass density.

Second modeling choice

The 3D stress tensor $\boldsymbol{\sigma}$ is defined as the spatial part of

$$\mathbf{\Sigma} = -2\rho_r\, g^{-1}(T\Psi)^{\star}\frac{\partial e}{\partial \mathbf{H}}(T\Psi)g^{-1} \qquad \big(\text{such as } \mathbf{\Sigma}.\mathbf{U}^{\flat} = 0\big),$$

and is given by

$$\boldsymbol{\sigma} := -\frac{2}{\gamma}\rho\,(g^{3D})^{\sharp}\mathbf{F}^{-\star}\frac{\partial e}{\partial \mathbf{H}}\mathbf{F}^{-1}(g^{3D})^{\sharp}.$$

Associated (3+1)-decomposition of $\mathbf{T} = L\mathbf{U}\otimes\mathbf{U} - \mathbf{\Sigma}$

$$\begin{cases} E_{\text{tot}} = \gamma^2 L - \frac{1}{c^2}\boldsymbol{u}^{\flat}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{u}^{\flat}, \\ \boldsymbol{p} = \gamma^2 L\,\boldsymbol{u} - \boldsymbol{\sigma}\cdot\boldsymbol{u}^{\flat}, \\ \mathbf{s} = \frac{\gamma^2}{c^2}L\,\boldsymbol{u}\otimes\boldsymbol{u} - \boldsymbol{\sigma}, \end{cases}$$

where $\gamma^2 L = \gamma^2\left(\rho_r c^2 + E\right) = \gamma\rho c^2 + E\big(1 + \frac{\gamma^2}{c^2}\left\|\boldsymbol{u}\right\|^2\big)$.

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RELATIVISTIC HYPERELASTICITY IN SCHWARZSCHILD SPACETIME

We will extend Souriau's results on Relativistic Hyperelasticity (in flat Minkowski spacetime) by taking into account gravity.

We neglect the influence of the continuous medium/the structure under study (represented by the matter field Ψ) as a source of the gravity field.

- The exterior Schwarzschild metric (1916) is a static solution g of Einstein equation $\mathbf{G}_g = 0$ in the vacuum (with $\Lambda = 0$).
- It is representative of the gravity field around a spherical and nonrotating planet (or a blackhole) of mass M and radius r_0, such as the Earth.
- In this model, the rotation of the Earth as a potential source of the gravitation field has been neglected.

SCHWARZSCHILD METRIC

In the so-called Cartesian isotropic coordinates $(x^0 = ct, x^i)$, with t the time,

$$g = -\left(\frac{1-\bar{r}_s/\bar{r}}{1+\bar{r}_s/\bar{r}}\right)^2 c^2 dt^2 + \left(1+\frac{\bar{r}_s}{\bar{r}}\right)^4 q, \qquad \bar{r} := \sqrt{\delta_{ij}x^i x^j},$$

$$g = -\mathcal{N}^2 c^2 dt^2 + g^{3D}, \qquad \mathcal{N} = \sqrt{-g_{00}} = \frac{1-\bar{r}_s/\bar{r}}{1+\bar{r}_s/\bar{r}},$$

- g^{3D} is the spatial (conformal) metric,

$$g^{3D} = k\,q = k\,\delta_{ij}dx^i dx^j, \qquad k = \left(1+\frac{\bar{r}_s}{\bar{r}}\right)^4.$$

- $\bar{r} = 0$ at the center of the planet, $\bar{r} \approx r_0$ on its surface,
- Reduced Schwarzschild radius ($\bar{r}_s \approx 2$ mm for the Earth):

$$\bar{r}_s := \frac{1}{4} r_s = \frac{GM}{2c^2}.$$

- The Minkowski spacetime is the limit case $M = 0$ (thus $\bar{r}_s = r_s = 0$).

Particularization for Schwarzschild spacetime

- 3D velocity $\boldsymbol{u}$ and Generalized Lorentz factor:

$$\boldsymbol{u} = -\frac{1}{\mathcal{N}}\,\mathbf{F}\,\frac{\partial \Psi}{\partial t}, \qquad \gamma = \frac{1}{\sqrt{1 - k\frac{\boldsymbol{u}^2}{c^2}}},$$

where $\boldsymbol{u}^2 := \|\boldsymbol{u}\|_q^2$ is the Euclidean squared norm and $k = \left(1 + \frac{\bar{r}_s}{\bar{r}}\right)^4$.

- Relativistic mass density: $\rho = \gamma \rho_r$.
- Conformation:

$$\mathbf{H} = \mathbf{F}^{-1}\left(\frac{1}{k} q^{-1} - \frac{1}{c^2}\boldsymbol{u} \otimes \boldsymbol{u}\right)\mathbf{F}^{-\star}.$$

- For the two choices of a 3D stress $\boldsymbol{\sigma}$, in the coordinate system (t, x^i),

$$\mathbf{T} = \begin{pmatrix} \frac{1}{c^2\mathcal{N}^2} E_{\text{tot}} & \frac{1}{c^2\mathcal{N}} \boldsymbol{p}^\star \\ \frac{1}{c^2\mathcal{N}} \boldsymbol{p} & \mathbf{s} \end{pmatrix}.$$

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Particularization for Schwarzschild spacetime

1. When the stress tensor $\boldsymbol{\sigma}$ is defined as the spatial part of $\mathbf{S}$:

$$
\begin{cases}
E_{\text{tot}} = \gamma\rho c^2 + E\left(1 + k\frac{\boldsymbol{u}^2}{c^2}\right) - \frac{1}{c^2}\boldsymbol{u}^\flat \cdot \boldsymbol{\sigma} \cdot \boldsymbol{u}^\flat, \\
\boldsymbol{p} = (\gamma\rho c^2 + E)\boldsymbol{u} - \boldsymbol{\sigma} \cdot \boldsymbol{u}^\flat, \\
\mathbf{s} = \gamma\rho\, \boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{\sigma}, \\
\boldsymbol{\sigma} = -\frac{2}{\gamma k^2}\rho\, q^{-1}\mathbf{F}^{-\star}\frac{\partial e}{\partial \mathbf{H}}\mathbf{F}^{-1}q^{-1} - \frac{\gamma^2 E}{c^2}\boldsymbol{u} \otimes \boldsymbol{u}.
\end{cases}
$$

2. When the stress tensor $\boldsymbol{\sigma}$ is defined as the spatial part of $\boldsymbol{\Sigma}$:

$$
\begin{cases}
E_{\text{tot}} = \gamma\rho c^2 + E\left(1 + k\gamma^2\frac{\boldsymbol{u}^2}{c^2}\right) - \frac{1}{c^2}\boldsymbol{u}^\flat \cdot \boldsymbol{\sigma} \cdot \boldsymbol{u}^\flat, \\
\boldsymbol{p} = \left(\gamma\rho c^2 + E\left(1 + k\gamma^2\frac{\boldsymbol{u}^2}{c^2}\right)\right)\boldsymbol{u} - \boldsymbol{\sigma} \cdot \boldsymbol{u}^\flat, \\
\mathbf{s} = \left(\gamma\rho + \frac{1}{c^2}E\left(1 + k\gamma^2\frac{\boldsymbol{u}^2}{c^2}\right)\right)\boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{\sigma}, \\
\boldsymbol{\sigma} = -\frac{2}{\gamma k^2}\rho\, q^{-1}\mathbf{F}^{-\star}\frac{\partial e}{\partial \mathbf{H}}\mathbf{F}^{-1}q^{-1}.
\end{cases}
$$

PARTICULARIZATION FOR SCHWARZSCHILD SPACETIME

- Conservation law for the current of matter

$$\operatorname{div}^g \mathbf{P} = \frac{1}{\mathcal{N}}\frac{\partial \rho}{\partial t} + \operatorname{div}^{g^{3D}}(\rho \boldsymbol{u}) + \rho \boldsymbol{u}\cdot \mathrm{d}\ln\mathcal{N} = 0.$$

- Conservation laws for the stress-energy tensor $\mathbf{T}$,

$$(\operatorname{div}^g \mathbf{T})^t = \frac{1}{c^2\mathcal{N}^2}\frac{\partial E_{\text{tot}}}{\partial t} + \frac{1}{c^2\mathcal{N}}\operatorname{div}^{g^{3D}}\boldsymbol{p} + 2\frac{\boldsymbol{p}}{c^2\mathcal{N}}\cdot \mathrm{d}\ln\mathcal{N} = 0.$$

$$(\operatorname{div}^g \mathbf{T})^\top = \frac{1}{c^2\mathcal{N}}\frac{\partial \boldsymbol{p}}{\partial t} + \operatorname{div}^{g^{3D}}\mathbf{s} + \mathbf{s}\cdot \mathrm{d}\ln\mathcal{N} + \frac{E_{\text{tot}}}{k}q^{-1}\mathrm{d}\ln\mathcal{N} = 0.$$

Remark

In York (1970) and Gourgoulhon (2012), these balance laws are expressed for the more general case of a non zero shift vector $\beta = g_{0i}\,\mathrm{d}x^i$, in so-called $(3+1)$ formalism in General Relativity.

Outline

1. Geometric frameworks for Continuum Mechanics
2. Lagrangian formulation of Relativistic Hyperelasticity
3. Spacetime structures
4. Relativistic Hyperelasticity in Schwarzschild spacetime
5. 4D Newton-Cartan Galilean limit of Relativistic Hyperelasticity
6. Conclusion

Galilean structure and Newton-Cartan limit

Dixon (1975), Havas (1964), Künzle (1976), Duval–Künzle (1977), Duval (1985)

- A Galilean structure on $\mathscr{M}$ is a pair $(\mathfrak{g}, \theta)$, where
  - $\mathfrak{g}$: a symmetric 2nd-order contravariant tensor of signature $(0, +, +, +)$,
  - θ: a one-form which spans the kernel of $\mathfrak{g}$.

  The structure is integrable if $\mathrm{d}\theta = 0$ (and thus, at least locally, $\theta = \mathrm{d}\hat{t}$).
- A Galilean structure $(\mathfrak{g}, \theta)$ can be obtained as a limit of a one-parameter family of smooth Lorentz metrics $\overset{\lambda}{g}$, such that

$$\overset{\lambda}{g}^{-1} = \mathfrak{g} + \lambda \boldsymbol{\kappa} + O(\lambda^2),$$

  with $\mathfrak{g}$ of signature $(0, +, +, +)$, and $\theta \in \ker \mathfrak{g}$.
- If $\mathrm{d}\theta = 0$, then, ∇^λ converges to a symmetric covariant derivative ∇^0 (Künzle, 1976), with $\nabla \mathfrak{g} = 0$ and $\nabla \theta = 0$.

Newton-Cartan limit of Schwarzschild metric

$$\overset{\lambda}{g} = -\mathscr{N}^2 c^2 \mathrm{d}t^2 + k\, q, \qquad \lambda := \frac{1}{c^2}.$$

where

$$\mathscr{N} = 1 - \frac{1}{c^2}\frac{GM}{\bar{r}} + O(1/c^4), \qquad k = 1 + \frac{1}{c^2}\frac{GM}{\bar{r}} + O(1/c^4).$$

- Co-metric

$$\begin{aligned} \overset{\lambda}{g}^{-1} &= \mathfrak{g} + \lambda\, \boldsymbol{\kappa} + O(\lambda^2) \\ &= q^{-1} - \frac{1}{c^2}\left((\partial_t)^2 + 2\frac{GM}{\bar{r}} q^{-1} \right) + O(1/c^4). \end{aligned}$$

- Riemannian volume form associated with the metric $\overset{\lambda}{g}$

$$\mathrm{vol}_{\overset{\lambda}{g}} = cf\, \mathrm{d}t \wedge \mathrm{vol}_q, \quad \text{where} \quad f = \mathscr{N} k^{\frac{3}{2}} = 1 + \frac{2}{c^2}\frac{GM}{\bar{r}} + O(1/c^4).$$

Galilean structure limit of Schwarzschild spacetime

$$\mathfrak{g} = q^{-1}, \qquad \boldsymbol{\kappa} = -(\partial_t)^2 - 2\frac{GM}{\bar{r}}q^{-1}, \qquad \theta = \mathrm{d}t.$$

- This structure integrable, the time function is the same, in either the relativistic context or the Galilean one.
- The foliation by the hypersurfaces Ω_t is common to both structures.
- However, the relativistic normal $\mathbf{N}^\lambda \to 0$ as $c \to \infty$.

The conformation vs Right Cauchy–Green tensor $\mathbf{C} := \mathbf{F}^\star q\,\mathbf{F}$

The conformation converges to

$$\lim_{c\to\infty} \overset{\lambda}{\mathbf{H}} = \mathbf{F}^{-1} q^{-1}\,\mathbf{F}^{-\star} = \mathbf{C}^{-1},$$

the inverse of the right Cauchy–Green tensor.

Covariant derivative ∇^λ and its limit ∇^0

The Christoffel symbols of the Newtonian limit ∇^0 are all vanishing except

$$\Gamma^i_{tt} = -\mathrm{g}^i, \qquad \mathbf{g} := -\frac{GM}{\bar{r}^2}\frac{x^i}{\bar{r}}\partial_{x^i} \ \text{(Newtonian gravity)}.$$

Divergences

$$\overset{\lambda}{\operatorname{div}}\,\mathbf{P} = \frac{\partial P^t}{\partial t} + \frac{\partial P^j}{\partial x^j} + \frac{2}{c^2}\delta_{ij}\mathrm{g}^i P^j + O(1/c^4),$$

$$(\overset{\lambda}{\operatorname{div}}\,\mathbf{T})^t = \frac{\partial T^{tt}}{\partial t} + \frac{\partial T^{tj}}{\partial x^j} + O(1/c^4),$$

$$(\overset{\lambda}{\operatorname{div}}\,\mathbf{T})^i = \frac{\partial T^{it}}{\partial t} + \frac{\partial T^{ij}}{\partial x^j} - \mathrm{g}^i T^{tt} + O(1/c^2).$$

Current of matter for the metric $\overset{\lambda}{g}$

The 3-form $\Psi^*\mu$ does not depend on c (the matter field Ψ and the mass measure μ do not depend on c, which is only introduced through the metrics).

Since

$$\Psi^*\mu = i_{\mathbf{P}^\lambda} \mathrm{vol}_{\overset{\lambda}{g}} = cf\, i_{\mathbf{P}^\lambda}\, \mathrm{d}t \wedge \mathrm{vol}_q = i_{cf\mathbf{P}^\lambda}\, \mathrm{d}t \wedge \mathrm{vol}_q = i_{(c\mathbf{P})^0}\, \mathrm{d}t \wedge \mathrm{vol}_q,$$

we get

$$c\overset{\lambda}{\mathbf{P}} = \frac{1}{f}(c\mathbf{P})^0, \quad \text{where} \quad \frac{1}{f} = 1 - \frac{2}{c^2}\frac{GM}{\bar{r}} + O(1/c^4).$$

Spacetime decompositions of $c\mathbf{P}$

$$c\overset{\lambda}{\mathbf{P}} = \overset{\lambda}{\rho}\,\partial_t + \overset{\lambda}{\rho}\,\overset{\lambda}{\boldsymbol{u}} \longrightarrow (c\mathbf{P})^0 = \overset{0}{\rho}\,\partial_t + \overset{0}{\rho}\,\overset{0}{\boldsymbol{u}}.$$

$$\overset{\lambda}{\rho} = \frac{\rho}{\mathcal{N}} = \frac{1}{f}\overset{0}{\rho} \to \overset{0}{\rho}, \qquad \overset{\lambda}{\boldsymbol{u}} = \mathcal{N}\boldsymbol{u} = \overset{0}{\boldsymbol{u}} \to \overset{0}{\boldsymbol{u}}, \qquad \overset{\lambda}{\rho}\,\overset{\lambda}{\boldsymbol{u}} = \rho\,\boldsymbol{u} = \frac{1}{f}\overset{0}{\rho}\,\overset{0}{\boldsymbol{u}} \to \overset{0}{\rho}\,\overset{0}{\boldsymbol{u}}.$$

Stress-energy tensor

In the coordinate system (t, x^i)

$$\overset{\lambda}{\mathbf{T}} = \begin{pmatrix} \dfrac{1}{c^2\mathcal{N}^2}E_{\text{tot}} & \dfrac{1}{c^2\mathcal{N}}\boldsymbol{p}^\star \\ \dfrac{1}{c^2\mathcal{N}}\boldsymbol{p} & \mathbf{s} \end{pmatrix} \longrightarrow \overset{0}{\mathbf{T}} = \begin{pmatrix} \overset{0}{\rho} & \overset{0}{\rho}\,\overset{0}{\boldsymbol{u}}^\star \\ \overset{0}{\rho}\,\overset{0}{\boldsymbol{u}} & \overset{0}{\rho}\,\overset{0}{\boldsymbol{u}} \otimes \overset{0}{\boldsymbol{u}} - \overset{0}{\boldsymbol{\sigma}} \end{pmatrix}.$$

if we assume that

$$\lim_{c\to\infty} (E/c^2) = 0.$$

The energy density E_{tot}, linear momentum $\boldsymbol{p}$ and spatial part $\mathbf{s}$ behave as

$$\frac{E_{\text{tot}}}{c^2\mathcal{N}} = \overset{\lambda}{\rho} + \frac{1}{c^2}\left(\overset{0}{E} + \frac{1}{2}\overset{0}{\rho}\,\overset{0}{\boldsymbol{u}}^2\right) + O(1/c^4),$$

$$\frac{\boldsymbol{p}}{c^2} = \overset{\lambda}{\rho}\,\overset{\lambda}{\boldsymbol{u}} + \frac{1}{c^2}\left(\left(\overset{0}{E} + \frac{1}{2}\overset{0}{\rho}\,\overset{0}{\boldsymbol{u}}^2\right)\overset{0}{\boldsymbol{u}} - \overset{0}{\boldsymbol{\sigma}}\cdot\overset{0}{\boldsymbol{u}}^\flat\right) + O(1/c^4),$$

$$\mathbf{s} = \overset{\lambda}{\rho}\,\overset{\lambda}{\boldsymbol{u}} \otimes \overset{\lambda}{\boldsymbol{u}} - \overset{0}{\boldsymbol{\sigma}} + O(1/c^2),$$

LIMITS OF THE BALANCE LAWS

1. At order 0 in λ: $\overset{\lambda}{\operatorname{div}}(c\overset{\lambda}{\mathbf{P}}) = \frac{\partial\overset{\lambda}{\rho}}{\partial t} + \operatorname{div}(\overset{\lambda}{\rho}\,\overset{\lambda}{\boldsymbol{u}}) + \frac{2}{c^2}\mathbf{g}\cdot\overset{\lambda}{\rho}\,\overset{\lambda}{\boldsymbol{u}} + O(1/c^4) = 0$, and $(\operatorname{div}^{\lambda}\mathbf{T}^{\lambda})^t = 0$ converge to usual (3D) expression of mass conservation

$$\frac{\partial\overset{0}{\rho}}{\partial t} + \operatorname{div}(\overset{0}{\rho}\,\overset{0}{\boldsymbol{u}}) = 0.$$

2. At order 0 in λ: the spatial part $(\overset{\lambda}{\operatorname{div}}\,\mathbf{T}^{\lambda})^{\top} = 0$ converges to the linear momentum balance,

$$\frac{\partial}{\partial t}(\overset{0}{\rho}\,\overset{0}{\boldsymbol{u}}) + \operatorname{div}\left(\overset{0}{\rho}\,\overset{0}{\boldsymbol{u}}\otimes\overset{0}{\boldsymbol{u}} - \overset{0}{\boldsymbol{\sigma}}\right) - \overset{0}{\rho}\,\mathbf{g} = 0.$$

3. At order 1 in λ: $c^2\left[\mathscr{N}(\overset{\lambda}{\operatorname{div}}\,\overset{\lambda}{\mathbf{T}})^t - \overset{\lambda}{\operatorname{div}}(c\overset{\lambda}{\mathbf{P}})\right]$ converges to usual (3D) energy balance,

$$\frac{\partial}{\partial t}\left(\overset{0}{E} + \frac{1}{2}\rho\overset{0}{\boldsymbol{u}}^2\right) + \operatorname{div}\left(\left(\overset{0}{E} + \frac{1}{2}\overset{0}{\rho}\,\overset{0}{\boldsymbol{u}}^2\right)\overset{0}{\boldsymbol{u}} - \overset{0}{\boldsymbol{\sigma}}\cdot\overset{0}{\boldsymbol{u}}^{\flat}\right) - \overset{0}{\rho}\,\mathbf{g}\cdot\overset{0}{\boldsymbol{u}} = 0.$$

Limits at $c \to \infty$ give the usual expression of Classical Continuum Mechanics

Omitting the superscripts 0:

1. is mass conservation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{u}) = 0.$$

2. recasts as the linear momentum balance

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \nabla_{\boldsymbol{u}} \boldsymbol{u} \right) = \operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{g},$$

   where ∇ is the covariant derivative for the Euclidean metric q.

3. recasts as the internal energy balance

$$\rho \left(\frac{\partial e}{\partial t} + \nabla_{\boldsymbol{u}} e \right) = \boldsymbol{\sigma} : \mathbf{d}, \qquad \mathbf{d} := \frac{1}{2} \left(\nabla \boldsymbol{u}^{\flat} + (\nabla \boldsymbol{u}^{\flat})^{\star} \right),$$

   with no heat transfer, and where $e := E/\rho$ is the specific internal energy.

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Conclusion

- Souriau's Lagrangian General relativity formulation of Hyperelasticity
- Mindset: Gauge theory, Geometrization of Physics
- Key role of the body $\mathscr{B}$ (of the mass measure μ and the fixed metric γ_0)
- Key role of the conformation (as strain)
- Proper definition of a stress
- Even if the full theory is 4D, Hyperelasticity constitutive laws are 3D
- Accounts for gravity (here in Schwarzschild spacetime)
- Newton-Cartan formulation of Galilean Relativity obtained as $c \to \infty$

Some authors introduce the orthogonal decomposition w.r.t. $\mathbf{U}$ of the metric,

$$g = h - \mathbf{U}^\flat \otimes \mathbf{U}^\flat, \qquad g^{-1} = h^\sharp - \mathbf{U} \otimes \mathbf{U}, \qquad h\mathbf{U} = 0.$$

Since $T\Psi.\mathbf{U} = 0$, h, $h^\sharp$ and the conformation $\mathbf{H}$ are simply related as

$$\mathbf{H} = (T\Psi)\, h^\sharp (T\Psi)^\star, \qquad h = g h^\sharp g = (T\Psi)^\star\, \mathbf{H}^{-1} (T\Psi).$$

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Relativistic perfect fluid

The stress-energy tensor of a Relativistic perfect fluid,

$$\mathbf{T} = (L + P)\,\mathbf{U} \otimes \mathbf{U} + P\,g^{-1}, \qquad L = \rho_r c^2 + E,$$

corresponds to an internal energy density of the form

$$E = \rho_r e(\rho_r),$$

where

$$P = \rho_r^2 e'(\rho_r).$$

$$\mathbf{\Sigma} = -Pg^{-1} - P\,\mathbf{U} \otimes \mathbf{U} = -P\,h^{\sharp},$$

$$\mathbf{S} = \mathbf{\Sigma} - E\,\mathbf{U} \otimes \mathbf{U} = -P\,g^{-1} - (E + P)\mathbf{U} \otimes \mathbf{U}.$$

Mass conservation in Hyperelasticity

Lemma (Souriau, 1958)

Let γ_0 be a fixed Riemannian metric on the body $\mathscr{B}$. Then, the rest mass density ρ_r can be written as

$$\rho_r = \rho_{\gamma_0}(\Psi)\sqrt{\det\left[\mathbf{H}(\gamma_0 \circ \Psi)\right]},$$

where Ψ is the matter field, $\mathbf{H}$ is the conformation, and $\rho_{\gamma_0} = \dfrac{\mu}{\mathrm{vol}_{\gamma_0}}$.

The fixed mass density ρ_{γ_0} (w.r.t. γ_0) is defined on the body $\mathscr{B}$.

Mass conservation in Classical Continuum Mechanics (on Ω_0)

$$\rho_0 = (\rho \circ \phi)J, \qquad J := \sqrt{\det(q^{-1}\mathbf{C})},$$

$\phi \colon \Omega_0 \to \Omega$ is the deformation
$\mathbf{C} := \phi^* q$ is the right Cauchy–Green tensor.