The Hamilton-de Donder-Weyl equations : from classical field theories to multisymplectic geometry

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Contents

In coordinates the **Hamilton-Volterra equations** $(*)$ for $\Psi : \mathbb{R}^n \supset \Sigma \to \mathbb{R}^N \times \mathbb{R}^{Nn}$ are given as follows :

$\forall \mu \in \{1, ..., n\}$ and $\forall a \in \{1, ..., N\}$,

$$\frac{\partial \mathcal{H}}{\partial q^a}(\Psi(x)) = \sum_{\mu=1}^{n} \frac{\partial (p_a^\mu \circ \Psi)(x)}{\partial x^\mu},$$

$$-\frac{\partial \mathcal{H}}{\partial p_a^\mu}(\Psi(x)) = \frac{\partial (q^a \circ \Psi)(x)}{\partial x^\mu}.$$

Time derivatives are replaced by partials in coordinate directions x^μ of Σ, replacing the time interval I of (time-dependant) mechanics.

Modern formulation of Volterra's equations for certain variational problems, here for n-dimensional sources and fields Ψ with values in a N-dimensional "configuration space" Q (or better appropriate sections) :

$$\Sigma^n \xleftarrow{\pi} \Sigma^n \times Q = E \leftarrow \Lambda_2^n T^*E = M(\pi)$$

where $\Lambda_2^n T^*E = \{\eta \in \Lambda^n T^*E \mid \iota_v \iota_w \eta = 0 \, \forall v, w \in \ker \pi_* \subset TE\}$ is the (twisted affine) dual of the first jet bundle of π with tautological n-form θ and $\omega = -d\theta$.

Equivalent to the Hamilton-Volterra equations is the

dynamical Hamilton-De Donder-Weyl equation:

$$(**) \quad (\psi_*)(\gamma^{\Sigma}) \lrcorner\, \omega_{\psi(t)} = -dH_{\psi(t)} \text{ typically with } H = \mathcal{H} - p.$$

Here γ^{Σ} is a co-volume on Σ, and $\mathcal{H}$ is a function on $P(\pi) = \Lambda^n_2 T^*E/\Lambda^n_1 T^*E$, with p a fibre coordinate on $M(\pi) = \Lambda^n_2 T^*E \to \Lambda^n_2 T^*E/\Lambda^n_1 T^*E = P(\pi)$, and ψ a section of $M(\pi) \to \Sigma$.

Remark. $(*) \Leftrightarrow (**)$.

References for the preceding : [Volterra]; [Román-Roy]; [Wagner and Wurzbacher 1]

Definition

A "multisymplectic" manifold (M, ω) is a pair, where M is a manifold, $n \geq 1$ and $\omega \in \Omega^{n+1}(M)$ is a closed differential form satisfying the following non-degeneracy condition: The map

$$\iota_\bullet \omega : TM \to \Lambda^n T^* M, \quad v \mapsto \iota_v \omega = v \lrcorner \omega$$

is injective. For fixed degree $n+1$ of the form such manifolds are also called "n-plectic".

Remarks. A 1-plectic manifold is a symplectic manifold. $M(\pi) = \Lambda^n_2 T^* E$ is the "standard multisymplectic manifold", for $n = 1$ it equals $T^*(I \times Q)$, modelling time-dependent mechanics.

Dynamics are now expressed via

- an equation for an unknown multivector field X_H, given a function H on $M(\pi)$, the

  Hamilton-De Donder-Weyl (HDW) equation:

  $$X_H \lrcorner \omega = -dH.$$

  Solving them is not enough for finding a "solution", since if $n > 1$ going from HDW to a map is an extra step (that is trivially assured by the existence of flows of vector fields, as opposed to multivector fields)!
- generalization to $(Z, \eta) \in \mathfrak{X}^k(M) \oplus \Omega^{n-k}(M)$ s.th.

  $$Z \lrcorner \omega = -d\eta.$$
- the **dynamical Hamilton-De Donder-Weyl equation** (see above).

The Klein-Gordon equation.

Let $d \geq 1$, $\Sigma \subset \mathbb{R}^{d+1}$ an open subset with coordinates $(x^0, x^1, \ldots, x^d)$ and $\pi : E = \Sigma \times \mathbb{R} \to \Sigma$, i.e., $Q = \mathbb{R}$ with coordinate q. Furthermore $P(\pi) = \Lambda^n_2 T^*E/\Lambda^n_1 T^*E \cong E \times \mathbb{R}^{d+1} = \Sigma \times \mathbb{R} \times \mathbb{R}^{d+1}$ with coordinates $(x^0, x^1, \ldots, x^d, q, p^0, p^1, \ldots, p^d)$. We take

$$\mathcal{H}_{KG}(x, q, p) = \frac{1}{2}\Big(- (p^0)^2 + \sum_{k=1}^{d} (p^k)^2\Big) + V(q)\,,$$

where $V : \mathbb{R} \to \mathbb{R}$ is a differentiable function.

The associated Hamilton-Volterra equations are then :

$$\frac{\partial q}{\partial x^\mu} = \frac{\partial \mathcal{H}_{KG}}{\partial p^\mu} = \pm p^\mu$$

$$\sum_{\mu=0}^{d} \frac{\partial p^\mu}{\partial x^\mu} = -\frac{\partial \mathcal{H}_{KG}}{\partial q} = -\frac{\partial V}{\partial q}\,.$$

Upon relabelling $x^0 = t$ we get the Klein-Gordon equation :

$$\Box q = -\frac{\partial^2 q}{\partial t^2} + \sum_{k=1}^{d} \frac{\partial^2 q}{\partial (x^k)^2} = -\frac{\partial V}{\partial q}\,.$$

The Poisson equation.

Let $\Sigma \subset \mathbb{R}^2$ an open subset with coordinates (x^1, x^2) and $Q = \mathbb{R}$ with coordinate q, i.e., $\pi : E = \Sigma \times \mathbb{R} \to \Sigma$. Then $P(\pi) = \Lambda^3_2 T^*E/\Lambda^3_1 T^*E \cong \Sigma \times \mathbb{R} \times \mathbb{R}^2$ with coordinates (x^1, x^2, q, p^1, p^2). We take

$$\mathcal{H}_{Poisson}(x^\mu, q, p^\nu, p) = \frac{1}{2}((p^1)^2 + (p^2)^2) - f(x^1, x^2)q\,,$$

where $f : \Sigma \to \mathbb{R}$ is a differentiable function.

A direct calculation expresses the Hamilton-Volterra equations as follows

$$\frac{\partial q}{\partial x^\mu} = \frac{\partial \mathcal{H}_{KG}}{\partial p^\mu} = p^\mu$$

$$\frac{\partial p^1}{\partial x^1} + \frac{\partial p^2}{\partial x^2} = f(x^1, x^2)\,,$$

i.e., we get Poisson's equation $\Delta_x q(x^1, x^2) = f(x^1, x^2)\,.$

More examples. (Compare also [Gieres].)

- (time-dependent) Hamilton equations on a symplectic manifold, notably for $\mathcal{H} : I \times T^*Q \to \mathbb{R}$
- Schrödinger equation on $\Omega \subset \mathbb{R}^3$ open subset; here $\Sigma = \mathbb{R} \times \Omega, E = \Sigma \times \mathbb{R}^2$ and $M_S = \{p_1^0 = q^2, p_2^0 = -q^1\} \subset \Lambda_2^4 T^*E \cong \mathbb{R} \times \Omega \times \mathbb{R}^2 \times \mathbb{R}^8$ and the ensueing HDW equation yields for $\psi = q : \Sigma \to \mathbb{C} \cong \mathbb{R}^2$:

$$i \frac{\partial \psi}{\partial t} = \Delta_x \psi + V(x) \cdot \psi$$

- minimal surface equation
- Korteweg-de Vries equation
- Maxwell's equations and Yang-Mills equations
- Einstein's equation
- "Exotic" equations (see below for the case of the six-sphere).

A $2k$-dimensional symplectic (or now 1-plectic) manifold (M, ω) enjoys locally a standard normal form : for every $x \in M$ there is a local chart (U, ϕ) around x such that

$$\phi^* \left(\sum_{j=1}^{k} dq^j \wedge dp_j \right) = \omega|_U \, .$$

Such a "Darboux theorem" does not hold in general on a n-plectic, m-dimensional manifold. A first obstruction is given by the "linear type", i.e., the equivalence class of the (non-degenerated) $(n+1)$-form ω_x on the tangent space of $x \in M$ must be constant in x. But this is not enough : there are higher-order obstructions beyond closedness of the form ω (unlike in the symplectic case !) References : [Ryvkin and Wurzbacher]; [Ryvkin]

If U is open in M and allows for a Darboux chart ϕ, the HDW equations are given in terms of minors of the Jacobi matrix of a differentiable map $\mathbb{R}^n \supseteq \Sigma \xrightarrow{\Psi} \phi(U) \subseteq \mathbb{R}^m$, i.e., for certain real numbers a^r_J,

$$\sum_J a^r_J \cdot \det{}_J(D\Psi)_x) = \frac{\partial H}{\partial y^r}(\Psi(x)) \quad \forall r = 1, \ldots, m\,,$$

where J runs over the subsets of cardinality n of $\{1, \ldots, m\}$, and $\det_J(D\Psi)_x$ is the $(n \times n)$-minor of the Jacobi-matrix $(D\Psi)_x$ (of size $m \times n$) defined by J, and H the Hamiltonian function on M, with coordinates $\phi = (y^1, \ldots, y^m)$ on U.

The resulting fully nonlinear inhomogeneous Jacobian minor equations can be solved in certain cases by geometric methods.

A case study : 2-plectic dynamics on the six-sphere

Reference for this section : [Wagner and Wurzbacher 2]

- on $\mathbb{R}^7$ we have a vector product from the octonions and a 3-form $\tilde{\omega}(u, v, w) = < u, v \times w >$
- the inclusion $j : S^6 \hookrightarrow \mathbb{R}^7$ yields an exact 2-plectic form $\omega = j^*(\tilde{\omega})$ on the six-sphere
- ω is compatible with the standard almost complex structure on S^6 (defined by $J_p(u) = p \times u$), i.e., $\omega(Ju, v, w) = \omega(u, Jv, w)$ for all $p \in S^6$ and $u, v, w \in T_pS^6$

Result 1.
(i) ω has constant linear type
(ii) (S^6, ω) has nowhere Darboux coordinates.

Result 2.
The automorphisms of (S^6, ω) are given by the compact finite-dimensional Lie group G_2.

Result 3.
A vector field $X \in \mathfrak{X}^1(S^6)$ solves the HDW equation on (S^6, ω) if and only if $X = X_\xi$ is a fundamental vector field associated to ξ in $\mathrm{Lie}(G_2)$.

Result 4.
Let H be a constant (Hamilton) function on S^6. Then $\forall X \in \mathfrak{X}^1(S^6)$ one has $\iota_{X \wedge JX}\omega = dH$, i.e., the couple $(X \wedge JX, 0)$ solves the HDW equation.

Result 5.

Let H be a constant (Hamilton) function on S^6. Then

(i) $\forall X \in \mathfrak{X}^1(S^6)$ with $\mathcal{L}_X J = 0$ there is a singular J-holomorphic foliation given by $D_p = << X_p, J_p(X_p) >>_{\mathbb{R}}$ for all p in S^6, whose leaves are the images of the solutions of the dynamical HDW equation associated to $(X \wedge JX, 0)$

$$F_p^{X,JX}(t^1, t^2) = \phi_{t^2}^{JX} \circ \phi_{t^1}^{X}(p)$$

(ii) if ξ, η are in Lie(G_2) and $H_{\xi,\eta} = \iota_{X_\xi \wedge X_\eta}\theta$ then $(X_\xi \wedge X_\eta, H_{\xi,\eta})$ is a solution of the HDW equation.

If furthermore $[\xi, \eta] = 0$, then for all $p \in S^6$,

$$F_p^{\xi,\eta}(t^1, t^2) = \phi_{t^2}^{X_\eta} \circ \phi_{t^1}^{\xi}(p)$$

solves the dynamical HDW equation.

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