Éclatement des feuilletages singuliers
à travers des algébroïdes de Lie infinies universelles

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Introduction

"Differential equations"

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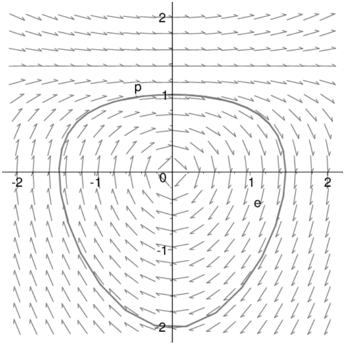
"Differential equations"= "vector fields"



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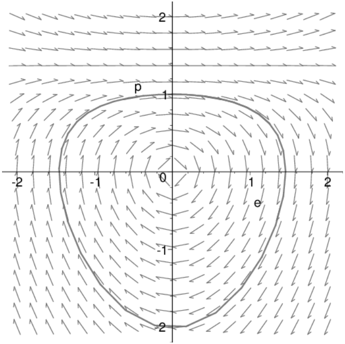

Many differential equations induce a partition of manifold:

$x \sim y$ are reachable using the differential equation

Introduction

In differential geometry, Singular foliations arise as

- Lie group actions
- Symplectic leaves of a Poisson manifold
- $\cdots$

Singular foliations

Definition

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- **Stability under $+$ and multiplication by functions $C^\infty(M)$**.
- **Locally finitely generateness**:

$$X \in \mathcal{F} \Longrightarrow X = \sum_i f_i X_i$$

Example

- Consider a polynomial function $\varphi \in C[x_1, \ldots, x_n]$.

$$\mathcal{F} = \left\{ \frac{\partial \varphi}{\partial x_i} \frac{\partial}{\partial x_j} - \frac{\partial \varphi}{\partial x_i} \frac{\partial}{\partial x_j}, 1 \leq i < j \leq n \right\}$$

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- **(Concentric spheres)**. The the singular foliation generated by the vector fields,

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},\ \ y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},\ \ z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}.$$

Anchored bundle

Definition

An anchored vector bundle is a pair (A, ρ) made of a vector bundle $A \to M$, and a vector bundle morphism called its *anchor map*.

$$
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A & \xrightarrow{\rho} & TM \\
\downarrow & & \downarrow \\
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Definition (Anchored bundle over $\mathcal{F}$)

Let $\mathcal{F}$ be a singular foliation on M. We say that an anchored bundle (A, ρ) is over $\mathcal{F}$ if $\rho(\Gamma(A)) = \mathcal{F}$.

Proposition

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Definition (Almost Lie algebroids)

Let (A, ρ) be an anchored vector bundle over M. We call *almost-Lie algebroid bracket* a skew-symmetric bilinear (over $\mathbb{K}$) map

$$[\cdot\,,\cdot]_A : \Gamma(A) \wedge \Gamma(A) \longrightarrow \Gamma(A)$$

that satisfies the *Leibniz identity*,

$$[x, fy]_A = \rho(x)[f]y + f[x, y]_A, \quad \text{for all } x, y \in \Gamma(A), f \in C^\infty(M) \tag{1}$$

and the *anchor condition*:

$$\rho([x, y]_A) = [\rho(x), \rho(y)], \quad \text{for all } x, y \in \Gamma(A). \tag{2}$$

Proposition

Every anchored vector bundle (A, ρ) over M such that $\rho(\Gamma(A)) = \mathcal{F}$ can be endowed with an almost-Lie algebroid bracket.

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kind	Alsmost kind	Wicked,ugly
Regular	Debord=projective	Irregular with plenty of relations

We will need the following!

Let $E \to M$ a vector bundle of rank d over a manifold M (or an algebraic variety).

Definition (Tautological subbundle)

The Grassmann bundle $\Pi\colon \mathrm{Gr}_{-r}(E) \longrightarrow M$ comes equipped with a canonical vector bundle $\mathcal{V}^E \subset \Pi^* E$ over it whose fiber over $(V, x) \in \mathrm{Gr}_{-r}(E|_x)$ is (V, x) itself.

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Also, it comes with a universal quotient bundle $A^E \simeq \Pi^* E / \mathcal{V}^E$ together with the exact sequence

$$0 \longrightarrow \mathcal{V}^E \longrightarrow \Pi^* E \longrightarrow A^E \longrightarrow 0. \quad (3)$$

Blowup a singular vector field

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- Diffeomorphism on regular points
- Less singular
- The same number of leaves

A blowup procedure

Let $(A, [\,\cdot,\cdot]_A, \rho)$ be an almost Lie algebroid over $\mathfrak{F}$. Denote by $M_{\mathrm{reg},\mathfrak{F}}$ be the set of regular points of $\mathfrak{F}$

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A construction

1. Consider the natural section defined by :

$$\sigma\colon M_{\mathrm{reg},\mathfrak{F}} \longrightarrow \mathrm{Gr}_{-r}(A),\ x \longmapsto (\ker \rho_x, x)$$

2. Let $\widetilde{M} := \overline{\sigma(M_{\mathrm{reg},\mathfrak{F}})}$ be the closure of the graph of σ in $\mathrm{Gr}_r(A)$.

Idea

1. M becomes (maybe) weird!
2. $\mathfrak{F}$ becomes easy

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A surprising result! : Under one blowup, any singular foliation becomes projective!

Geometric resolutions of a singular foliation

Let (A, ρ) be an anchord bundle such that $\rho(\Gamma(A)) = \mathcal{F}$.

1. Consider the kernel $\ker(\rho)$ of

$$\rho \colon \Gamma(A) \longrightarrow \mathfrak{X}(M).$$

2. Under some good conditions, there exists a second vector bundle $B \to M$ and a vector bundle morphism $\mathrm{d} \colon B \longrightarrow A$ such that

$$\mathrm{d}\left(\Gamma(B)\right) = \ker(\rho)$$

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In particular, we have $\rho \circ \mathrm{d} = 0$, and

$$\begin{array}{ccccc} B & \xrightarrow{\mathrm{d}^{(1)}} & A & \xrightarrow{\rho} & TM \\ \downarrow & & \downarrow & & \downarrow \\ M & = & M & = & M \end{array}$$

Geometric resolutions of a singular foliation

Definition

Let $\mathcal{F} \subseteq \mathfrak{X}(M)$ be a singular foliation on a manifold M. A complex of vector bundles

$$\begin{array}{ccccccccccccc}
\longrightarrow & E_{-i-1} & \xrightarrow{\mathrm{d}^{(i+1)}} & E_{-i} & \xrightarrow{\mathrm{d}^{(i)}} & E_{i-1} & \longrightarrow & \cdots\cdots & \xrightarrow{\mathrm{d}^{(2)}} & E_{-1} & \xrightarrow{\rho} & TM \\
 & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
=\!=\!= & M & =\!=\!= & M & =\!=\!= & M & =\!=\!= & \cdots\cdots & =\!=\!= & M & =\!=\!= & M
\end{array}$$

is said to be a geometric resolution of $\mathcal{F}$ if the following complex of is exact:

$$\longrightarrow \Gamma(E_{-i-1}) \xrightarrow{\mathrm{d}^{(i+1)}} \Gamma(E_{-i}) \xrightarrow{\mathrm{d}^{(i)}} \Gamma(E_{-i+1}) \longrightarrow \cdots \longrightarrow \Gamma(E_{-1}) \xrightarrow{\rho} \mathcal{F}.$$

More structures behind a singular foliation

Theorem (LGL, LLS)

Denote by $\mathfrak{F}$ be a singular foliation over M. Any geometric resolution

$$\cdots \xrightarrow{\mathrm{d}} \Gamma(E_{-3}) \xrightarrow{\mathrm{d}} \Gamma(E_{-2}) \xrightarrow{\mathrm{d}} \Gamma(E_{-1}) \xrightarrow{\rho} \mathfrak{F} \longrightarrow 0 \tag{4}$$

comes equipped with a unique (up to homotopy) almost graded Lie algebroid structure over $\mathfrak{F}$ whose unary bracket is $\ell_1 := \mathrm{d}$.

Blowup procedures for singular foliations

Settings: Let $\mathfrak{F}$ be a singular foliation that admits a geometric resolution.

$$(E, \mathrm{d}, \rho): \quad \cdots \overset{\ell_1 = \mathrm{d}^{(4)}}{\longrightarrow} E_{-3} \overset{\ell_1 = \mathrm{d}^{(3)}}{\longrightarrow} E_{-2} \overset{\ell_1 = \mathrm{d}^{(2)}}{\longrightarrow} E_{-1} \overset{\rho = \mathrm{d}^{(1)}}{\longrightarrow} TM.$$

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1. Consider the natural section $M_{\mathrm{reg},\mathfrak{F}}$ defined by :

$$\sigma_i \colon M_{\mathrm{reg},\mathfrak{F}} \longrightarrow \mathrm{Gr}_{-r_i}(E_{-i}),\ x \longmapsto \left(\mathrm{im}(\mathrm{d}_x^{(i+1)}), x\right) \tag{5}$$

2. Let $\widetilde{M}_i := \overline{\sigma_i(M_{\mathrm{reg},\mathfrak{F}})}$ be the closure of the graph of σ_i in $\mathrm{Gr}_{-r_i}(E_{-i})$.

Theorem (Part I)

1. *For each* i*, the projection* $\pi_i \colon \widetilde{M}_i \to M$ *is invertible on an open dense subset, proper and onto.*
2. *For all* $x \in M$*,* $\pi_i^{-1}(x) \subseteq \mathrm{Gr}_{r_i - \mathrm{rk}\left(\mathrm{d}_x^{(i)}\right)}(H^{-i}(\mathfrak{F}, x))$
3. *For each* $i \geq 1$*,* $\widetilde{M}_i$ *does not depend on the choice of a geometric resolution of* $\mathfrak{F}$*.*

Theorem (Part II)

Let $\widetilde{M}_i$ be as Theorem (Part I) for $i \geq 1$. If $\mathfrak{F}$ is a polynomial singular foliation on $M \in \{\mathbb{C}^N, \mathbb{R}^N\}$, then $\widetilde{M}_i$ is a locally affine variety. Moreover,

1. *for every $i \geq 1, \mathfrak{F}$ is lifted to a singular foliation $\widetilde{\mathfrak{F}}_i$ on the locally affine variety $\widetilde{M}_i$,*

Theorem (Part II)

Let $\widetilde{M}_i$ be as Theorem (Part I) for $i \geq 1$. If $\mathfrak{F}$ is a polynomial singular foliation on $M \in \{\mathbb{C}^N, \mathbb{R}^N\}$, then $\widetilde{M}_i$ is a <u>*locally affine*</u> *variety. Moreover,*

1. *for every $i \geq 1$, $\mathfrak{F}$ is* <u>*lifted*</u> *to a singular foliation $\widetilde{\mathfrak{F}}_i$ on the locally affine variety $\widetilde{M}_i$,*

Also,

- *$\widetilde{\mathfrak{F}}_1$ rises as the action of a Lie algebroid over $\widetilde{M}_1$ whose anchor is injective on an open dense subset.*

Proof. It uses the 2-ary bracket & the canonical bundle $A^{E_{-1}}$.

Thank you for your attention!