Numerical analysis of the non-smooth dynamics of a visco-elastic beam (plate) impacting stops

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Content

1. Context : contact problems in Framatome ?
2. Numerical and theoretical results beam's impact problems : time/ spaces schemes, energy conservation ?
3. Birfurcations and chaos : Numerical comparisons of resolution methods
4. From beams to plates : impacts problems ?
5. Conclusions and perspectives

framatome

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1. Contact problems in Framatome ?

- Industrial problems
- Academic problems
- A simple example
- Solving methods : penalty, aug. Lag., Nitsche
- Difficulties – issues ?

Context: Industrial problem

- We aim to predict the dynamics of beams in contact within the industrial framework of rod control usage.
- Appearance of non-repeatabilities **due to contact,** and parameter related-bifurcations (dynamic change induced by parameter change).
- The numerous open problems concerning modeling and resolution techniques for managing contact, impact and friction phenomena between deformable solids can lead industries to numerical solutions with non-physical behaviors.
-

Context: Industrial problem

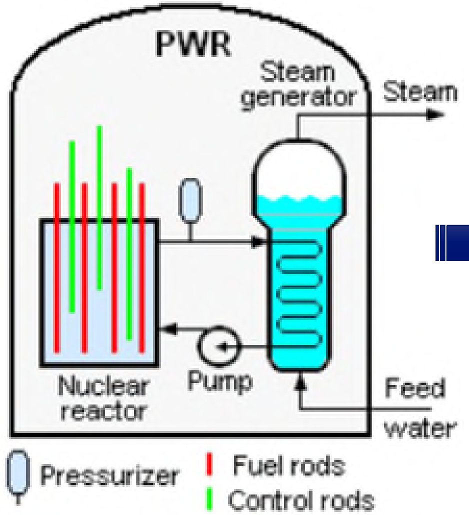
- Difficulties in accounting for impacts in implicit dynamic calculations based on Finite Element Models (FEM) Open mathematical problems:

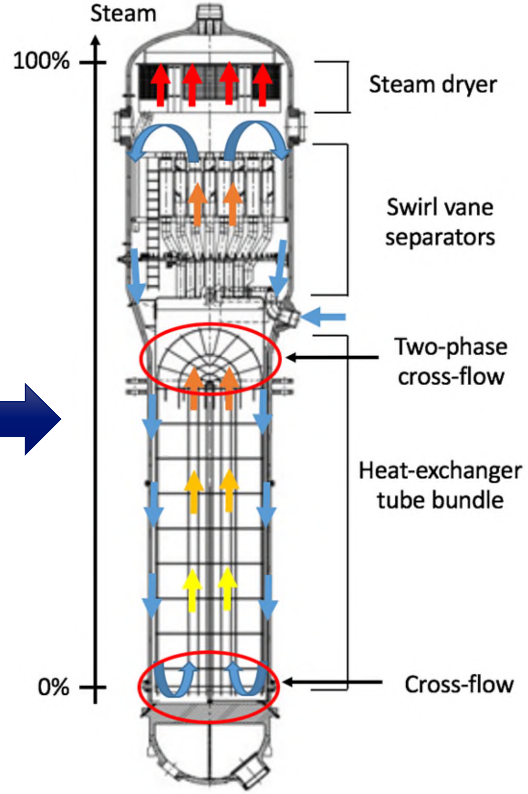
  uniqueness, stability, convergence of numerical schemes

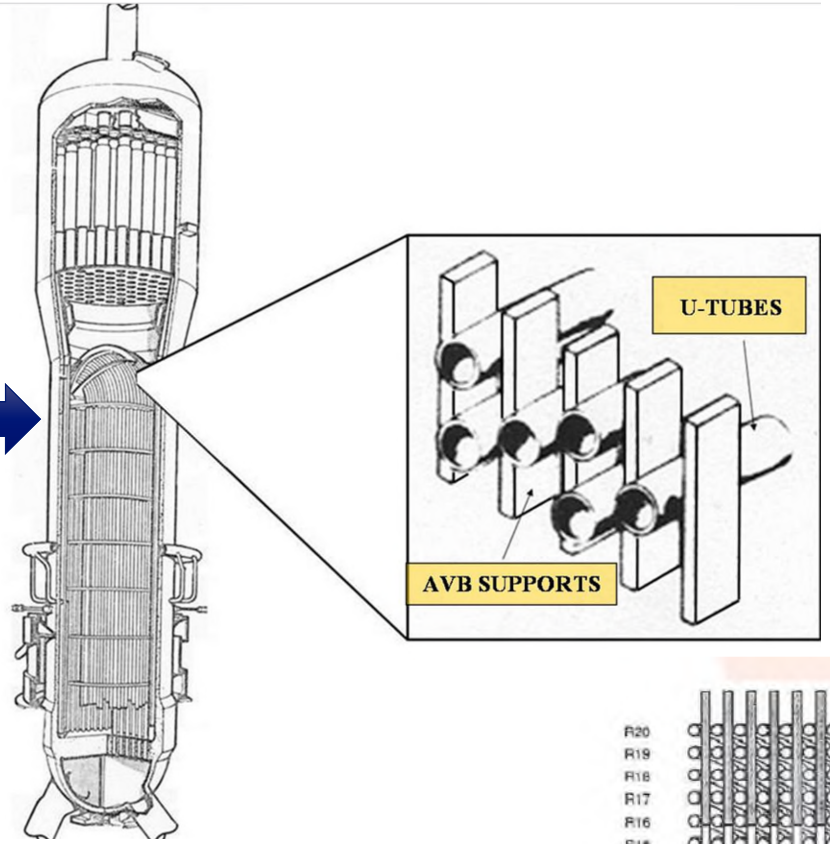
- Appearance of **non-repeatabilities due to contact**
- *Numerical bifurcations*
- *Possible presence of chaotic regimes*
- *Spurious oscillations*
- *Creation or dissipation of energy*

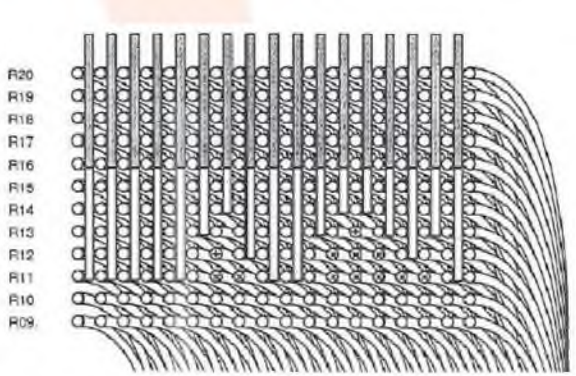
Context: Industrial problem

In a nuclear steam generator, the heat transfer tubes are supported by the contact with support plates and anti-vibration bars. The two-phase flow flows over the tubes and causes a vibration when operating, and then contact problems occurs on these tubes.



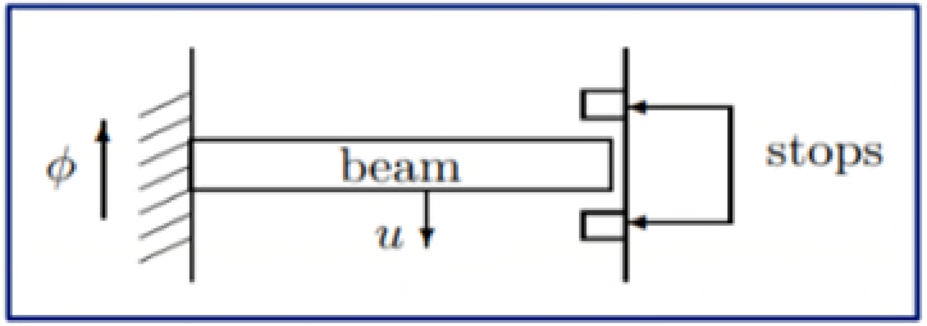






Context: Study framework

- Model of an **Euler-Bernoulli beam (simplest model)** between two rigid (non-elastic) and extended obstacles.
- The **unknowns of the problem** are the displacement $u(x,t)$, the velocity $v(x,t)$, and the contact reaction $r(x,t)$.
- Generation of **bifurcation diagrams** (characterization of the dynamic regime).
- Study and comparison of the distributions of contact reactions for different regimes.

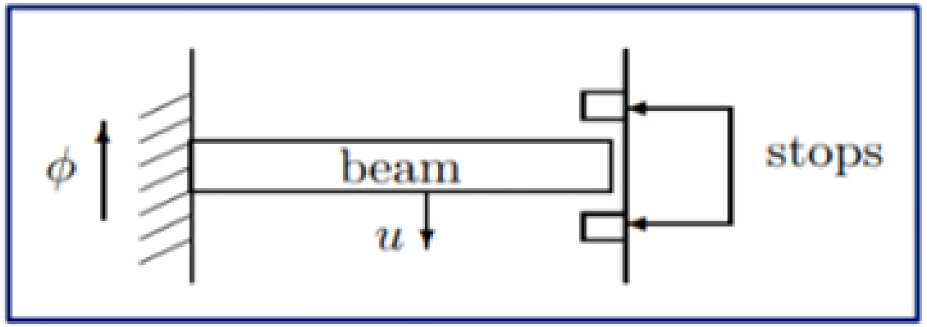


Motivations:

- identify the origin of the phenomena of numerical “non-repeatability”

- how to construct num. schemes (time and space) allowing to avoid the phenomena of “non-repeatability”?

Context: Study framework



α damping paramater

$$
\begin{cases}
\rho S \dfrac{\partial^2 u}{\partial t^2}(x,t) + \alpha E I \dfrac{\partial^5 u}{\partial x^4 \partial t}(x,t) + E I \dfrac{\partial^4 u}{\partial x^4}(x,t) = f(x,t) & \forall (x,t) \in]0,L[\times]0,T[\\
u(x,0) = u_0(x), \quad \dot{u}(x,0) = v_0(x) & \forall x \in [0,L] \\
u(0,t) = \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial^2 u}{\partial x^2}(L,t) = \dfrac{\partial^3 u}{\partial x^3}(L,t) = 0 & \forall t \in [0,T]
\end{cases}
$$

Linear dynamic equations (clamped-free E-B beam)

$+$

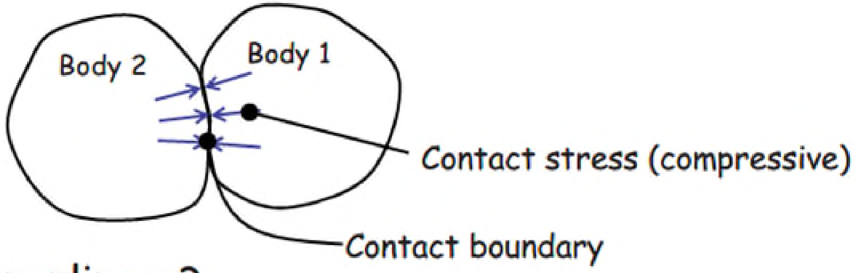
$$
\begin{cases}
u_n \leq 0 & (i) \\
\sigma_n(\mathbf{u}) \leq 0 & (ii) \\
\sigma_n(\mathbf{u})\, u_n = 0 & (iii) \\
\sigma_t(\mathbf{u}) = 0 & (iv)
\end{cases}
$$

Non-penetration conditions (frictionless)

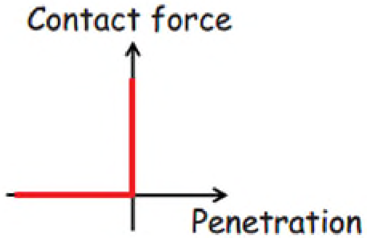
Position of the contact problem

Contact Problem – Boundary Nonlinearity

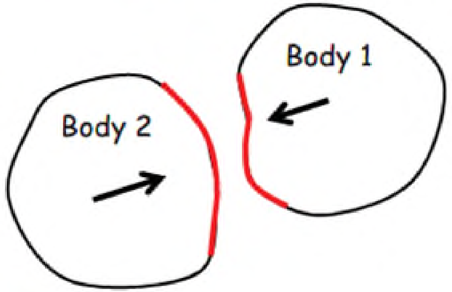
- Contact problem is categorized as boundary nonlinearity
- Body 1 cannot penetrate Body 2 (impenetrability)



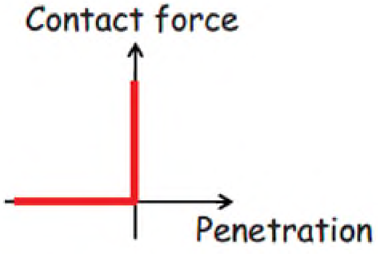
- Why nonlinear?
  - Both contact boundary and contact stress are unknown!!!
  - Abrupt change in contact force (difficult in NR iteration)



Why are contact problems difficult?

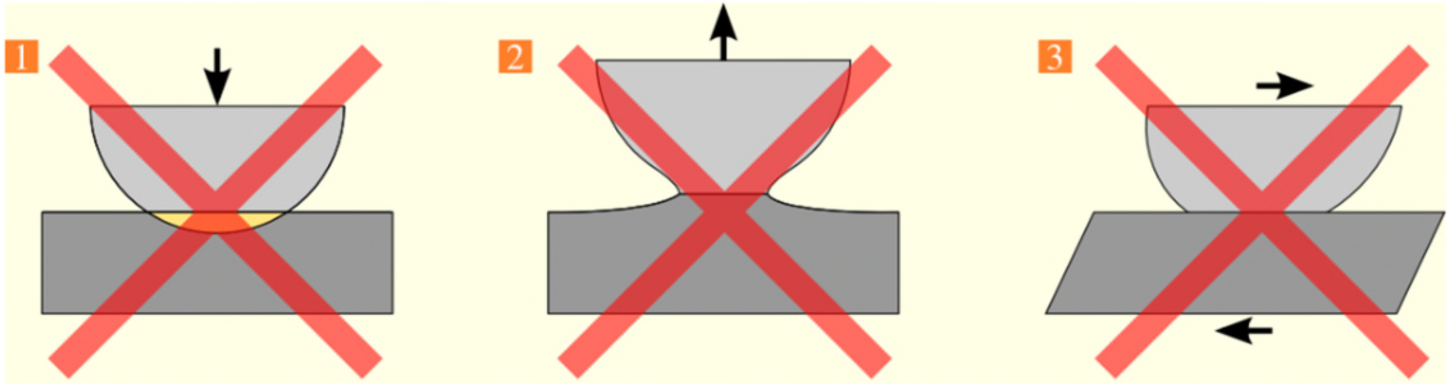


- Unknown boundary
  - Contact boundary is unknown a priori
  - It is a part of solution
  - Candidate boundary is often given
- Abrupt change in force
  - Extremely discontinuous force profile
  - When contact occurs, contact force cannot be determined from displacement
  - Similar to incompressibility (Lagrange multiplier or penalty method)



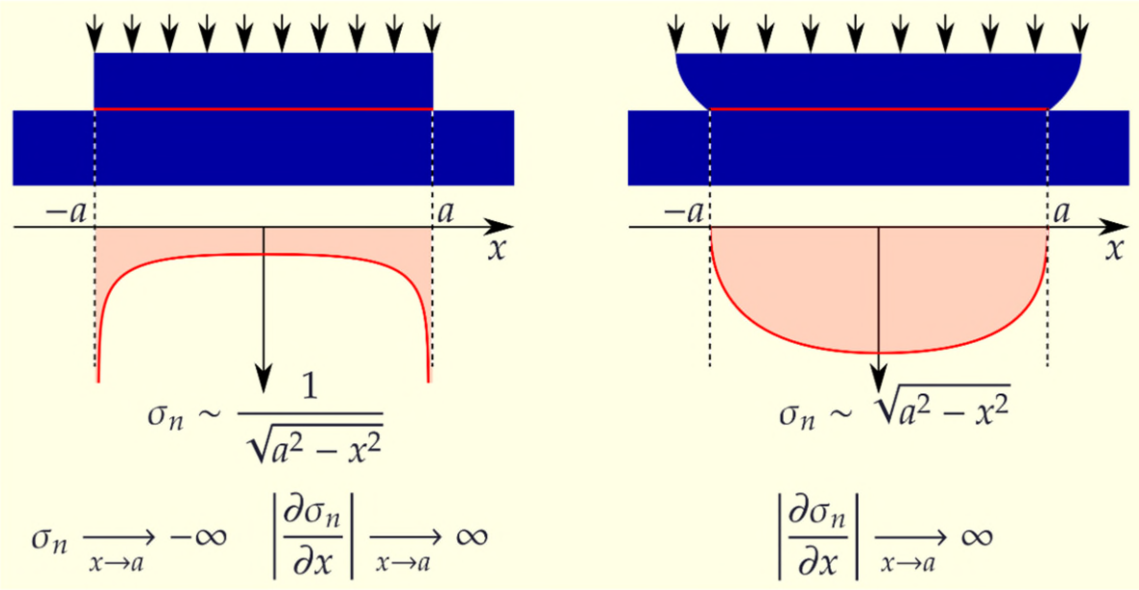
- Discrete boundary
  - In continuum, contact boundary varies smoothly
  - In numerical model, contact boundary varies node to node
  - Very sensitive to boundary discretization

Position of the static contact problem

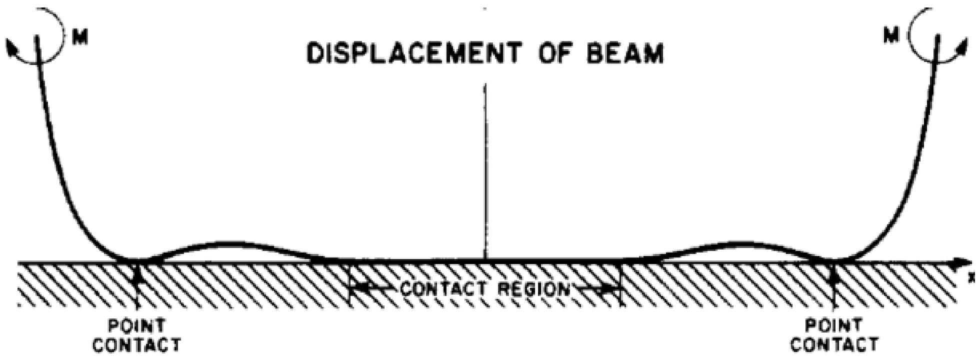


Frictionless contact conditions (*intuitive*)

1. No penetration
2. No adhesion
3. No shear transfer



Infinite contact pressure and/or its derivative

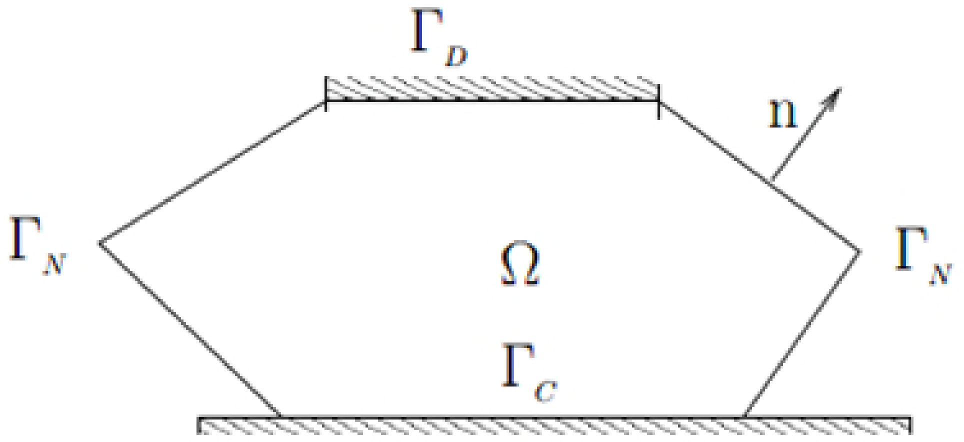


NON C^1 -SMOOTH SOLUTIONS !

Position of the static contact problem

$\mathbf{u} : \Omega \to \mathbb{R}^2$ verifying the equations and conditions (1)–(2):

$$(1)\begin{cases} \mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f} = \mathbf{0} & \text{in } \Omega, \\ \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{A}\,\boldsymbol{\varepsilon}(\mathbf{u}) & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_D, \\ \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \mathbf{g} & \text{on } \Gamma_N, \end{cases} \qquad (2)\begin{cases} u_n \le 0 & (i) \\ \sigma_n(\mathbf{u}) \le 0 & (ii) \\ \sigma_n(\mathbf{u})\,u_n = 0 & (iii) \\ \sigma_{\mathbf{t}}(\mathbf{u}) = 0 & (iv) \end{cases}$$



$$\mathbf{K} := \{\mathbf{v} \in \mathbf{V} \,:\, v_n = \mathbf{v}\cdot\mathbf{n} \le 0 \text{ on } \Gamma_C\}.$$

$$\mathbf{V} := \left\{\mathbf{v} \in \left(H^1(\Omega)\right)^d \,:\, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\right\}.$$

$$a(\mathbf{u},\mathbf{v}) := \int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v})\, d\Omega,$$

$$L(\mathbf{v}) := \int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\Omega + \int_{\Gamma_N} \mathbf{g}\cdot\mathbf{v}\, d\Gamma,$$

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{K} \text{ such that:} \\ a(\mathbf{u}, \mathbf{v}-\mathbf{u}) \ge L(\mathbf{v}-\mathbf{u}), \qquad \forall \mathbf{v} \in \mathbf{K}. \end{cases}$$

$\longleftrightarrow$ $Min_{\,\mathbf{u}\in\mathbf{K}}\, f(\mathbf{u})$

Stampacchia's Theorem ensures that elastic problem in small def. admits a unique solution

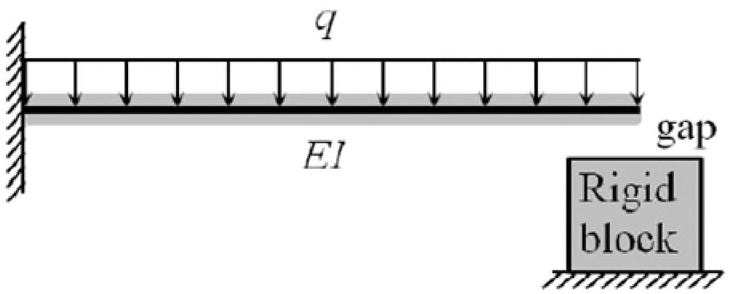
A simple static elastic example

Contact of a Cantilever Beam with a Rigid Block

- q = 1 kN/m, L = 1 m, $EI = 10^5$ N·m², initial gap δ = 1 mm
- Trial-and-error solution
  - First assume that the deflection is smaller than the gap

$$v_N(x) = \frac{qx^2}{24EI}(x^2 + 6L^2 - 4Lx), \quad v_N(L) = \frac{qL^4}{8EI} = 0.00125\text{m}$$

  - Since $v_N(L) > \delta$, the assumption is wrong, the beam will be in contact



- Solution using contact constraint
  - Treat both contact force and gap as unknown and add constraint

$$v(x) = \frac{qx^2}{24EI}(x^2 + 6L^2 - 4Lx) - \frac{\lambda x^2}{6EI}(3L - x)$$

  - When $\lambda = 0$, no contact. Contact occurs when $\lambda > 0$. $\lambda < 0$ impossible
  - Gap condition: $g = v_{tip} - \delta \leq 0$

- Solution using contact constraint cont.
  - Contact condition

No penetration: $g \leq 0$

Positive contact force: $\lambda \geq 0$

Consistency condition: $\lambda g = 0$

A simple static elastic example

Cantilever Beam Contact with a Rigid Block cont.

- Solution using contact constraint cont.
  - Contact condition

    No penetration: $g \le 0$

    Positive contact force: $\lambda \ge 0$

    Consistency condition: $\lambda g = 0$
  - Lagrange multiplier method

$$\lambda g = \lambda \left(0.00025 - \frac{\lambda}{3 \times 10^5} \right) = 0$$

  - When λ = 0N → g = 0.00025 > 0 → violate contact condition
  - When λ = 75N → g = 0 → satisfy contact condition

Lagrange multiplier, λ, is the contact force

- Penalty method
  - Small penetration is allowed, and contact force is proportional to it
  - Penetration function

$$\phi_N = \frac{1}{2}(|g| + g)$$

$\phi_N = 0$ when $g \le 0$
$\phi_N = g$ when $g > 0$

  - Contact force

$$\lambda = K_N \phi_N$$

K_N: penalty parameter

  - From $g = v_{tip} - \delta \le 0$

$$g = 0.00025 - \frac{K_N}{3 \times 10^5} \frac{1}{2}(|g| + g)$$

  - Gap depends on penalty parameter

Penalty parameter	Penetration (m)	Contact force (N)
3×10^5	1.25×10^{-4}	37.50
3×10^6	2.27×10^{-5}	68.18
3×10^7	2.48×10^{-6}	74.26
3×10^8	2.50×10^{-7}	74.92
3×10^9	2.50×10^{-8}	75.00

Gap depends on penalty parameter

Solving contact problem

In order to take into a count **non-penetration condition** (frictionless)

$$
\begin{cases}
u_n \leq 0 & (i) \\
\sigma_n(\mathbf{u} \leq 0 & (ii) \\
\sigma_n(\mathbf{u})u_n \leq 0 & (iii) \\
\sigma_t(\mathbf{u}) = 0 & (iv)
\end{cases}
$$

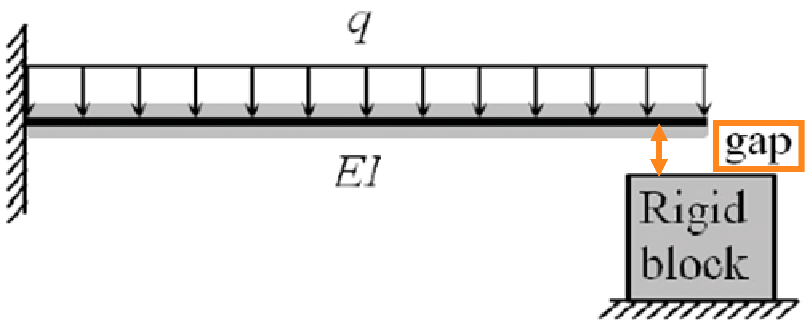
⇒ Classical methods : **Penality**, **Aug. Lagrangian**, Nitsche (3D)

$$[a]_+ = \frac{a + |a|}{2}$$

- Penality : $f_p(x) = f(x) + K_N[-gap(x)]_+^2$

- Augmented Lagrangian : $\mathcal{L}(x, \lambda) = f(x) + K_N[-gap(x)]_+^2 + \lambda \cdot gap(x)$

- Nitsche (3D) : $\sigma_n(u) = -\frac{1}{\gamma}[u_n - \gamma\sigma_n(u)]_+$



- Extented Nitsche methods (1D, 2D) to thin structures : Matthieu Schorsch (Phd Studient Insa lyon- Framatome)

A simple static elastic example

Penalty method : Functional to be minimized *f*(x) under constraint *g*(x) ≥ 0

$f_p(x) = f(x) + K_N[-g(x)]_+{}^2$ Where K_N is the penalty parameter.

- Stationary point must satisfy

$\nabla f_p(x) = \nabla f(x) + 2 \langle -g(x) \rangle \nabla g(x) = 0$

- Solution tends to the precise solution as $K_N \rightarrow \infty$

simple physical interpretation no additional degrees of freedom;

smooth functional

solution is not exact:

too small penalty → large penetration;

too large penalty → ill-conditioning of the global matrix;

user has to choose penalty K_N properly

Parameter K_N Depends on the mesh size (space disc.) and time step (time disc.)

The lake of consistency of the penalty method is generally illustrated on the normal displacement graph where we can note a larger interpenetration compared to the other methods.

A simple static elastic example

On convergence of the penalty method for unilateral contact problems

Franz Chouly[a,*], Patrick Hild[b]

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[b] *Institut de Mathématiques de Toulouse - UMR 5219 (CNRS/INSAT/UT1/UT2/UT3), Université Paul Sabatier (UT3), 118 route de Narbonne, 31062 Toulouse Cedex 9, France.*

Abstract

We present a convergence analysis of the penalty method applied to unilateral contact problems in two and three space dimensions. We first consider, under various regularity assumptions on the exact solution to the unilateral contact problem, the convergence of the continuous penalty solution as the penalty parameter ε vanishes. Then, the analysis of the finite element discretized penalty method is carried out. Denoting by h the discretization parameter, we show that the error terms we consider give the same estimates as in the case of the constrained problem when the penalty parameter is such that $\varepsilon = h$.

Keywords: unilateral contact, variational inequality, finite elements, penalty method, a priori error estimates.
AMS Subject Classification: 65N12, 65N30, 35J86, 74M15.

https://hal.archives-ouvertes.fr/hal-00688641

Theorem : K_N = 1/h !

3D error analysis :
optimal CVG of order 1

A simple static elastic example

Pure Lagrangian method :

$$\mathcal{L}(x,\lambda) = f(x) + \boxed{\lambda g(x)} \rightarrow \text{Saddle point} \rightarrow \min_x \max_\lambda \mathcal{L}(x,\lambda)$$

Exact solution
no adjustable parameters

Non smooth:
Additional degrees of freedom increase the problem (primal/ dual)
Hard optimization problem
not fully unconstrained: $\lambda \leq 0$

Aug. Lagrangian method :

$$\mathcal{L}(x,\lambda) = f(x) + K_N[-g(x)]_+{}^2 + \boxed{\lambda g(x)} \rightarrow \min_x \max_\lambda \mathcal{L}(x,\lambda)$$

Uzawa algorithm $\boxed{\lambda^{i+1} = \lambda^i - 2K_N[-g(x)]_+}$ and λ^0=0

Good physical solution
smoothed functional
no additional degrees of
Freedom (primal problem)
Iterative algorithm (maybe slow)
Chattering may appear
Regularizing parameter K_N Depends (weekly) on the mesh size (space disc.) and time step (time disc.)

Ref. : [Hestnes 1969], [Powell 1969], [Glowinski, Le Tallec 1989], [Alart, Curnier 1991], [Simo, Laursen 1992]

Nitsche method (rigid obstacle)

$$[a]_+ = \frac{a+|a|}{2}$$

$\mathbf{u} : \Omega \to \mathbb{R}^2$ verifying the equations and conditions (1)–(2):

$$(1)\begin{cases} \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f} = \mathbf{0} & \text{in } \Omega, \\ \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{A}\,\boldsymbol{\varepsilon}(\mathbf{u}) & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_D, \\ \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \mathbf{g} & \text{on } \Gamma_N, \end{cases} \qquad (2)\begin{cases} u_n \leq 0 & (i) \\ \sigma_n(\mathbf{u}) \leq 0 & (ii) \\ \sigma_n(\mathbf{u})\, u_n = 0 & (iii) \\ \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) = 0 & (iv) \end{cases}$$

Let $\gamma > 0$. The contact conditions (2) (i)-(iii) on Γ_C are equivalent to:

$$\sigma_n(\mathbf{u}) = -\frac{1}{\gamma}[u_n - \gamma\sigma_n(\mathbf{u})]_+ .$$

$$a(\mathbf{u}, \mathbf{v}) - \int_{\Gamma_C} \sigma_n(\mathbf{u})\, v_n \, d\Gamma = L(\mathbf{v}).$$

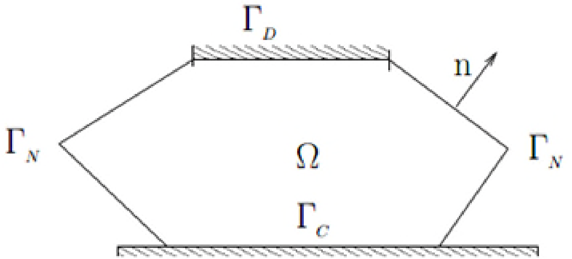
$$a(\mathbf{u}, \mathbf{v}) - \int_{\Gamma_C} \gamma\, \sigma_n(\mathbf{u})\, \sigma_n(\mathbf{v})\, d\Gamma + \int_{\Gamma_C} \frac{1}{\gamma}[u_n - \gamma\sigma_n(\mathbf{u})]_+ (v_n - \gamma\sigma_n(\mathbf{v}))\, d\Gamma = L(\mathbf{v}).$$

Our Nitsche-based method then reads:

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ such that:} \\ A_\gamma(\mathbf{u}^h, \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma} [P_\gamma(\mathbf{u}^h)]_+ P_\gamma(\mathbf{v}^h)\, d\Gamma = L(\mathbf{v}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h. \end{cases}$$

F. Chouly, P. Hild, **A Nitsche-based method for unilateral contact problems: numerical analysis**, 2012

- Consistency
- Stability
- Convergence
- Robustness
- Error analysis
- No added variable
- Independent to Nitsche parameter γ



Difficulties – Issues ?

Numerical problems encountered:

"chattering", locking, non-convergence, non-respect of the physics associated with inequalities, distorted energy balance, spurious oscillations, very small integration time step required, numerical chaos…

Theorical issues:

Existence and uniqueness SDH and **static elasticity without friction**: ok but nothing in general (**Stamapacchia's** theorem)

Existence only uniqueness SDH and elasticity WITH quasi static friction (coeff. Friction « small coeff », P. Ballard, G. Jarusek)

Bifurcations in the case of Signorini problem (Very difficult path-following of non-differentiable curves)

Sensitivity of the solution to CI or CB : OPEN problems

Even more delicate **dynamic contact problems** : *rigid solids theory* of JJ Moreau (uniqueness for a given impact coefficient),

sporadic results for *deformable solids cases* (L. Paoli, M. Schatzmann,Y. Dumont, A. Petrov, C. Pozzolini, Y. Renard, M. Salaun)

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2. Numerical and theoretical results beam's impact problems : time/ spaces schemes, energy conservation ?

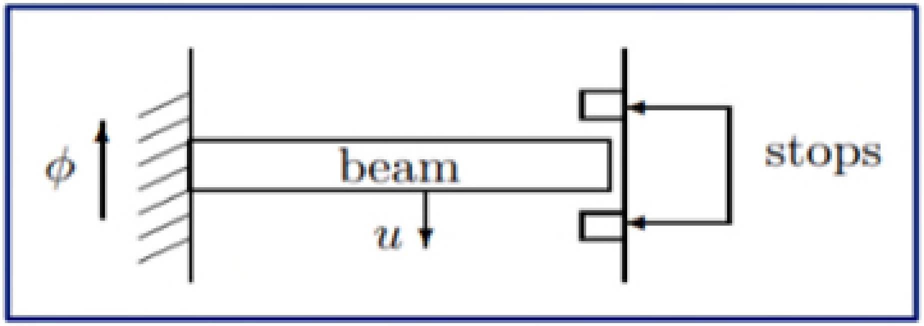
- Continuous beam model: extension on distributed obstacles from L.Kuttler & M.Shillor (2001) and Y. Dumond & Paoli (2006)
- Singular Mass Mathod : Y. Renard (2010)
- Vibro-impacts of a beam on rigid obstacles, Pozzolini, Salaun (2010)
- Energy conserving schemes, Pozzolini, Renard, Salaun (2013)
- Semi-discretized model: taking account of viscosity A. Khaddari (2025)

Euler-Bernoulli beam model with complementarity conditions

Find $u: [0, L] \times [0, T] \to \mathbb{R}$ such that:

$$\begin{cases} \rho S \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(\alpha EI \frac{\partial^3 u}{\partial x^2 \partial t}\right) + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 u}{\partial x^2}\right) = f - r, & \\ u(x,0) = u_0(x), \quad \dot{u}(x,0) = v_0(x), & \forall x \in [0,L], \\ u(0,t) = \phi(t), \; \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial^k u}{\partial x^k}(L,t) = 0, \quad \forall k \in \{2,3\}, & \forall t \in [0,T] \\ [r]_- \perp (u - g_1) \geq 0, \quad [r]_+ \perp (g_2 - u) \geq 0 & \end{cases}$$

$$[a]_+ = \frac{a + |a|}{2}$$



Beam model with penalization method

Find $u: [0, L] \times [0, T] \to \mathbb{R}$ such that:

$$\begin{cases} \rho S \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(\alpha E I \frac{\partial^3 u}{\partial x^2 \partial t}\right) + \frac{\partial^2}{\partial x^2}\left(E I \frac{\partial^2 u}{\partial x^2}\right) = f - r, \\ u(x,0) = u_0(x), \quad \dot{u}(x,0) = v_0(x), & \forall x \in [0, L], \\ u(0,t) = \phi(t), \ \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial^k u}{\partial x^k}(L,t) = 0, \quad \forall k \in \{2,3\}, & \forall t \in [0, T] \\ [r]_- \perp (u - g_1) \geq 0, \quad [r]_+ \perp (g_2 - u) \geq 0 \end{cases}$$

]

$$r(x,t) = \kappa \left[(u(x,t) - g_2(x))_+ - (g_1(x) - u(x,t))_+ \right.$$

Penalization

(relaxation method on Signorini conditions)

variationnal weak formulation

$$\begin{cases} \text{Find } u : [0,T] \in \mathbb{K} \text{ such that for almost every } t \in [0,T] \text{ and for every } w \in \mathbb{W} \\ \displaystyle\int_\Omega \left[\rho S \frac{\partial^2 u(t)}{\partial t^2}(w - u(t)) + \alpha EI \frac{\partial^3 u(t)}{\partial x^2 \partial t} \frac{\partial^2 (w-u(t))}{\partial x^2} + EI \frac{\partial^2 u(t)}{\partial x^2} \frac{\partial^2 (w-u(t))}{\partial x^2} \right] d\Omega \\ \qquad \geq \displaystyle\int_\Omega f(w - u(t))\, d\Omega \\ u(x,0) = u_0(x) \in \mathbb{K}, \quad \dot{u}(x,0) = v_0(x), \quad \forall \in \Omega \end{cases}$$

- Material domain: $\Omega = [0, \mathrm{L}]$
- Space of displacements: $\mathbb{W} = \{\mathrm{w} \in \mathrm{H}^2(\Omega; \mathbb{R}) | w(0) = 0, \partial_{\mathrm{x}} w(0) = 0\}$
- Space of velocities: $\mathbb{H} = \mathrm{L}^2(\Omega; \mathbb{R})$
- Set of admissible displacements: $\mathbb{K} = \{w \in \mathbb{W} | g_1(x) \leq w(\mathrm{x}) \leq g_2(x)\ a.e.\ x \in (0, L)\}$

Theorem K-S (2001) : existence if $\alpha = 0$, existence and unicity of a solution for $\alpha > 0$

K. L. Kuttler and M. Shillor, Vibrations of a beam between two stops. *Dyn. Contin. Discrete Impuls. Sys.* Ser B 8 (2001) 93–110.

semi-discretization FEM

$$(W - U(t))^T \left(M\, \ddot{U}(t) + \alpha\, K\, \dot{U}(t) + K\, U(t) \right) \geq (W - U(t))^T\, F(t), \quad \forall W \in \mathbb{K}^h,$$

$$\boldsymbol{U} = \begin{bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \\ \vdots \\ u_N \\ \theta_N \end{bmatrix} \qquad \boldsymbol{F} = \begin{bmatrix} f_u(\phi) \\ f_\theta(\phi) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

semi-discretization FEM : singular mass

Projection spaces:

$$\mathbb{W}^h = \text{vect}\{\phi_i \mid 1 \le i \le N_W\}, \quad \dim(\mathbb{W}^h) = N_W,$$

$$\mathbb{H}^h = \text{vect}\{\psi_i \mid 1 \le i \le N_H\}, \quad \dim(\mathbb{H}^h) = N_H.$$

$$N_H < N_W$$

Velocity conversion:

$$\sum_{j=1}^{N_H} v_j(t) \int_0^L \rho S \psi_j \psi_i \, dx = \sum_{s=1}^{N_W} \frac{\partial u_s}{\partial t}(t) \int_0^L \rho S \phi_s \psi_i \, dx. \Rightarrow CV(t) = B\dot{U}(t)$$

(U,V)-coupled equation:

$$B^T \dot{V}(t) + KU(t) = F(t) + G^T\lambda(t)$$

⬇

$$M\ddot{U}(t) + KU(t) = F(t) + G^T\lambda(t)$$

Singular mass matrix: $M = B^T C^{-1} B$

Surjectivity lemma: G is surjective on $\mathbb{F} = \text{Ker}(B)$
$\Rightarrow \exists \mathbb{F}^c \subset \text{Ker}(G) | \mathbb{F} \oplus \mathbb{F}^c = \mathbb{R}^{Nw}$

Yves Renard's theorem: Existence and uniqueness of solution & Conservation of energy.

elasto-dynamic 2nd order hyperbolic 1D/2D contact problem

$$\begin{cases} \text{Find } u : [0,T] \to K \text{ such that for a.e. } t \in (0,T], \\ \langle \frac{\partial^2 u}{\partial t^2}(t), w - u(t) \rangle_{W',W} + a(u(t), w-u(t)) \geq l(w-u(t)) \quad \forall w \in K, \\ u(0) = u_0, \quad \frac{\partial u}{\partial t}(0) = v_0. \end{cases}$$

second order hyperbolic operator (membrane, bar …)

$$\begin{cases} \text{Find } U : [0,T] \to \overline{K}^h \text{ such that} \\ (W - U(t))^T (M\ddot{U}(t) + AU(t)) \geq (W - U(t))^T L. \quad \forall W \in \overline{K}^h, \forall t \in (0,T], \\ U(0) = U_0, \quad B\dot{U}(0) = CV_0. \qquad M = B^T C^{-1} B, \; V(t) = C^{-1} B\dot{U}(t). \end{cases} \quad \text{(P)}$$

Theorem : If we put $M = B^T C^{-1} B$ then Problem (P) admits a unique solution. Moreover, this solution is Lipschitz-continuous with respect to t. And then the solution *U (t)* to Problem (P) is energy conserving in the sense that the discrete energy is constant with respect to t.

Y. Renard. The singular dynamic method for constrained second order hyperbolic equations. application to dynamic contact problems. *J. Comput. Appl. Math*, 2010.

elasto-dynamic 1D/2D contact problem

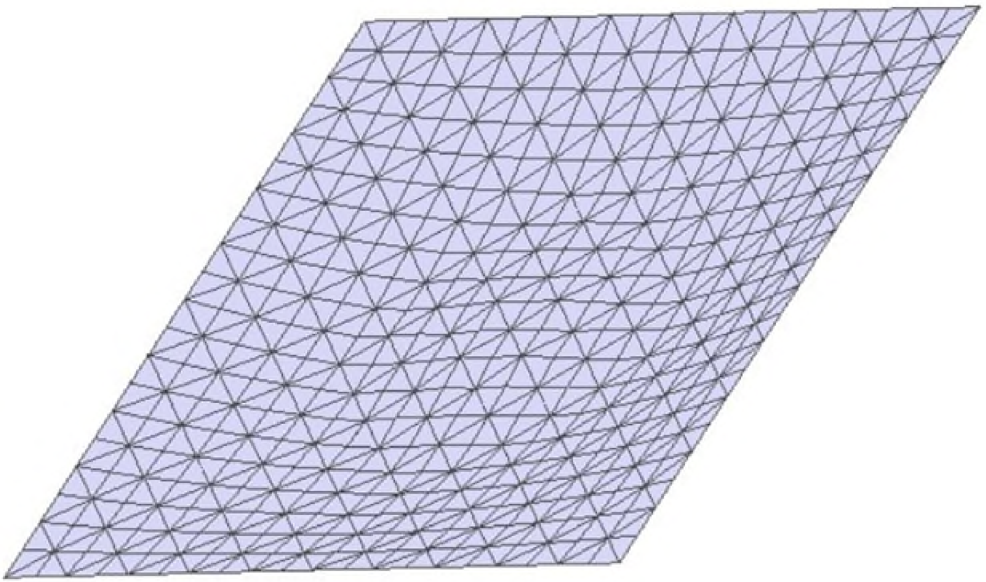


Figure 1: $h = 0.05$.

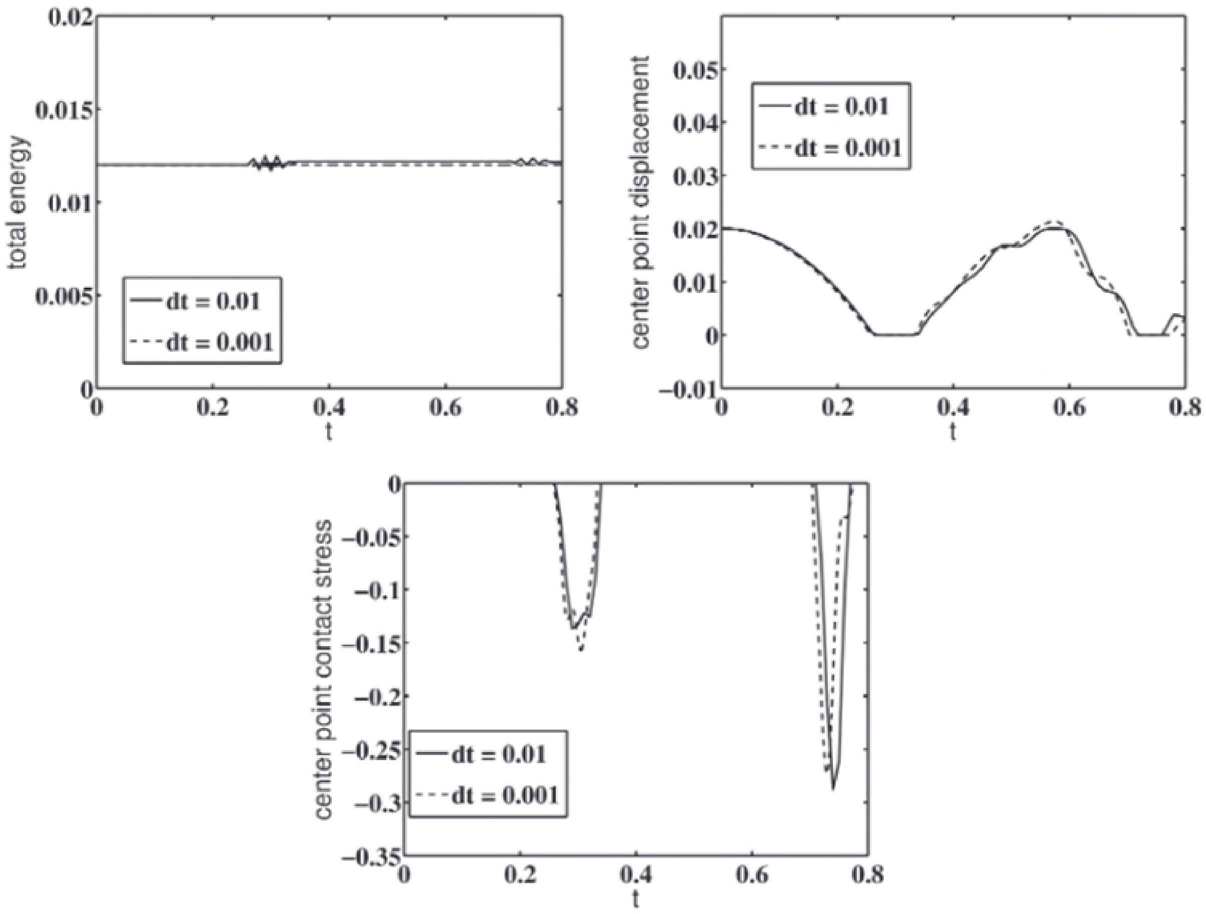


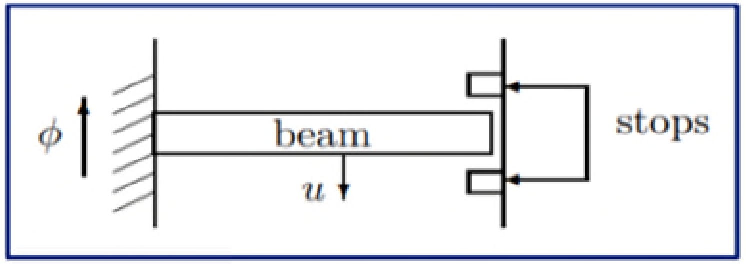
Figure 6: *Evolution of the energy, the displacement at the center point* $(0.5, 0.5)$ *and the contact stress at the center point for a* P_2/P_1 *method, a midpoint scheme and with* $h = 0.1$.

Y. Renard. The singular dynamic method for constrained second order hyperbolic equations. application to dynamic contact problems. *J. Comput. Appl. Math*, 2010.

elasto-dynamic 4th order hyperbolic 1D/2D contact problem

$$\begin{cases} \text{Find } u \;:\; [0,T] \to \mathbb{K} \text{ such that for almost every } t \in [0,T] \text{ and for every } w \in \mathbb{W} \\ \int_\Omega \left[\rho S \, \frac{\partial^2 u}{\partial t^2}(t)\,(w - u(t)) \;+\; EI\,\frac{\partial^2 u}{\partial x^2}(t)\,\frac{\partial^2 (w-u(t))}{\partial x^2} \right] d\Omega \;\geq\; \int_\Omega f\,(w-u(t))\,d\Omega \\ u(x,0) = u_0(x) \in \mathbb{K}, \quad \dot{u}(x,0) = v_0(x), \quad \forall x \in \Omega. \end{cases}$$

Assuming that $f \in L^2$, Kuttler and Schillor proved existence (2001).



$$\rho S \frac{\partial^2 u}{\partial t^2}(x,t) + EI \frac{\partial^4 u}{\partial x^4}(x,t) = f(x,t)$$

fourth order operator (Euler-Bernouilli BEAM)

if no − *damping* $\alpha = 0$

$$\mathbf{C}\; V(t) = \mathbf{B}\; \dot{U}(t).$$

Since $\mathbf{C}$ is always invertible, we obtain $V(t) = \mathbf{C}^{-1}\;\mathbf{B}\;\dot{U}(t)$ and, then, $\dot{V}(t) = \mathbf{C}^{-1}\;\mathbf{B}\;\ddot{U}(t)$, which allows to eliminate V. So the semi-discretized problem (2.1) is equivalent to

$$\begin{cases} \text{Find } U \;:\; [0,T] \to \mathbb{K}^h \text{ and } V \;:\; [0,T] \to \mathbb{H}^h \text{ such that for all } t \in (0,T] \\ (W - U(t))^T\;(\mathbf{M}\ddot{U}(t) + \mathbf{K}U(t)) \;\geq\; (W-U(t))^T\;F, \quad \forall W \in \mathbb{K}^h, \\ \mathbf{C}\;V(t) = \mathbf{B}\;\dot{U}(t), \\ U(0) = U_0, \quad V(0) = V_0, \end{cases} \tag{2.2}$$

where $\mathbf{M}$ is the so-called singular mass matrix defined by

$$\mathbf{M} = \mathbf{B}^T\;\mathbf{C}^{-1}\;\mathbf{B}. \tag{2.3}$$

Elasto-dynamic 4th order hyperbolic 1D/2D contact problem

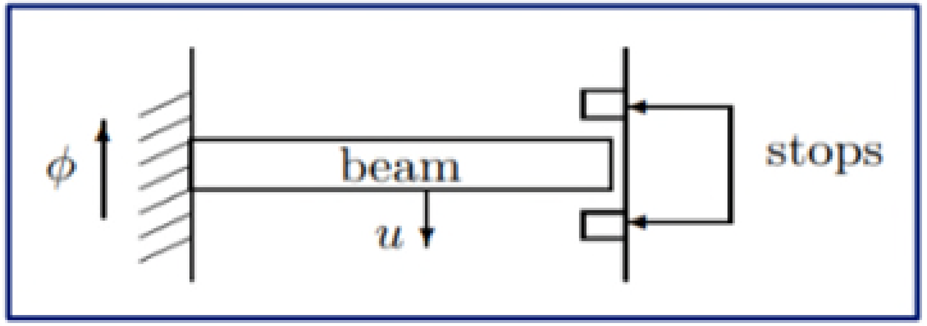
$$\mathbf{C}\ V(t) = \mathbf{B}\ \dot{U}(t).$$

Since $\mathbf{C}$ is always invertible, we obtain $V(t) = \mathbf{C}^{-1}\ \mathbf{B}\ \dot{U}(t)$ and, then, $\dot{V}(t) = \mathbf{C}^{-1}\ \mathbf{B}\ \ddot{U}(t)$, which allows to eliminate V. So the semi-discretized problem (2.1) is equivalent to

(P)
$$\begin{cases} \text{Find } U\ :\ [0,T] \to \mathbb{K}^h \text{ and } V\ :\ [0,T] \to \mathbb{H}^h \text{ such that for all } t \in (0,T] \\ (W - U(t))^T\ (\mathbf{M}\ddot{U}(t) + \mathbf{K}U(t))\ \geq\ (W - U(t))^T\ F,\quad \forall W \in \mathbb{K}^h, \\ \mathbf{C}\ V(t) = \mathbf{B}\ \dot{U}(t), \\ U(0) = U_0,\quad V(0) = V_0, \end{cases}$$

where $\mathbf{M}$ is the so-called singular mass matrix defined by

$$\mathbf{M} = \mathbf{B}^T\ \mathbf{C}^{-1}\ \mathbf{B}.$$



forth order operator (Euler-Bernouilli BEAM)

if no- damping $\alpha = 0$

Theorem : If we put $M = B^T C^{-1} B$ then Problem (P) admits a unique solution. Moreover, this solution is Lipschitz-continuous with respect to t. And then the solution *U (t)* to Problem (P) is energy conserving in the sense that the discrete energy is constant with respect to t.

C. Pozzolini, M. Salaun, Some energy conservative schemes for vibro-impacts of a beam on rigid obstacles, ESAIM: M2AN 45 (2011)

elasto-dynamic beam contact problem

$$\left\{\begin{array}{l} \text{Find } U^{n+1,e} = \dfrac{U^{n+1} + eU^{n-1}}{1+e} \in \mathbb{K}^h \text{ such that for all } W \in \mathbb{K}^h \\ (W - U^{n+1,e})^T \left(\mathbf{M}_r \dfrac{U^{n+1} - 2U^n + U^{n-1}}{\Delta t^2} + \mathbf{K} \left(\beta U^{n+1} + (1-2\beta)U^n + \beta U^{n-1}\right)\right) \\ \qquad \geq (W - U^{n+1,e})^T \; F^{n,\beta}. \end{array}\right.$$

MATLAB

Time scheme : Newmark-Dumont-Paoli-Schatzman with restitution coefficient $e \geq 0$

Contact inequality : aug. lag. Method (*quadprog :* **MATLAB**)

L. Paoli and M. Schatzman, Schéma numérique pour un modèle de vibrations avec contraintes unilatérales et perte d'énergie aux impacts, en dimension finie, C. R. Acad. Sci. Paris Sér. I Math., 317 (1993)

Y. Dumont and L. Paoli, Vibrations of a beam between obstacles: convergence of a fully discretized approximation. ESAIM: M2AN 40 (2006)

C. Pozzolini, M. Salaun, Some energy conservative schemes for vibro-impacts of a beam on rigid obstacles, ESAIM: M2AN 45 (2011)

Total discretisation following Dumont-Paoli (2006)

- e-Newmark-Dumont-Paoli $\left(\beta = \frac{1}{2}, \gamma = \frac{1}{2}\right)$ scheme

$\gamma = \frac{1}{2}, \beta = 0$: impl. central Diff.

$\gamma = \frac{1}{2}, \beta = \frac{1}{4}$ average acc. scheme

if no- damping $\alpha = 0$

$$\begin{cases} \text{Find } U^{n+1,e} \equiv \dfrac{U^{n+1} + eU^{n-1}}{1+e} \in \mathbb{K}^h \text{ such that for all } W \in \mathbb{K}^h \\ (W - U^{n+1,e})^T \left(\mathbf{M} \dfrac{U^{n+1} - 2U^n + U^{n-1}}{\Delta t^2} + \mathbf{K} \left(\beta U^{n+1} + (1-2\beta) U^n + \beta U^{n-1} \right) \right) \\ \qquad \geq (W - U^{n+1,e})^T \left(\beta \tilde{F}^{n+1} + (1-2\beta)\tilde{F}^n + \beta \tilde{F}^{n-1} \right) \end{cases}$$

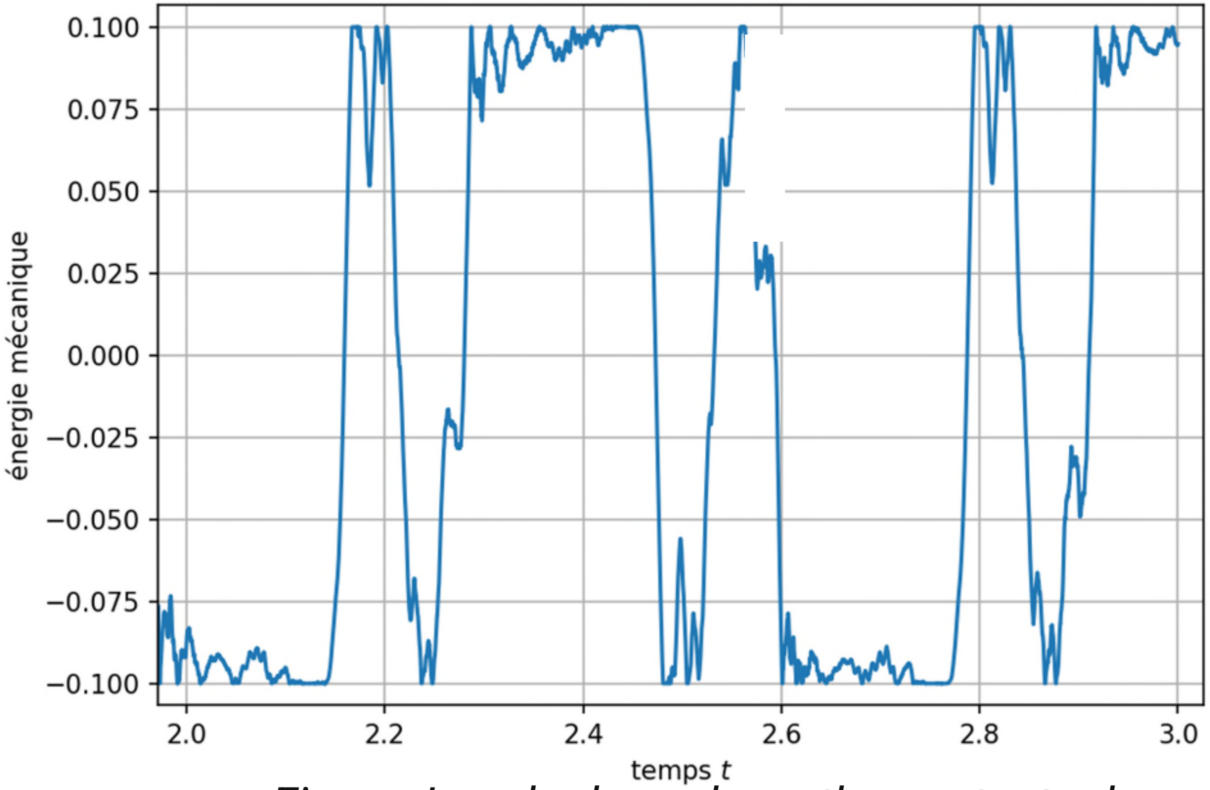
- Mid-point scheme

$$\begin{cases} \text{Find } U^{n+1/2} \in \mathbb{K}^h \text{ such that:} \\ (W - U^{n+1/2})^T (\mathbf{M}_r \ddot{U}^{n+1/2} + \mathbf{K} U^{n+1/2}) \geq (W - U^{n+1/2})^T F^n, \quad \forall W \in \mathbb{K}^h, \\ U^{n+1/2} = \dfrac{U^n + U^{n+1}}{2}, \quad \dot{U}^{n+1/2} = \dfrac{\dot{U}^n + \dot{U}^{n+1}}{2}, \end{cases}$$

Y. Dumont and L. Paoli, Vibrations of a beam between obstacles: convergence of a fully discretized approximation. ESAIM: M2AN 40 (2006)

Numerical Simulation of pipe impact : spurious oscillations

If no-damping $\alpha = 0$



<u>Figure:</u> Local rebounds on the contact edges

Some energy conservative schemes for vibro-impacts of a beam on rigid obstacles, C. Pozzolini, M. Salaun ESAIM: M2AN 45 (2011) 1163-1192

elasto-dynamic beam contact problem

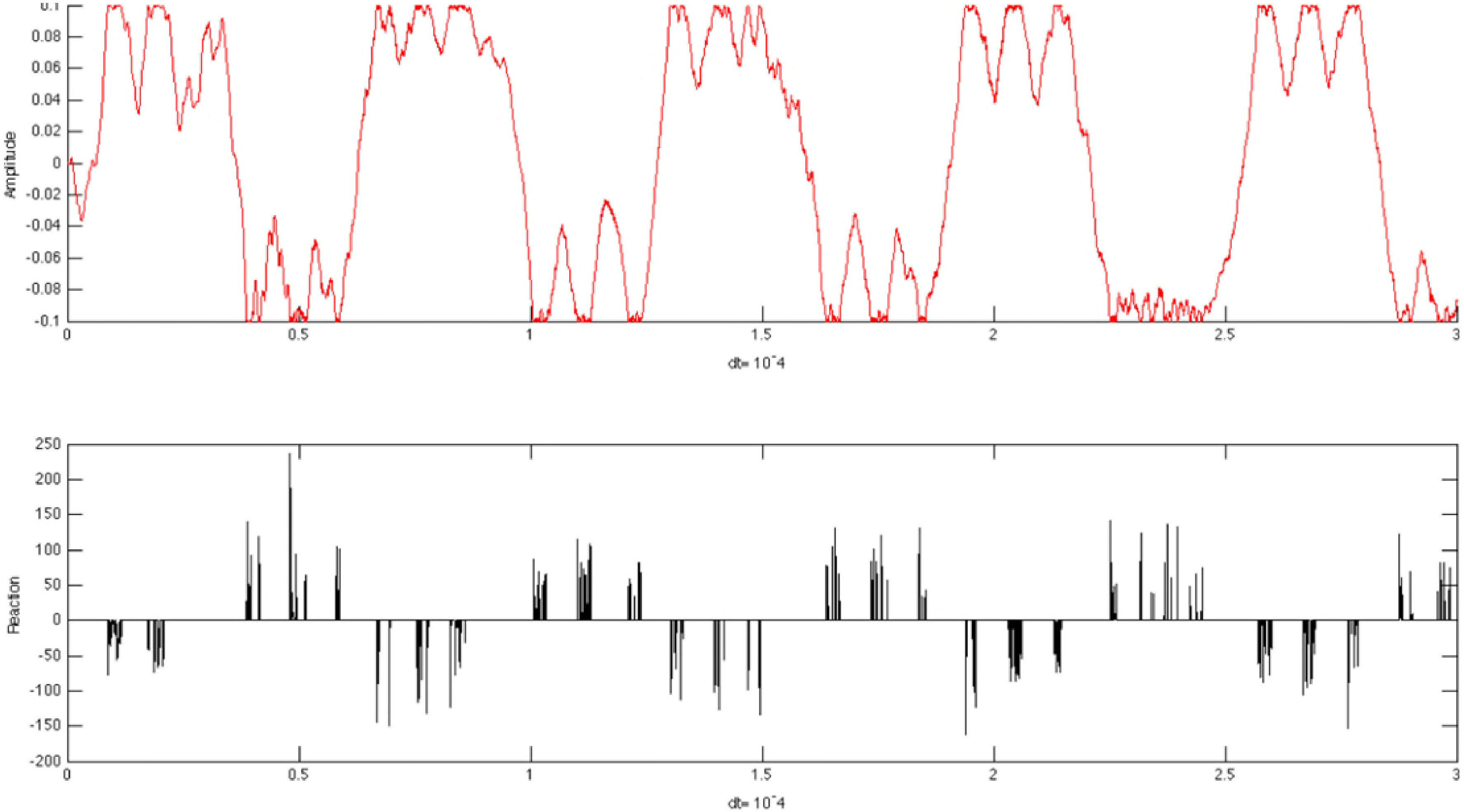


FIGURE 5. Displacement and reaction with NDP P3/P0 singular mass matrix. $e = 0$, $\Delta t = 10^{-4}$, 100 elements.

MATLAB

C. Pozzolini, M. Salaun, Some energy conservative schemes for vibro-impacts of a beam on rigid obstacles, ESAIM: M2AN 45 (2011)

elasto-dynamic beam contact problem

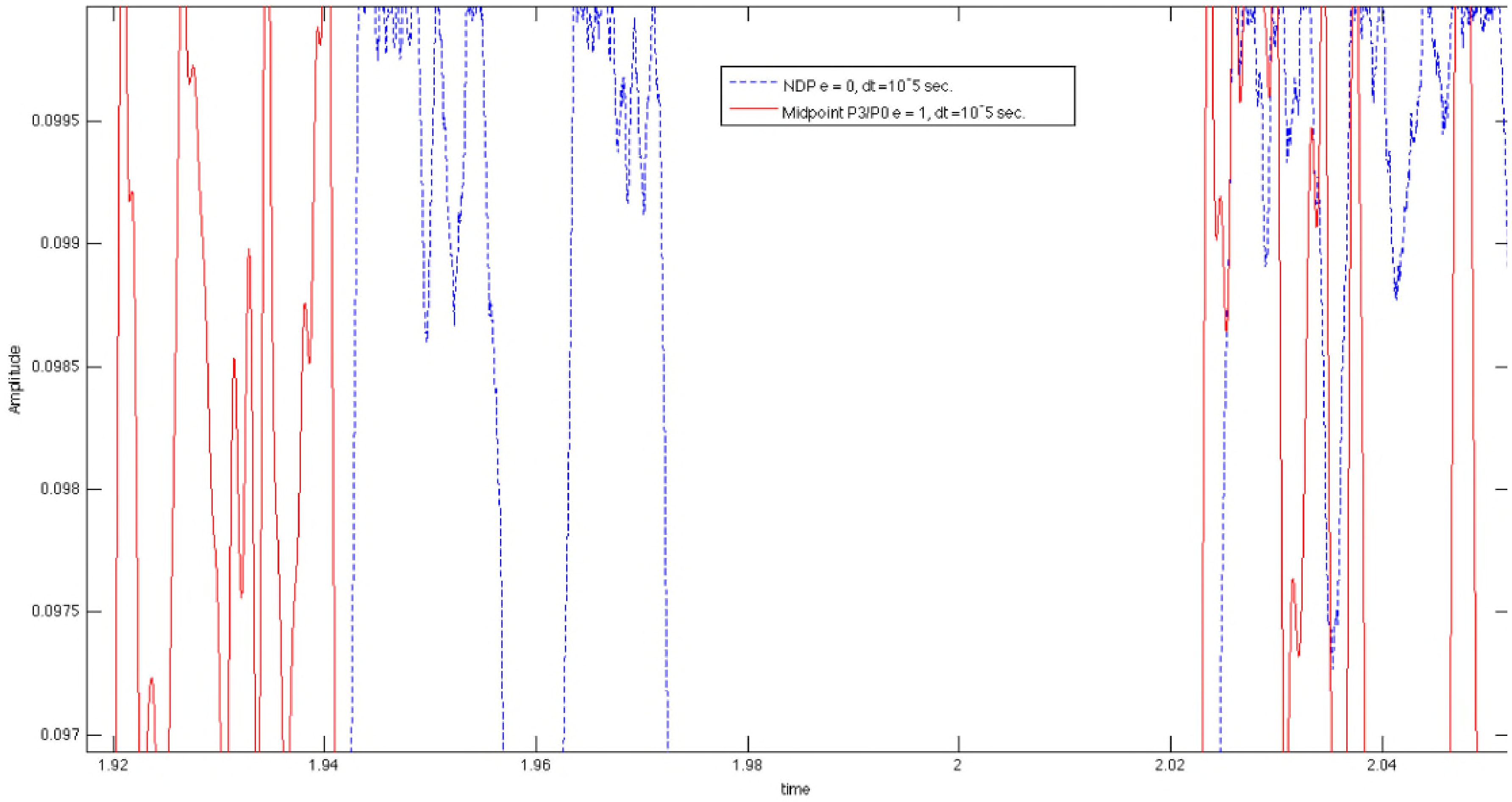


FIGURE 8. Zoom - Midpoint P3/P0 singular mass matrix ($e = 1$) *versus* NDP regular mass matrix ($e = 0$). $\Delta t = 10^{-5}$, 100 elements.

C. Pozzolini, M. Salaun, Some energy conservative schemes for vibro-impacts of a beam on rigid obstacles, ESAIM: M2AN 45 (2011)

MATLAB

elasto-dynamic beam contact problem

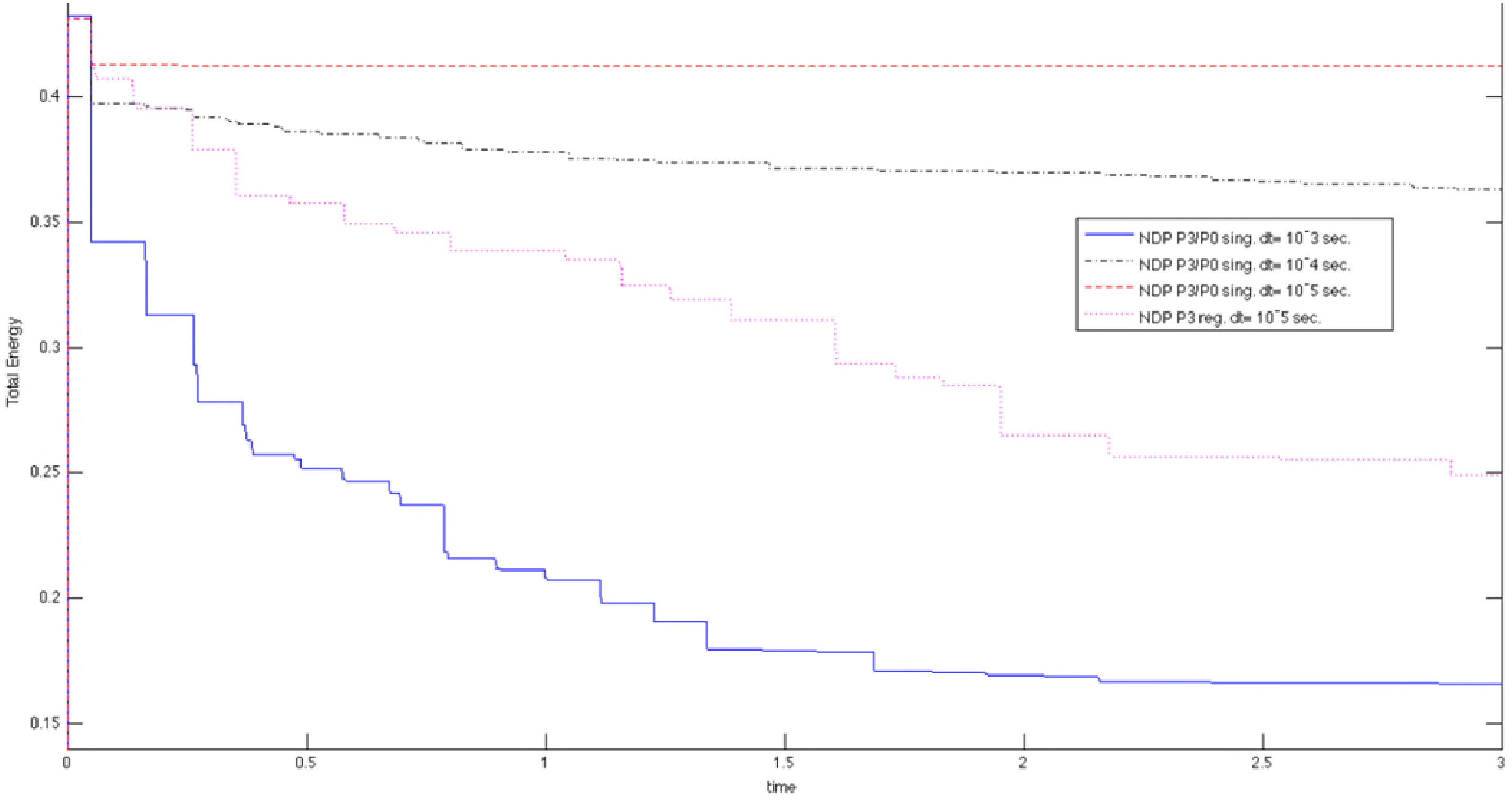


FIGURE 15. Energy for different time steps – Regular *versus* P3/P0 singular mass matrix for NDP scheme. $e = 0$, $\Delta t = 10^{-3}$, 10^{-4}, 10^{-5}, 39 elements.

C. Pozzolini, M. Salaun, Some energy conservative schemes for vibro-impacts of a beam on rigid obstacles, ESAIM: M2AN 45 (2011)

MATLAB

elasto-dynamic beam contact problem

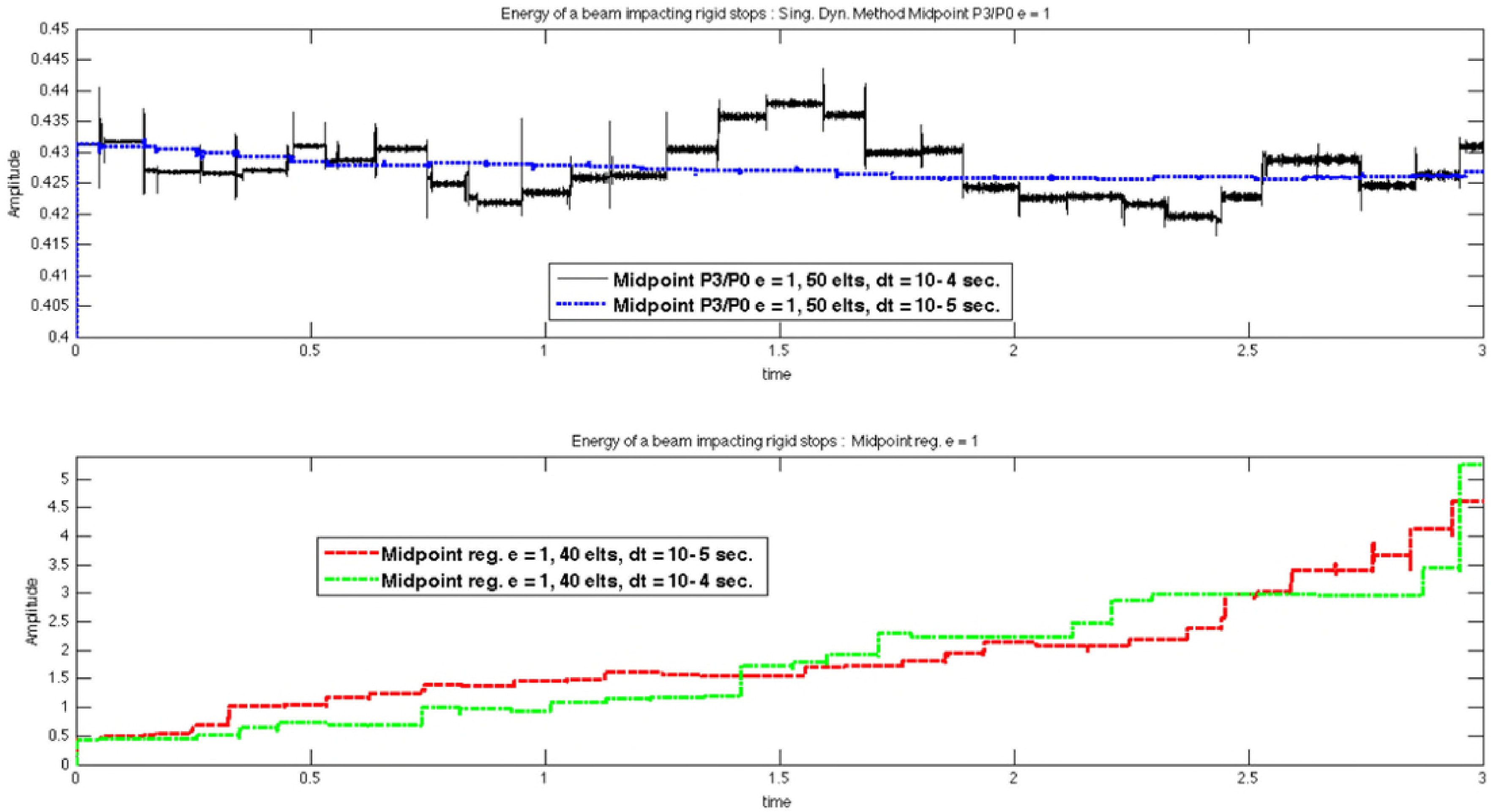


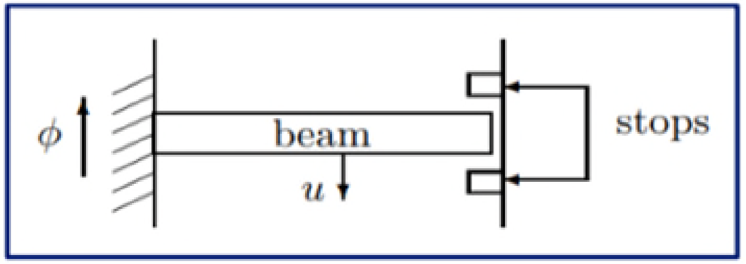
FIGURE 18. Regular *versus* P3/P0 singular mass matrix. Midpoint scheme. $e = 1$, $\Delta t = 10^{-4}$, 10^{-5}, 40 and 50 elements.

C. Pozzolini, M. Salaun, Some energy conservative schemes for vibro-impacts of a beam on rigid obstacles, ESAIM: M2AN 45 (2011)

Explaining singular mass matrix method for viscoelastic beam semi-discret impact problem

Work in progress Ayman Khaddari Phd studiant Framatome – Insa Lyon (with Yves Renard):

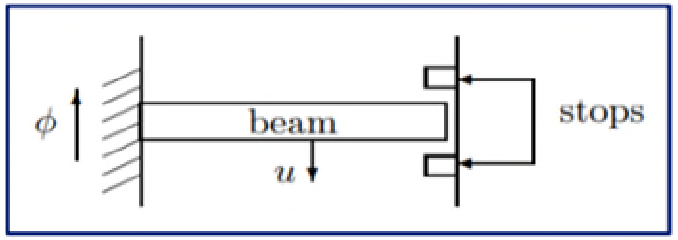
Extend the **Singular mass matrix theorem** to the viscous case.



$$(W - U(t))^T \left(M\ddot{U}(t) + \alpha K \dot{U}(t) + K U(t) \right) \geq (W - U(t))^T F(t), \quad \forall W \in \mathbb{K}^h,$$

$$\begin{cases} \text{Find } u : [0,T] \in \mathbb{K} \text{ such that for almost every } t \in [0,T] \text{ and for every } w \in \mathbb{W} \\ \displaystyle\int_\Omega \left[\rho S \frac{\partial^2 u(t)}{\partial t^2}(w - u(t)) + \alpha EI \frac{\partial^3 u(t)}{\partial x^2 \partial t} \frac{\partial^2 (w - u(t))}{\partial x^2} + EI \frac{\partial^2 u(t)}{\partial x^2} \frac{\partial^2 (w - u(t))}{\partial x^2} \right] d\Omega \\ \qquad \geq \displaystyle\int_\Omega f(w - u(t))\, d\Omega \\ u(x,0) = u_0(x) \in \mathbb{K}, \quad \dot{u}(x,0) = v_0(x), \quad \forall \in \Omega \end{cases}$$

semi-discretization FEM : beam impact problem



$$(W - U(t))^T \left(M\,\ddot{U}(t) + \alpha\, K\,\dot{U}(t) + K\,U(t) \right) \geq (W - U(t))^T\, F(t), \quad \forall W \in \mathbb{K}^h,$$

	Continuous Problem	Semi-discretized problem
Damping + Penalty	$\exists!\ u_{\alpha,\epsilon} \in L^\infty(0,T;\mathbb{W})\vert$ $u'_{\alpha,\epsilon} \in L^2(0,T;\mathbb{H}) \cap L^\infty(0,T;\mathbb{W})$ $u''_{\alpha,\epsilon} \in L^2(0,T;\mathbb{W}')$	$\exists!\ u_{\alpha,\epsilon} \in C^1(0,T;\mathbb{R}^{Nw})$
No Damping + Penalty	$\exists\ u_\epsilon \in L^\infty(0,T;\mathbb{W})\vert$ $u'_\epsilon \in L^\infty(0,T;\mathbb{H})$ $u''_\epsilon \in L^2(0,T;\mathbb{W}')$	$\exists!\ u_\epsilon \in C^1(0,T;\mathbb{R}^{Nw})$
Damping + Complementarity	$\exists\ u_\alpha \in L^2(0,T;\mathbb{K})\vert$ $u'_\alpha \in L^2(0,T;\mathbb{W})$	$\exists u \in W^{1,2}(0,T;\mathbb{K} \cap \mathbb{F}) + W^{2,2}(0,T;\ \mathbb{F}^c)$
No Damping + Complementarity	$\exists\ u \in L^2(0,T;\mathbb{K})\vert$ $u' \in L^2(0,T;\mathbb{H})$	$\exists!\ u \in W^{1,\infty}(0,T;\mathbb{K} \cap \mathbb{F}) + W^{3,\infty}(0,T;\ \mathbb{F}^c)$

Readapting Kuttler-Shillor's proofs

Yves Renard's proof

$\exists$: Existence

$\exists!$: Existence and uniqueness

A. Khaddari Phd studiant Framatome-Insa lyon

Total discretisation following Dumont-Paoli (2006)

if $\alpha > 0$ work in progress Ayman Khaddari Phd studiant Framatome – Insa Lyon (with Yves Renard):

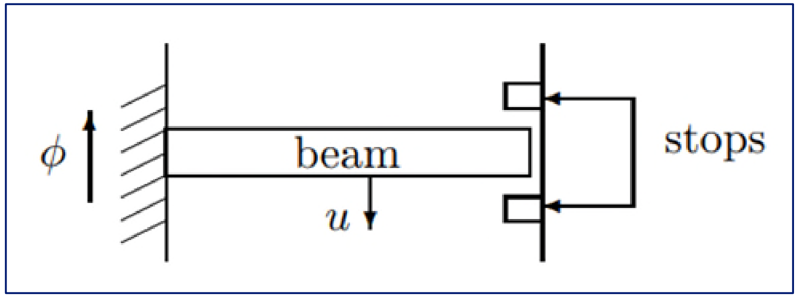
$$\begin{cases} \text{Find } U^{n+1,e} \equiv \dfrac{U^{n+1} + eU^{n-1}}{1+e} \in \mathbb{K}^h \text{ such that for all } W \in \mathbb{K}^h \\ (W - U^{n+1,e})^T \left(\mathbf{M} \dfrac{U^{n+1} - 2U^n + U^{n-1}}{\Delta t^2} + \alpha \mathbf{K} \dfrac{3U^{n+1} - 4U^n + U^{n-1}}{2\Delta t} \right. \\ \qquad \left. + \mathbf{K} \left(\beta U^{n+1} + (1-2\beta)U^n + \beta U^{n-1} \right) \right) \geq (W - U^{n+1,e})^T \, F^{n,\beta} \end{cases}$$

e-Newmark-Dumont-Paoli $\left(\beta = \frac{1}{2}, \gamma = \frac{1}{2}\right)$

$\gamma = \frac{1}{2}, \beta = 0$: impl. central Diff.

$\gamma = \frac{1}{2}, \beta = \frac{1}{4}$ average acc. scheme

Numerical Simulation of pipe impact



Déplacement imposed disp. :

$$\phi(t) = A \cdot sin(\omega t)$$

Properties :

$$\rho = 8000\ kg/m^3$$

$$E = 2 \cdot 10^{11}\ Pa$$

$$L = 1.501\ m$$

$$d_e = 1\ cm$$

$$d_i = 0.9\ cm$$

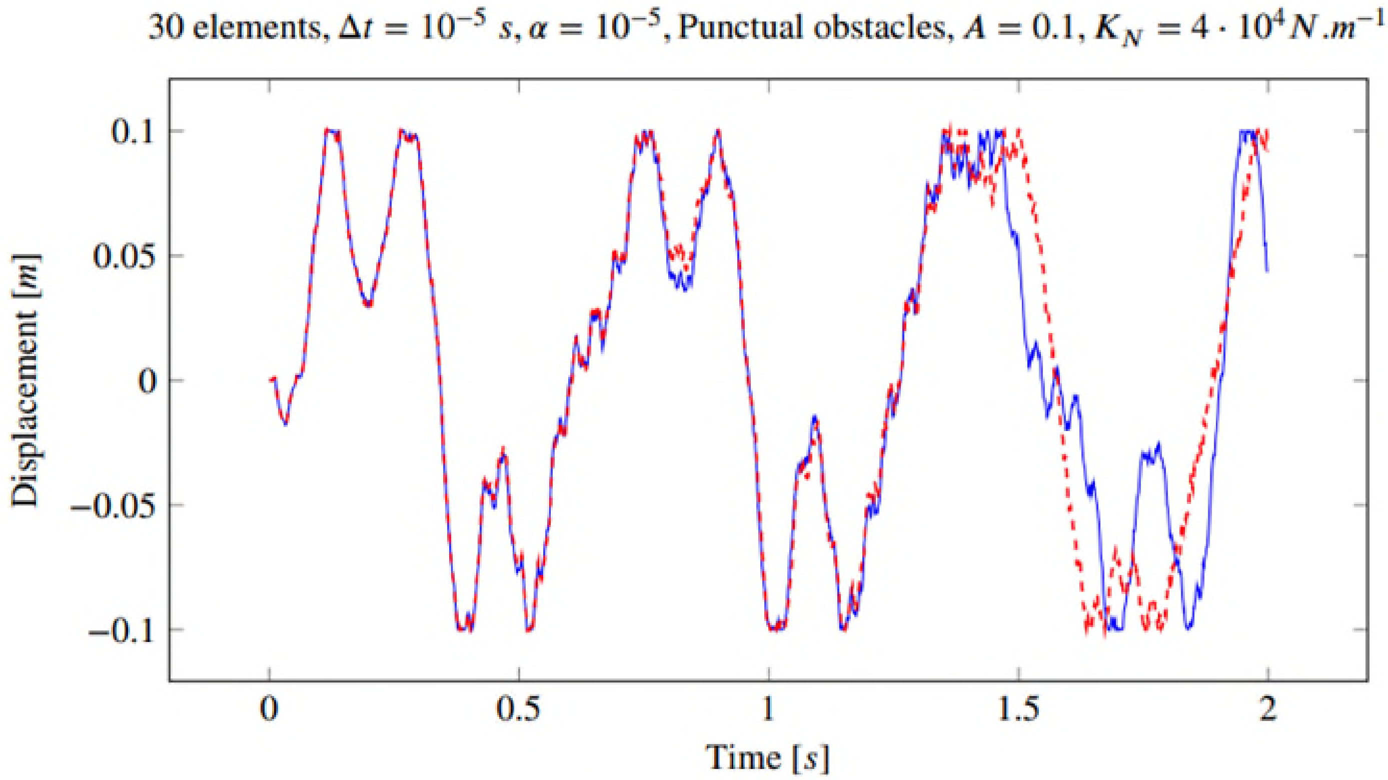
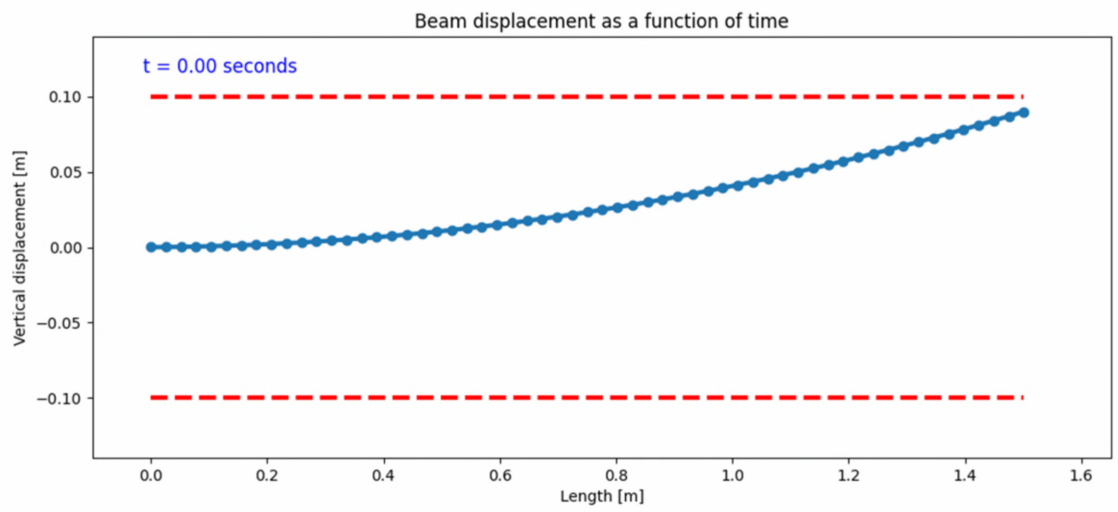


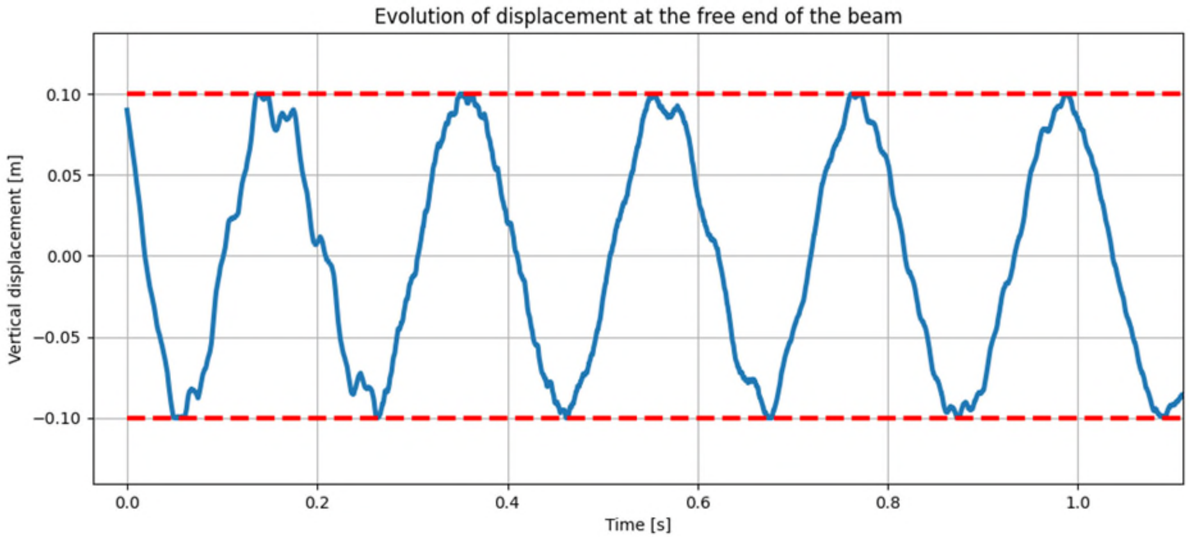
Figure 4: Comparison of the Augmented Lagrangian method (—) and the penalty method (—)

Numerical Simulation of pipe impact

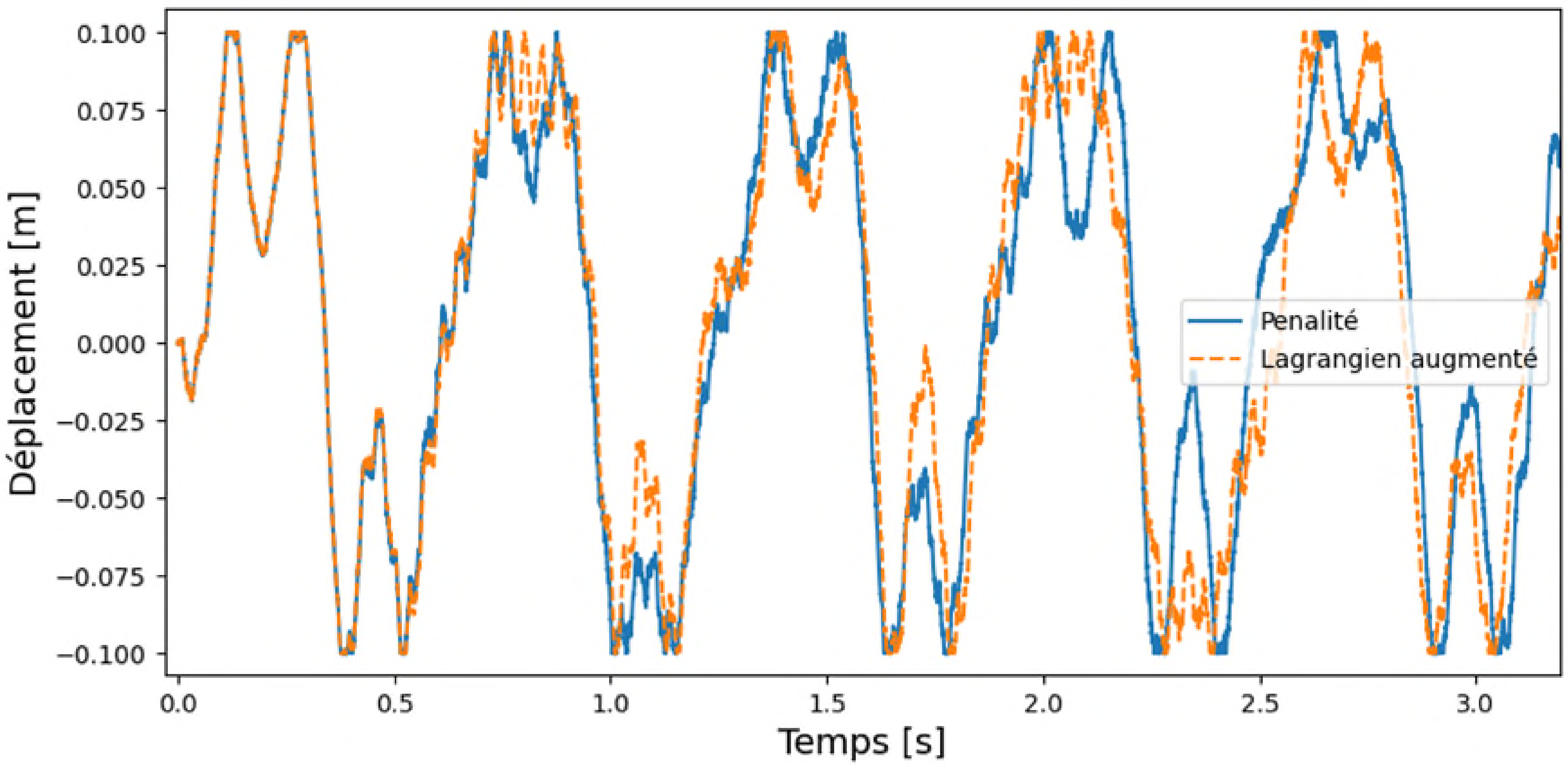


⇐ Animation of mouvement over time

Evolution of the displacement of the end of the beam ⇒



Numerical Simulation of pipe impact



Contact on one point

without damping

30 elements

dt = 10^-4

3

3. Birfurcations and chaos : Numerical comparisons of resolution methods

- Definitions bifurcations, diagrams, Poincaré’s sections and chaos
- Characterization of non-repeatability zones of vibro-impact of a beam NDP (M. Schorsch, intership in Framatome, 2023)
- Results comparison for diverse methods of impact simulations (A. Khaddari, Phd studiant, 2025)

3 - Problem defining constants

Beam Properties:

Young coefficient: $E = 2 \times 10^{11} Pa$
Geometry: Hollow cylinder
Moment of Inertia: $I = 1{,}7 \times 10^{-10} Kg.m^2$
Section: $S = 1{,}5 \times 10^{-5} m^2$
Density: $\rho = 8 \times 10^3 Kg.m^{-3}$
Damping: $\alpha = 1 \times 10^{-4} s$
Length: $L = 1{,}5\ m$

Scheme parameters:

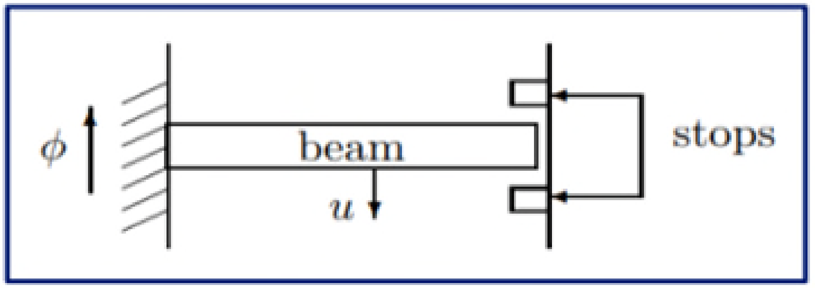
Mass matrix type: singular or regular
Contact type: Augmented Lagrangian method or penalization method
Number of elements: $Nbel$
Time-step: dt
Newmark-β,1/2-Dumont-Paoli Scheme parameters: β or $e = 0$

Obstacles:

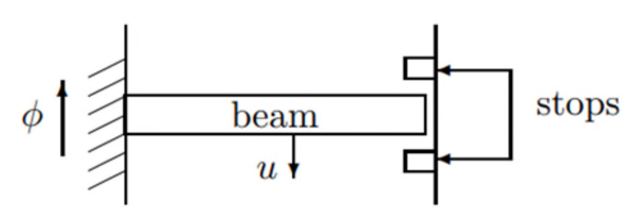
Upper bound: $+0.1\ m$
Lower bound: $-0.1\ m$

Clamping solicitation:

Amplitude: amp
Angular frequency: $omega = 10$



Definitions and objectives

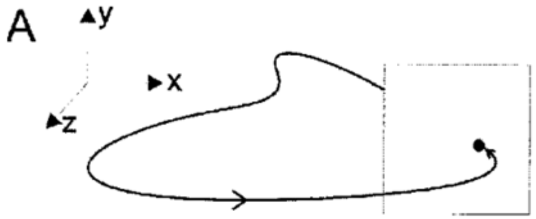


- Stationarization: use of the moving average over periods 'kT'.
- Poincaré section: the set of points of the solution (in phase portrait) of the free node on the times 'kT'.
- Lyapunov exponents: parameter λ of the gap evolution between the perturbed solution and the initial solution; i.e. $|u(t) - u_{pert}(t)| \approx Ae^{\lambda t}$
- Bifurcation: Small change in a parameter change in steady-state type (period, quasi-periodic, chaos)
- Bifurcation diagram: shows the evolution of the dynamics of a solution as a function of a system parameter.
- Characterize the regime encountered: periodic, quasi-periodic, chaotic
- Identify areas of non-repeatability: parametric analysis, bifurcation diagram

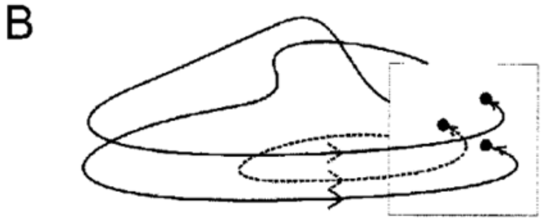
- Why do we plot the probability distributions of contact reactions?
  - Chaotic solutions are very sensitive to initial conditions.
  - Probability distributions are invariant over the steady state.

Methods for characterizing stationary regimes

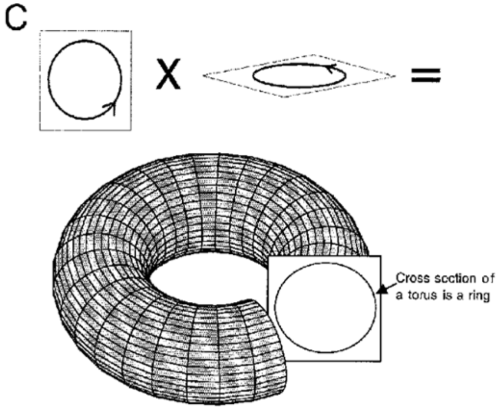
- Phase portrait : velocity versus displacement
- Poincaré Section



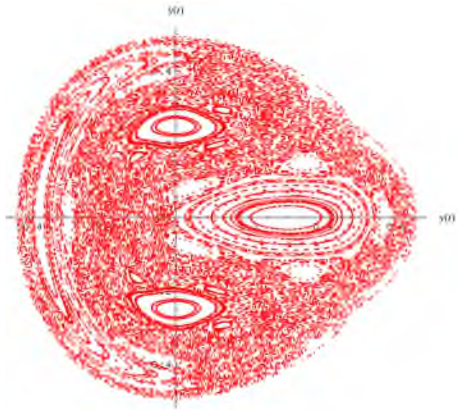
1 T periodic = 1 point



3 T periodic = 3 points



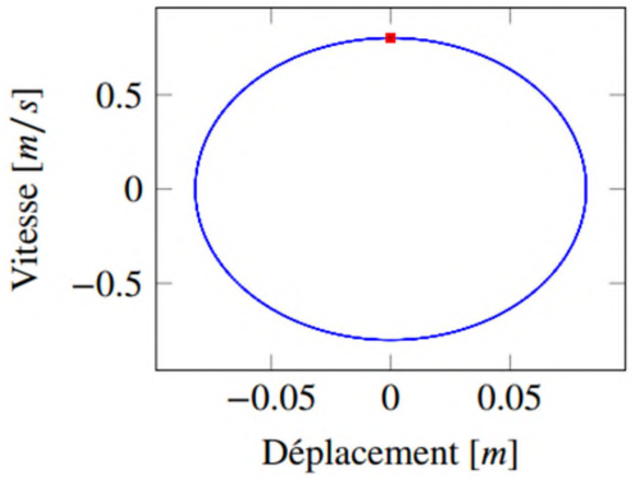
Quasi-periodic = 1 orbit



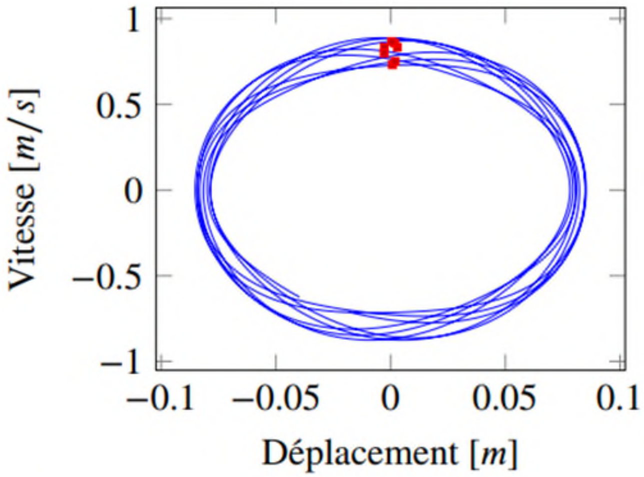
Chaos = dimension > 1

Methods for characterizing stationary regimes

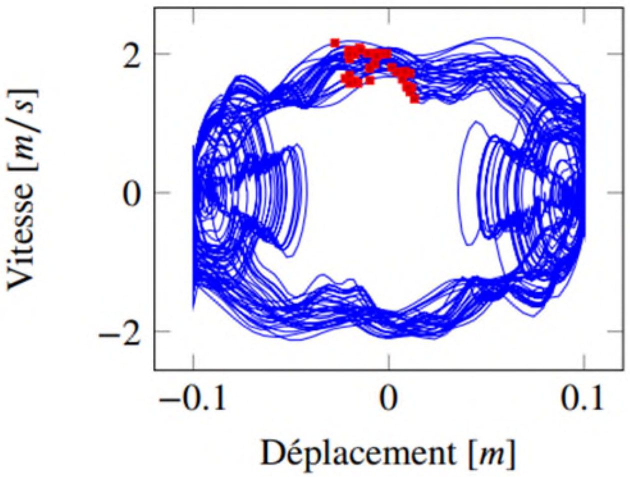
- Poincaré section : Intersection phase portrait obirts at each forcing period $T = \frac{1}{f} = \frac{2\pi}{\omega}$



(a) Régime périodique (6 périodes)



(b) Régime quasi-périodique (6 périodes)



(c) Régime chaotique (37 périodes)

Identifying k-periodicity by looking at the spacing of points in the Poincaré section

steady state ?

To differentiate a quasi-periodic regime from a chaotic regime:

Property of a chaotic regime:

A chaotic system exhibits extreme sensitivity to initial conditions.

Consequence :

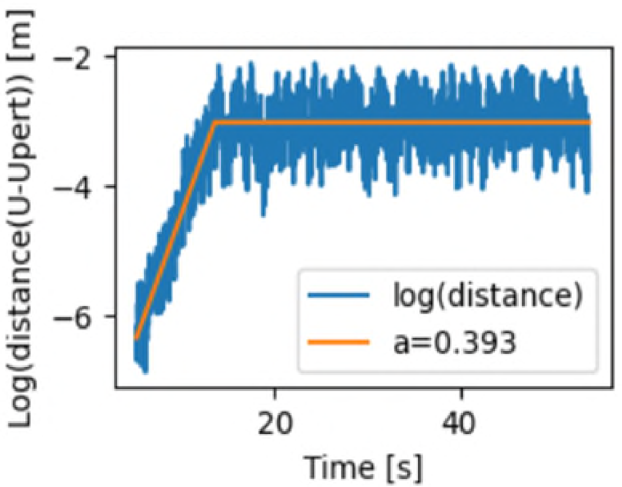
the distance $\delta U_d(t)$ between two infinitely close trajectories of a chaotic system (with an initial separation of δU_{d0}, increases exponentielly such that :

$$|\delta U_d(t)| \approx e^{\lambda t}|\delta U_{d0}|$$

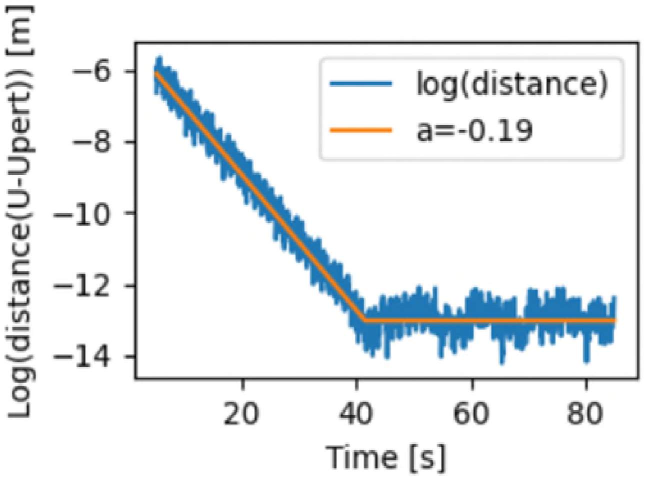
1) Growing exponentielly
2) Converge to a finite scalar

steady state ?

Exponential growth on (x,y) becomes a linear growth on diagram (x,log(y))



a) chaotic



b) quasi-periodic

Slope Coefficient of linear growth : $\boldsymbol{\lambda}$ **Lyapunov coefficient**

$$|\delta U_d(t)| \approx e^{\lambda t}|\delta U_{d0}|$$

$$\begin{cases} \lambda > 0 : & \text{Chaotic} \\ \lambda \leq 0 : & \text{Non-chaotic} \end{cases}$$

how to find a steady state ?

Steady state = energy balance in equilibrium

⇒ Moving average of the RMS value of the displacement (equivalent to an energy)

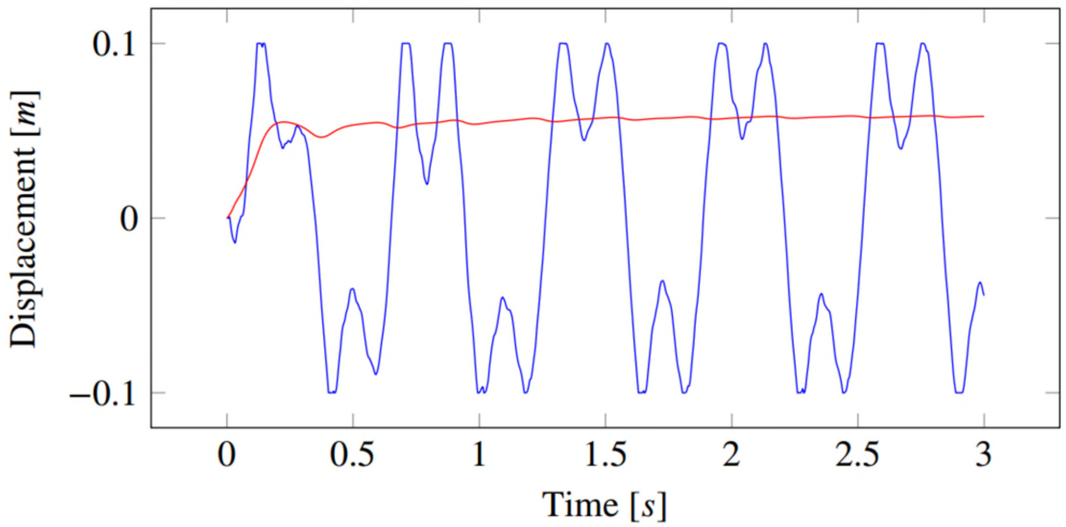
RMS value : $$\overline{\mathbf{U}_d} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} u_{d_i}^2}$$

Moving average :

$$\mathcal{MA}\left(\mathbf{U}_d\right)_{n+1} = \frac{1}{n+1}\left(n \cdot \mathcal{MA}\left(\mathbf{U}_d\right)_n + \overline{\mathbf{U}_d}\right)$$

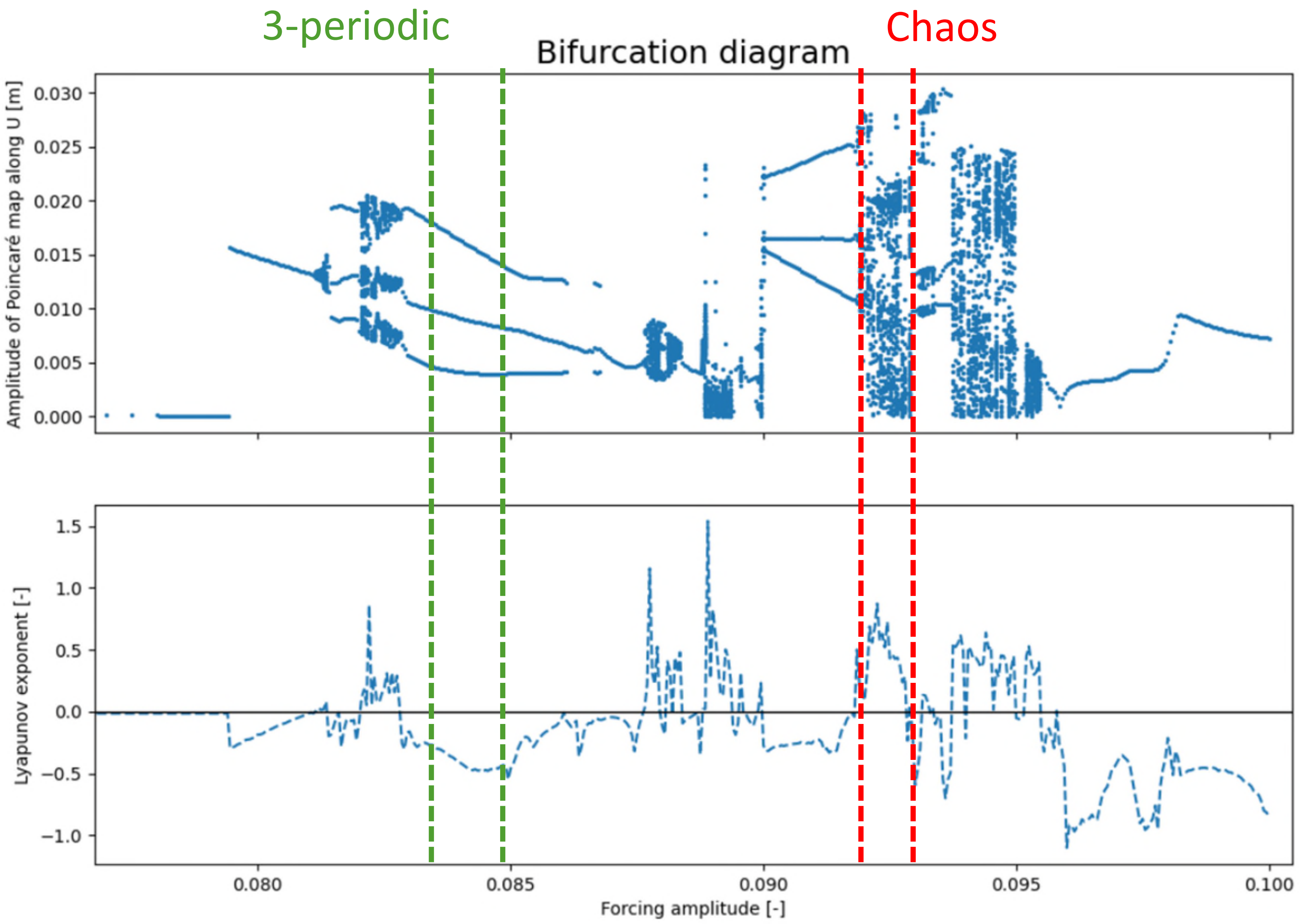
⇒ convergence test :

$$\frac{\sqrt{\left|\mathcal{MA}\left(\mathbf{U}_d\right)^2_{period\ k} - \mathcal{MA}\left(\mathbf{U}_d\right)^2_{period\ k-1}\right|}}{\mathcal{MA}\left(\mathbf{U}_d\right)^2_{period\ k-1}} \cdot 100 \leq \varepsilon$$



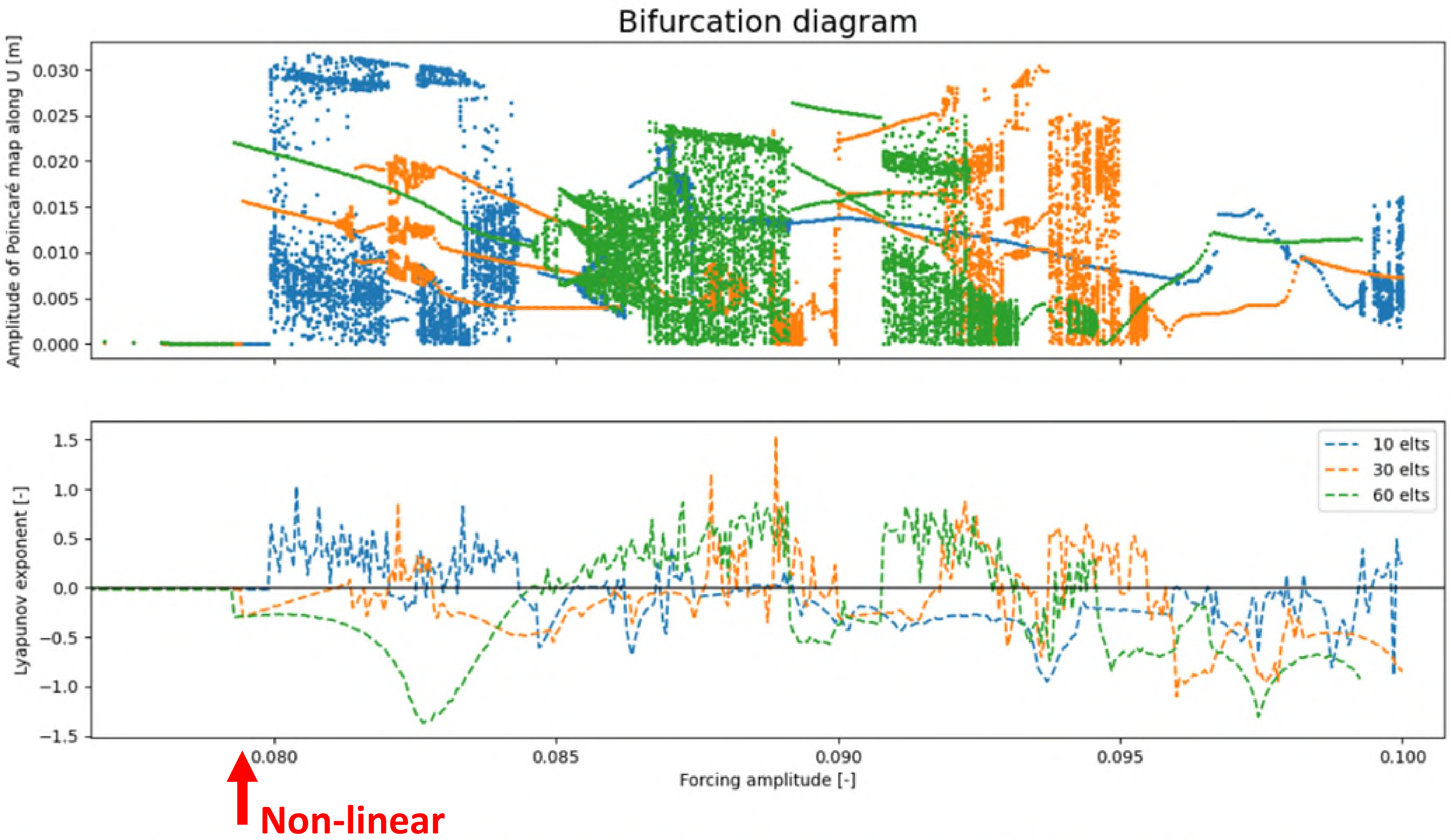
Evolution of the displacement of the last node of the beam (—) and the moving average of its RMS value (—) as a function of time

Resultats



Amplitude bifurcation diagram, 30 elements, $\alpha = 10^{-4}$, Augmented Lagrangian

mesh size influence

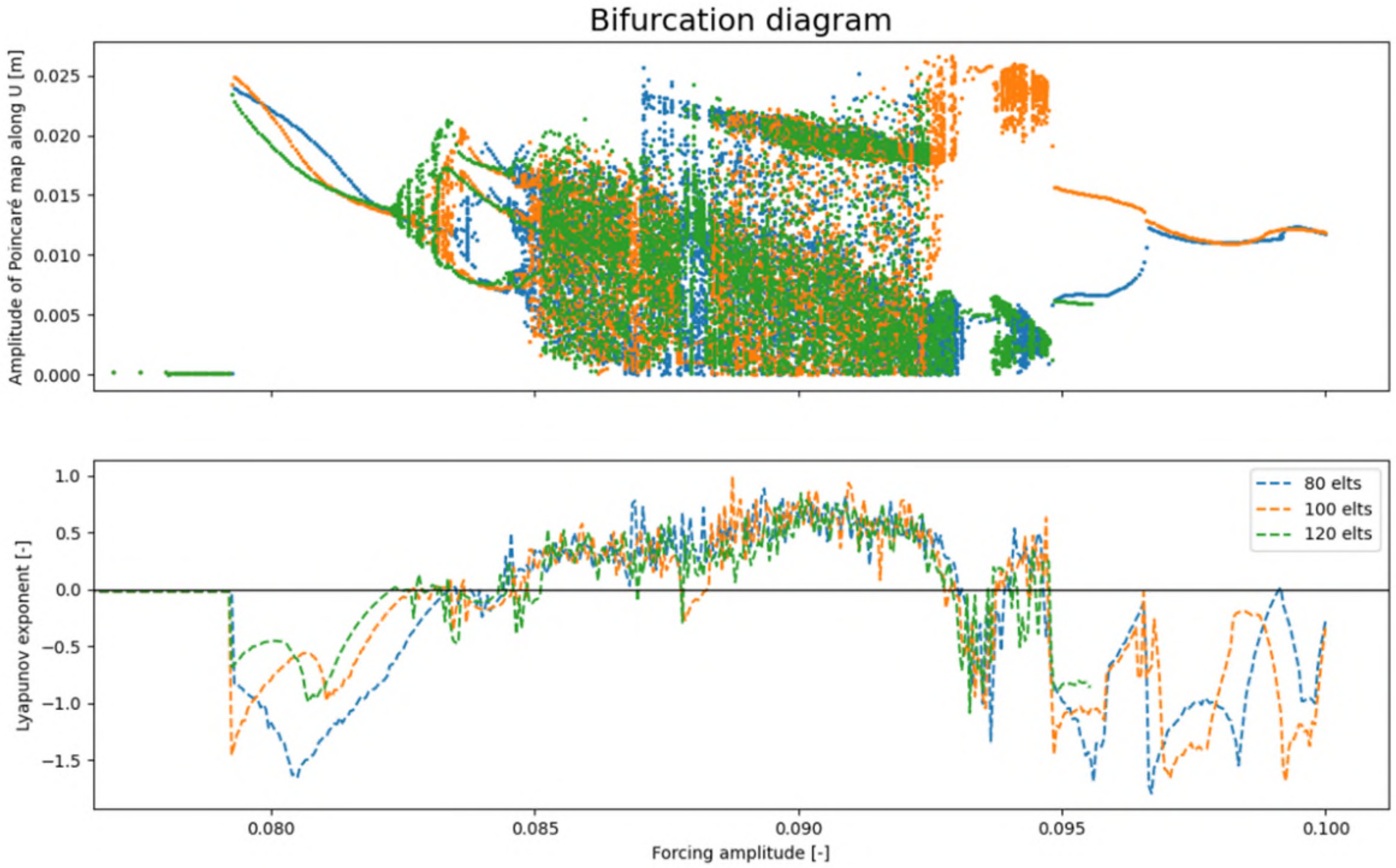


Amplitude bifurcation diagram, [10, 30, 60] elements, $\alpha = 10^{-4}$, Augmented Lagrangian

⇒ no convergence !

Warning !

Mesh size influence



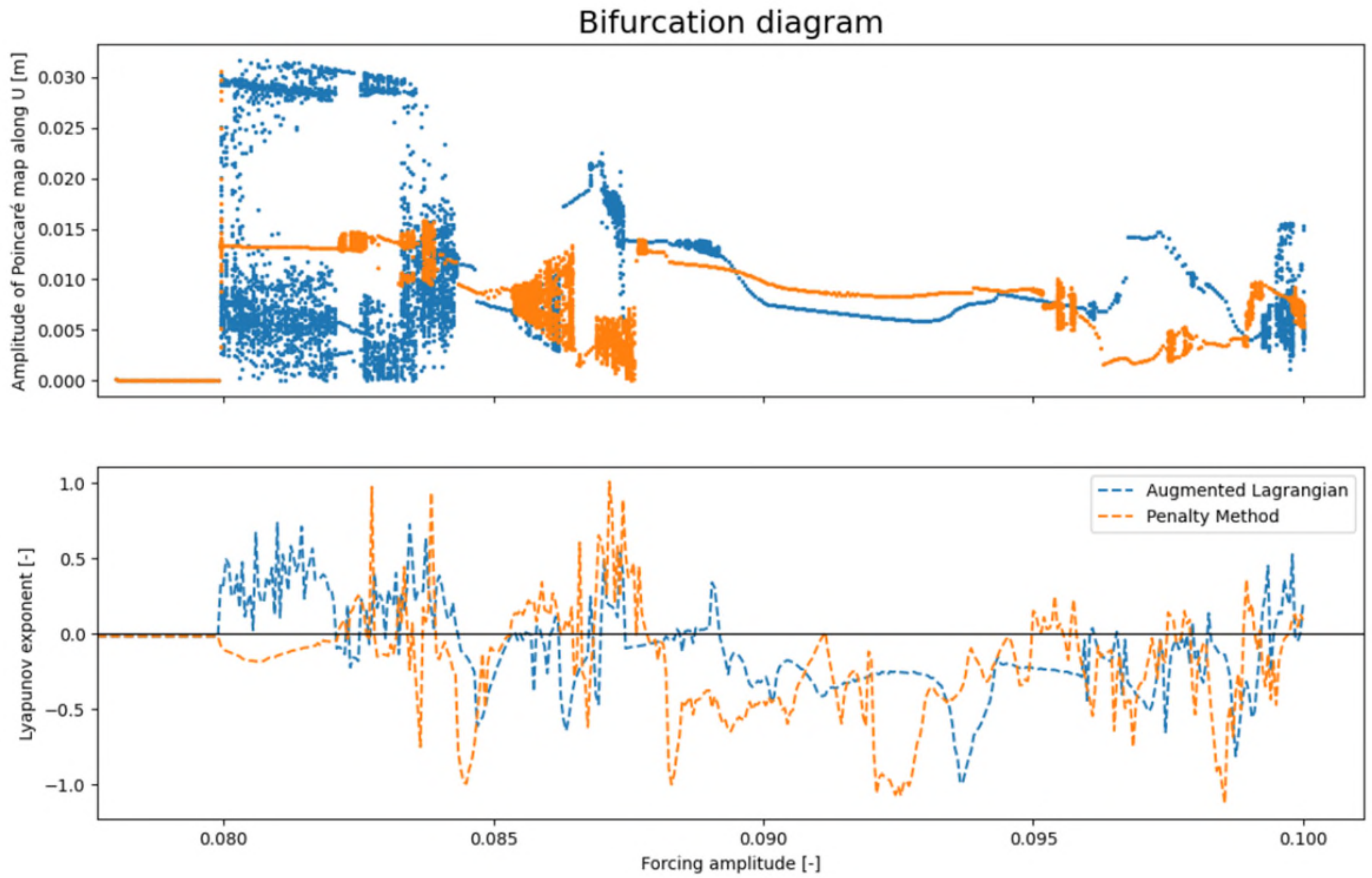
Amplitude bifurcation diagram, [80, 100, 120] elements, $\alpha = 10^{-4}$, Augmented Lagrangian

Seems to be converged …

Comparison between contact simulation methods

- We look at the augmented Lagrangian and penalty methods, as well as the different choices of mass matrix (regular and singular) in order to compare how the distribution of contact reactions is established for the different schemes.

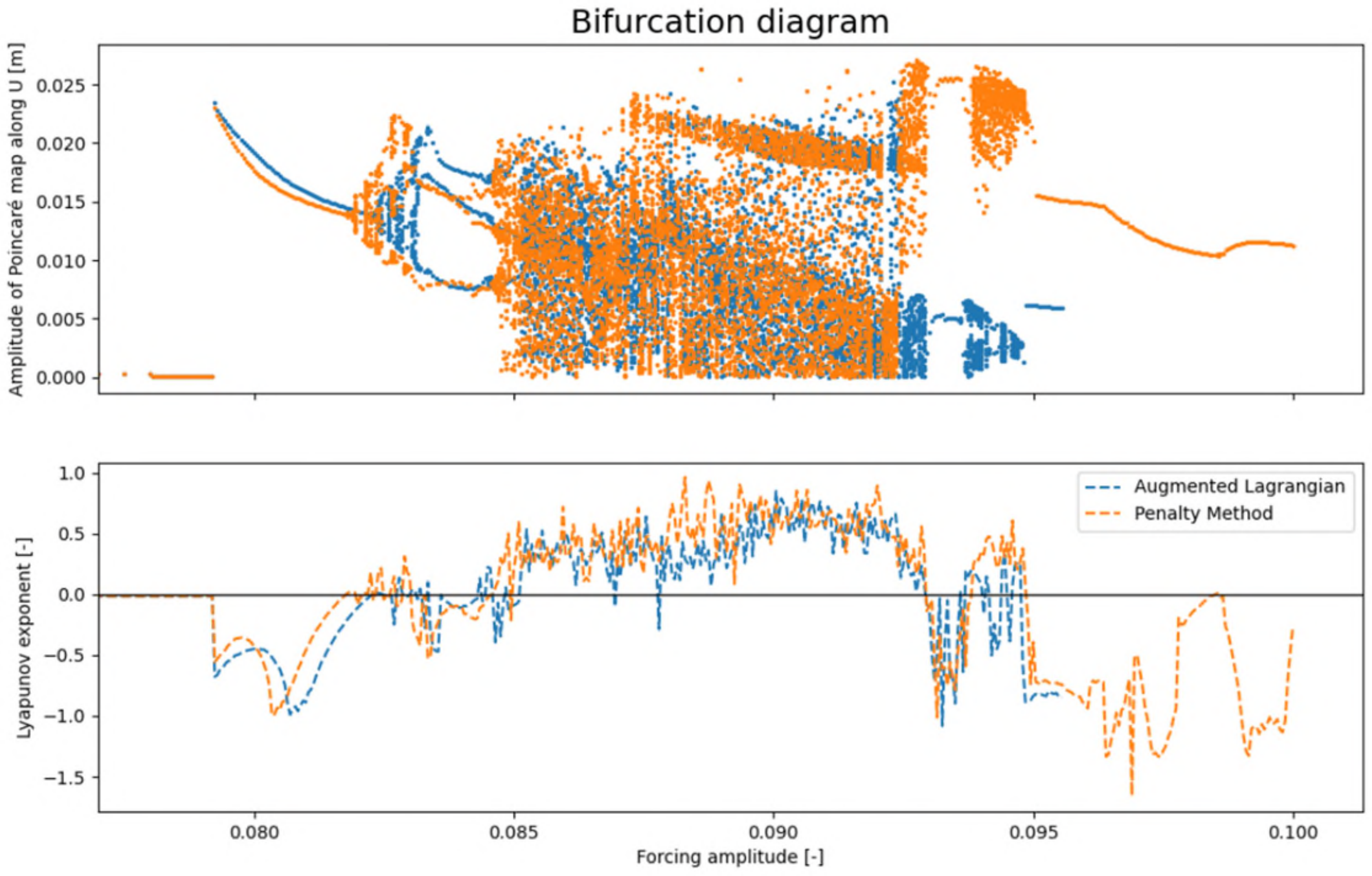
Influence of the penalty parameter



Amplitude bifurcation diagram, 10 elements, $\alpha = 10^{-4}$, Augmented Lagrangian & Penalty ($K_N = 4 \cdot 10^4\, N.m^{-1}$)

K_N penalty parameter

Influence of the penalty parameter



Amplitude bifurcation diagram, 120 elements, $\alpha = 10^{-4}$, Augmented Lagrangian & Penalty ($K_N = 4 \cdot 10^4 N.m{-}1$)

Convergence in number of elements of bifurcation diagrams

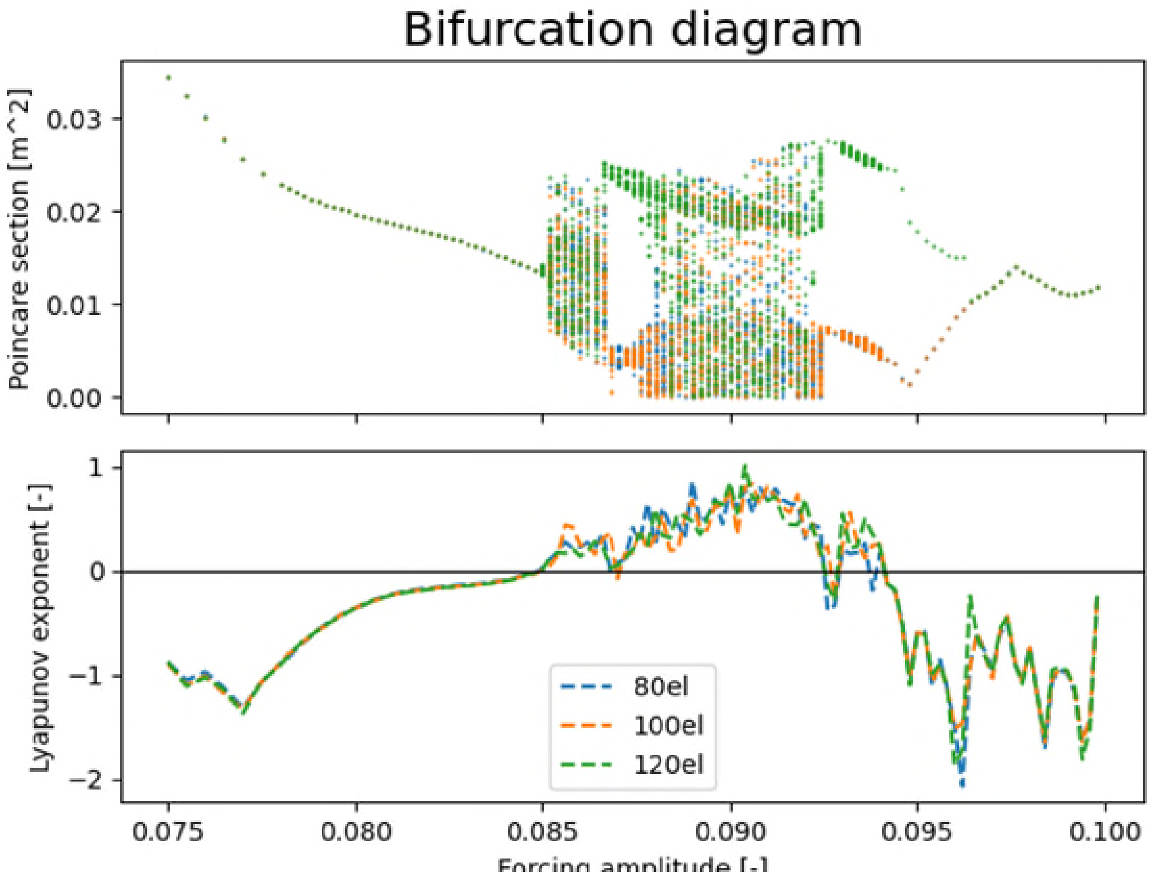


Figure 3 : Bifurcation diagrams (singular mass, augmented Lagrangian, beta=1/2, dt=10^-4)

- **Convergence (by eye) for 80 elements (using the method considered).**
- **Multi-stability for the Poincaré section (between 100 and 120).**

Convergence in time-step of bifurcation diagrams

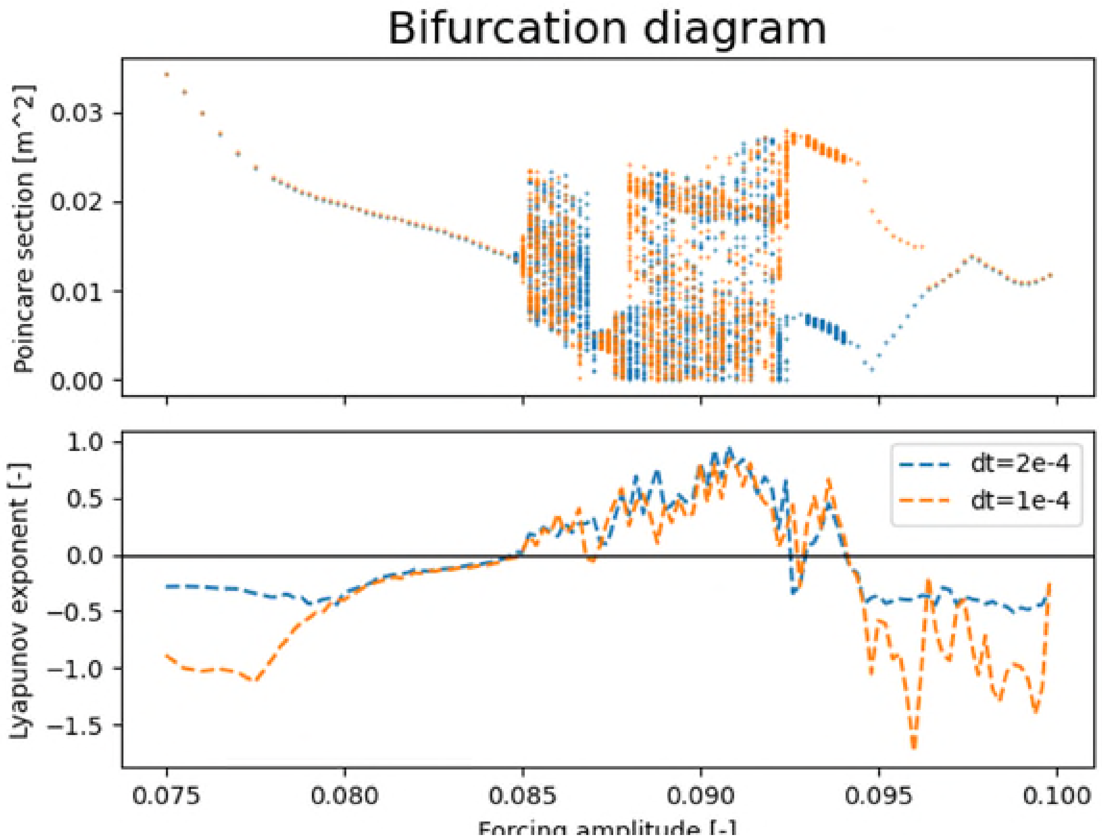


Figure: Bifurcation diagrams (singular mass, augmented Lagrangian, beta=1/4, Nbel=80)

- **High differences in stability of periodic solutions.**
- **Multi-stability for the Poincaré section.**

→ Needs more refinement in the time-step

Comparison of bifurcation diagrams for different contact simulation methods

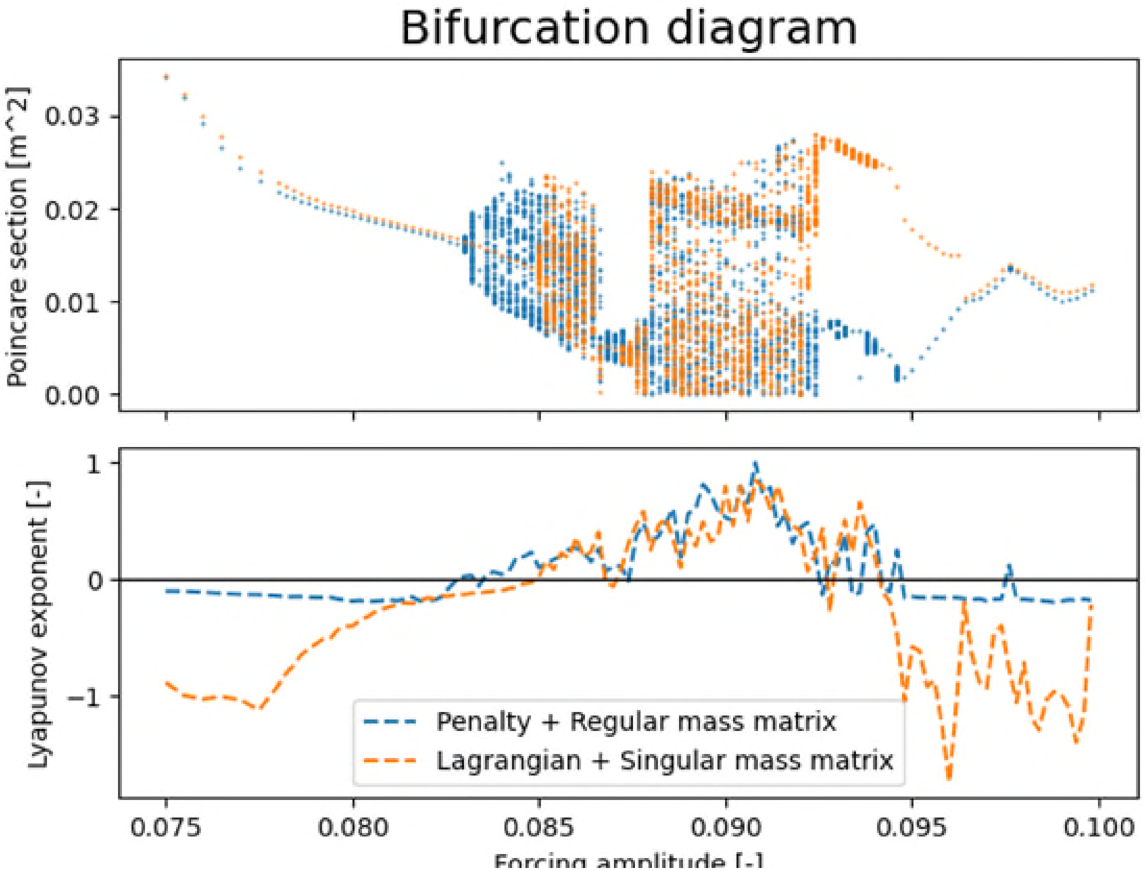
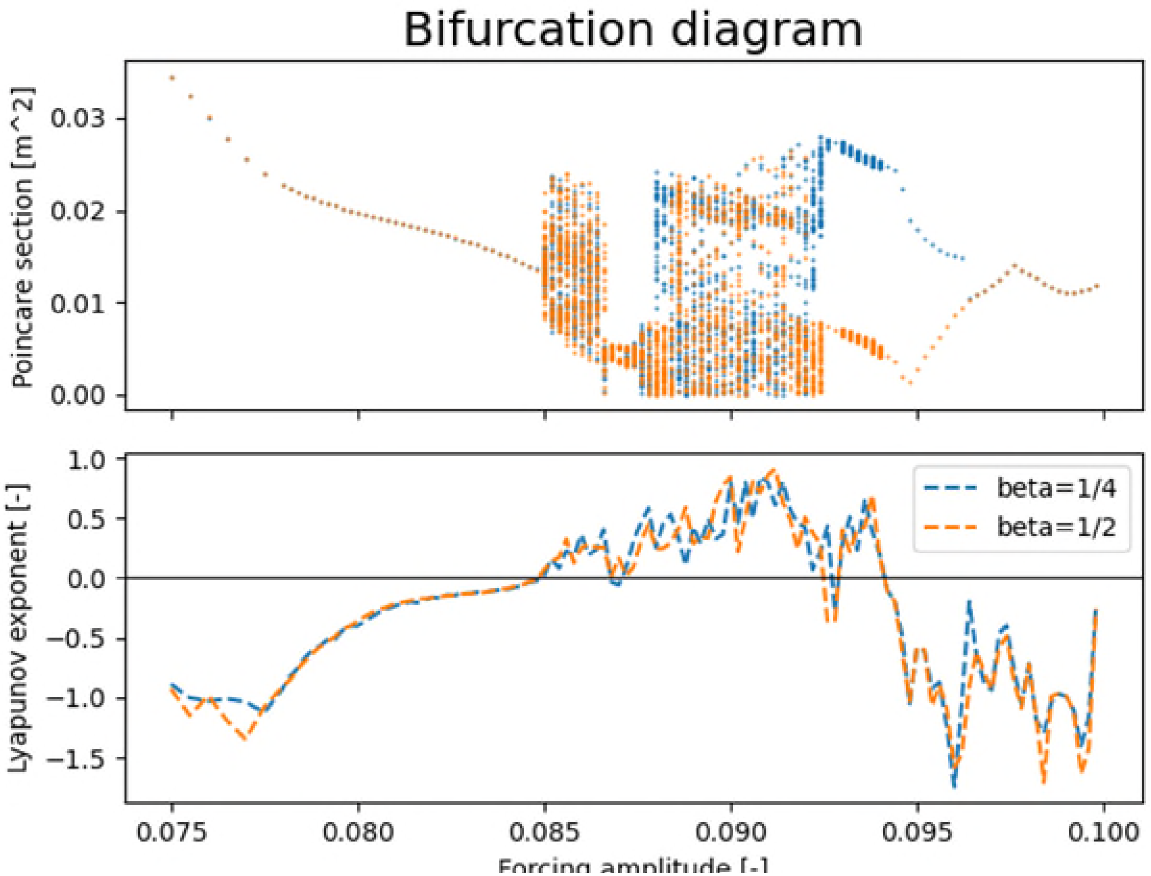


Figure: Bifurcation diagrams (singular mass, augmented Lagrangian, beta=1/4, Nbel=80 , dt=10^-4)

- **High differences in stability of periodic solutions.**
- **Multi-stability for the Poincaré section.**

Comparison of bifurcation diagrams for different β parameters



Dumont-Paoli-Newmark-(β,1/2)-(e=0) schemes:

- β =1/2: **implicit central difference scheme**
- β =1/4: **average acceleration scheme**

Figure: Bifurcation diagrams (singular mass, augmented Lagrangian, Nbel = 80, dt=10^-4)

Small differences in outcomes associated with chaotic regimes.

3 - Comparison of bifurcation diagrams for two mass matrix methods

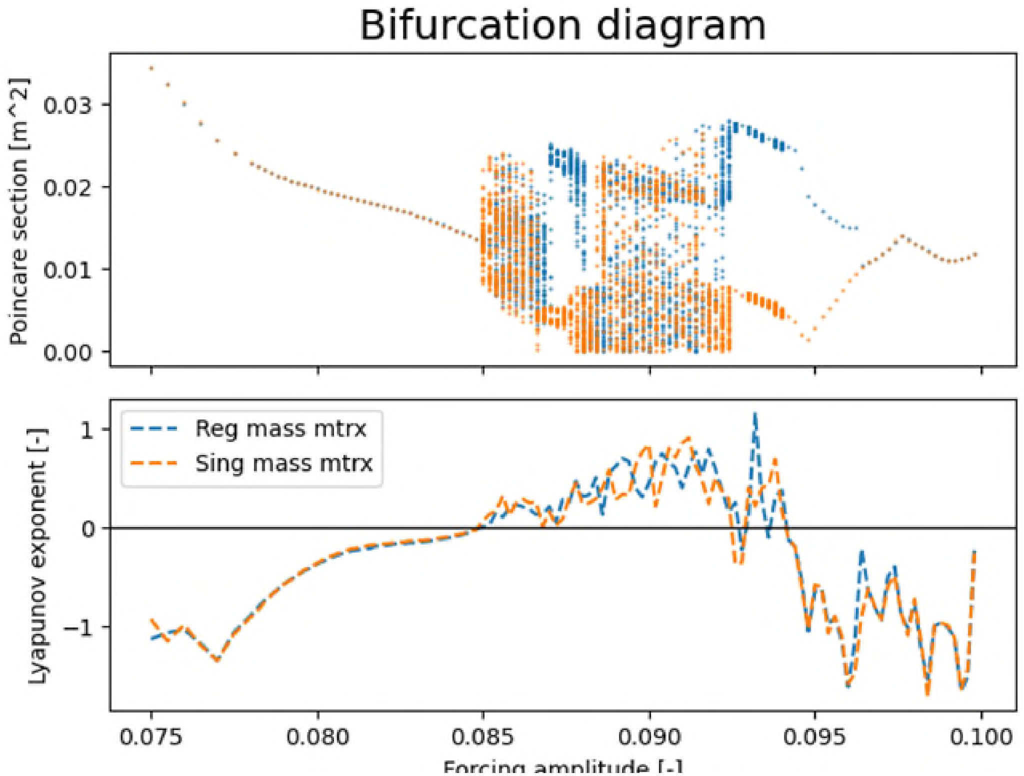


Figure: Bifurcation diagrams (Lagrangian, Nbel = 80, dt=10^-4, beta=1/2)

3 - Comparison of bifurcation diagrams for two penalty coefficients

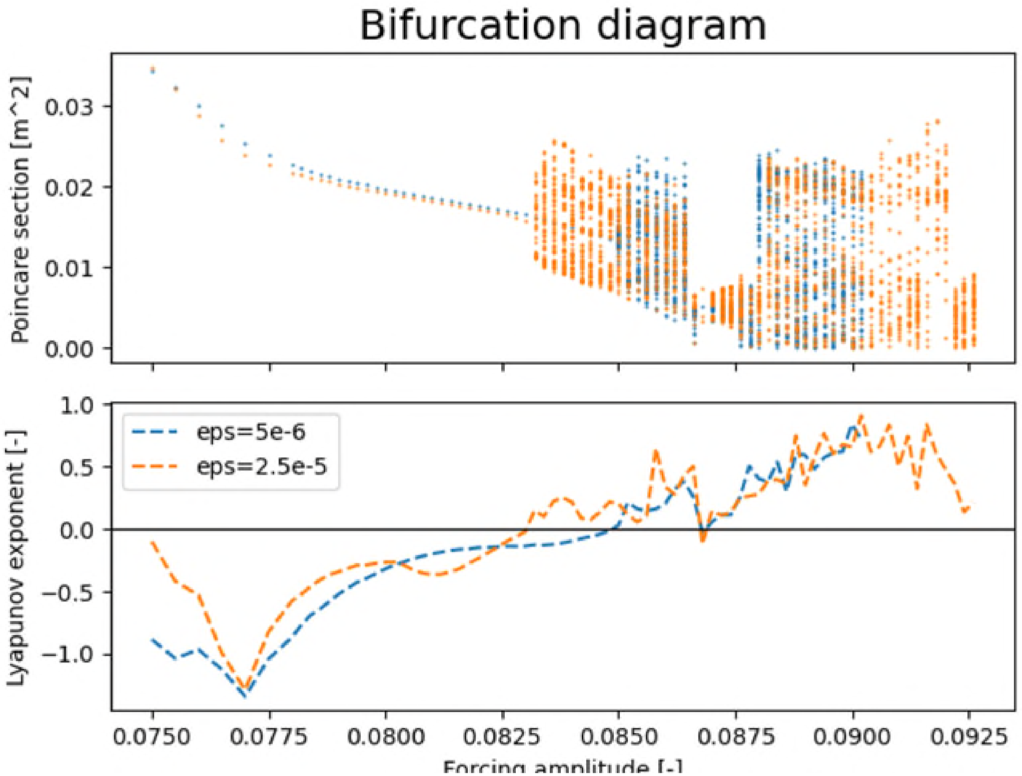


Figure: Bifurcation diagrams (regular mass, Penalty, Nbel = 120, dt=10^-4, beta=1/2)

Numerical results

Numerical tasks: Simulate – Collect – Classify – Interpret

Objective:

Investigate the nonlinear dynamic behavior of a beam model through numerical simulations. This involves:

- Studying the influence of scheme parameters (e.g., β in time integration), regime types, and numerical methods on the steady-state and transient dynamics of the system.
- Comparing different contact simulation approaches and assessing how the numerical setup affects the modeling outcomes.
- Analyzing the causal relationships between numerical scheme choices and the resulting probability distributions of key variables, such as the contact reaction force (initial focus).
- Constructing a comprehensive "zoology" or taxonomy of the expected dynamical behaviors arising from different parameter configurations and numerical regimes.

Conclusion for beam impact problems

- Although the penalty method approaches theoretically the complementarity, the manner in which the contact reactions are distributed seems to differ drastically meaning that their types of dynamics are distinct.

- The aim of Ayman Khaddari's thesis is to continue this analysis
  - to qualitatively characterize the areas of numerical non-repeatability,
  - propose a methodology for quatifying the probability of transition from one dynamic regime to another,
  - and select among the numerical methods those most robust with respect to the problem while remaining closest to real physics.

4

4. From beams to plates : impact problems

- Thm for bilaplacian (2D) Pozzolini, Renard, Salaun, 2013
- Sing. Mass Matrix for KL plate, Pozzolini, Renard, Salaun, 2016
- Nitsche’s method for plates models, Pozzolini, Fabre, Renard, 2021

From beam to elasto-dynamic plate impact-problems

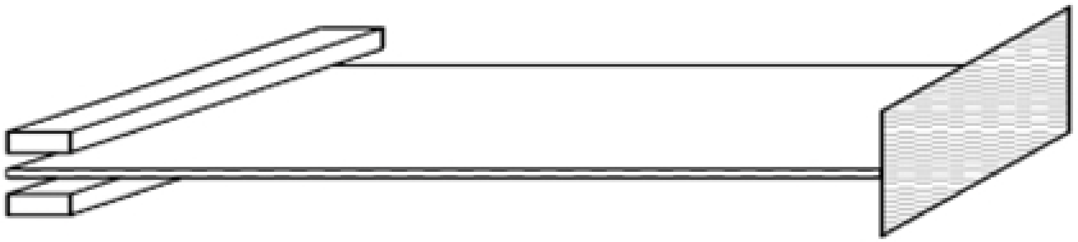


FIGURE 1. Example of bending clamped plate between rigid obstacles.

Theorem (C. Pozzolini, M. Salaun, Y. Renard (2013)) :

Existence of a solution for impact problem between a KL plate and a rigid obstacle

C. Pozzolini, Y. Renard, M. Salaün. Vibro-impact of a plate on rigid obstacles: existence theorem, convergence of a scheme and numerical simulations. *IMA. J. Numer. Anal.*, 33(1):261--294, 2013.

C. Pozzolini, Y. Renard, M. Salaün. Energy conservative finite element semi-discretization for vibro-impacts of plates on rigid obstacles. *ESAIM Math. Model. Numer. Anal.*, 50:1585--1613, 2016.

From beam to elasto-dynamic 2D contact problem

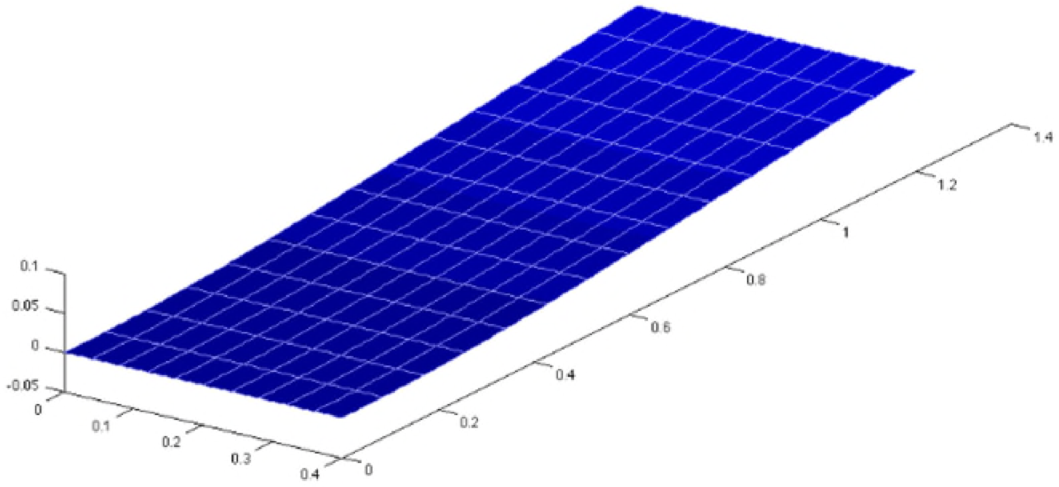


FIGURE 3. Bending clamped plate above a rigid obstacle: FVS quadrangular mesh.

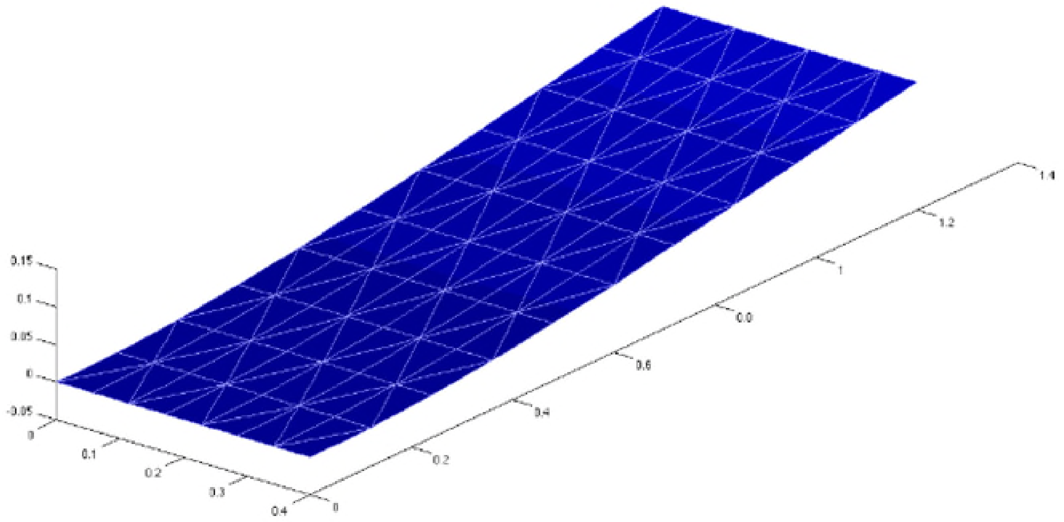


FIGURE 4. Bending clamped plate above a rigid obstacle: FVS quadrangular mesh.

C. Pozzolini, Y. Renard, M. Salaün. Vibro-impact of a plate on rigid obstacles: existence theorem, convergence of a scheme and numerical simulations. *IMA. J. Numer. Anal.*, 33(1):261--294, 2013.

C. Pozzolini, Y. Renard, M. Salaün. Energy conservative finite element semi-discretization for vibro-impacts of plates on rigid obstacles. *ESAIM Math. Model. Numer. Anal.*, 50:1585--1613, 2016.

From beam to elasto-dynamic 2D contact problem

$$\begin{cases} \text{Find } U^{n+1,e} = \dfrac{U^{n+1} + eU^{n-1}}{1+e} \in \mathbb{K}^h \text{ such that for all } W \in \mathbb{K}^h \\ (W - U^{n+1,e})^T \left(\mathbf{M}_r \dfrac{U^{n+1} - 2U^n + U^{n-1}}{\Delta t^2} + \mathbf{K} \left(\beta U^{n+1} + (1-2\beta)U^n + \beta U^{n-1} \right) \right) \\ \qquad \geq \left(W - U^{n+1,e} \right)^T F^{n,\beta}. \end{cases}$$

Time scheme : Newmark Dumond-Paoli-Schatzman with restitution coefficient $e \geq 0$, Contact inequality : aug. lag. method

L. Paoli and M. Schatzman, Schéma numérique pour un modèle de vibrations avec contraintes unilatérales et perte d’énergie aux impacts, en dimension finie, C. R. Acad. Sci. Paris Sér. I Math., 317 (1993)

C. Pozzolini, Y. Renard, M. Salaün. Vibro-impact of a plate on rigid obstacles: existence theorem, convergence of a scheme and numerical simulations. *IMA. J. Numer. Anal.*, 33(1):261--294, 2013.

C. Pozzolini, Y. Renard, M. Salaün. Energy conservative finite element semi-discretization for vibro-impacts of plates on rigid obstacles. *ESAIM Math. Model. Numer. Anal.*, 50:1585--1613, 2016.

From beam to elasto-dynamic 2D contact problem

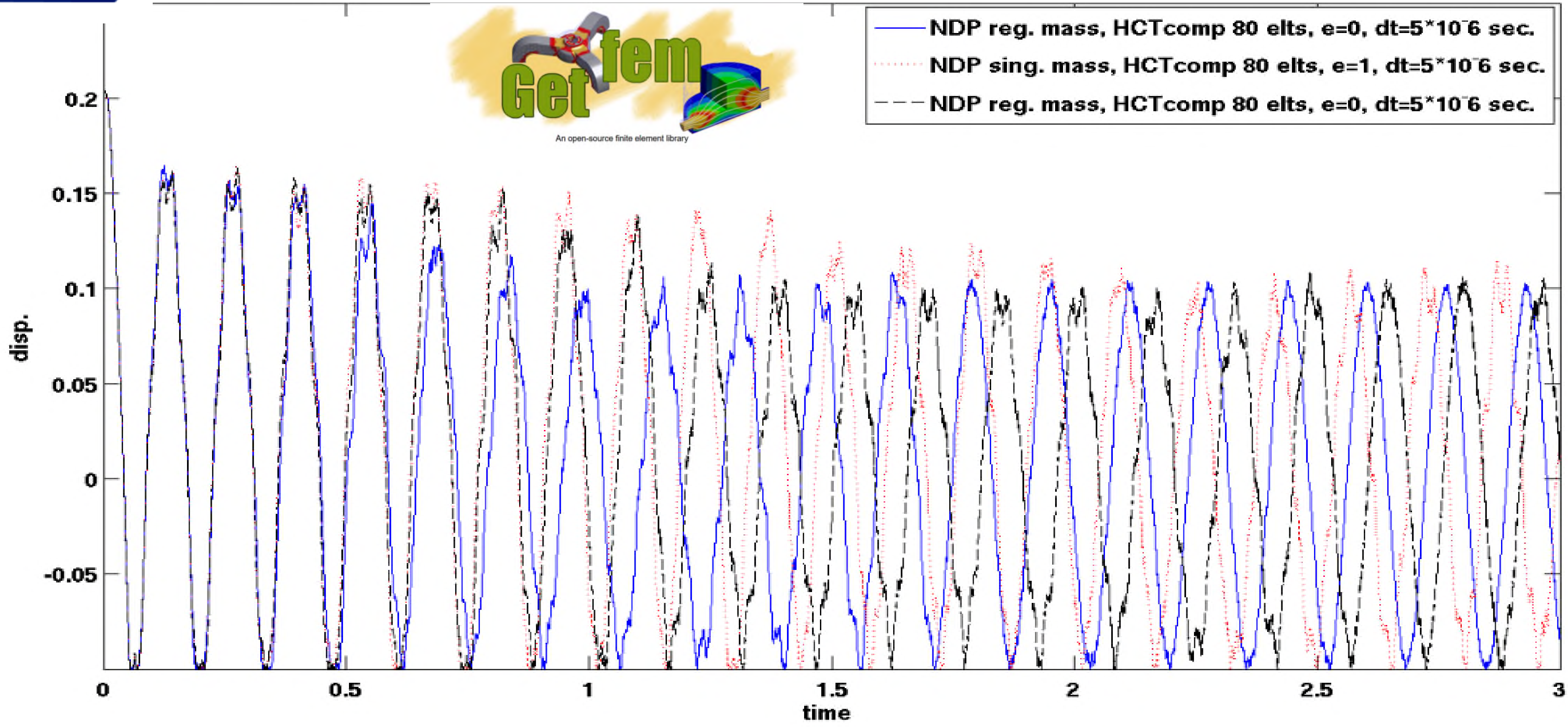


Figure 6. Complete HCT, 80 triangles. e-β-Newmark scheme. $\Delta t = 5 \times 10^{-6}$. $e = 0$ or $e = 1$ regular or singular mass matrices.

From beam to elasto-dynamic 2D contact problem

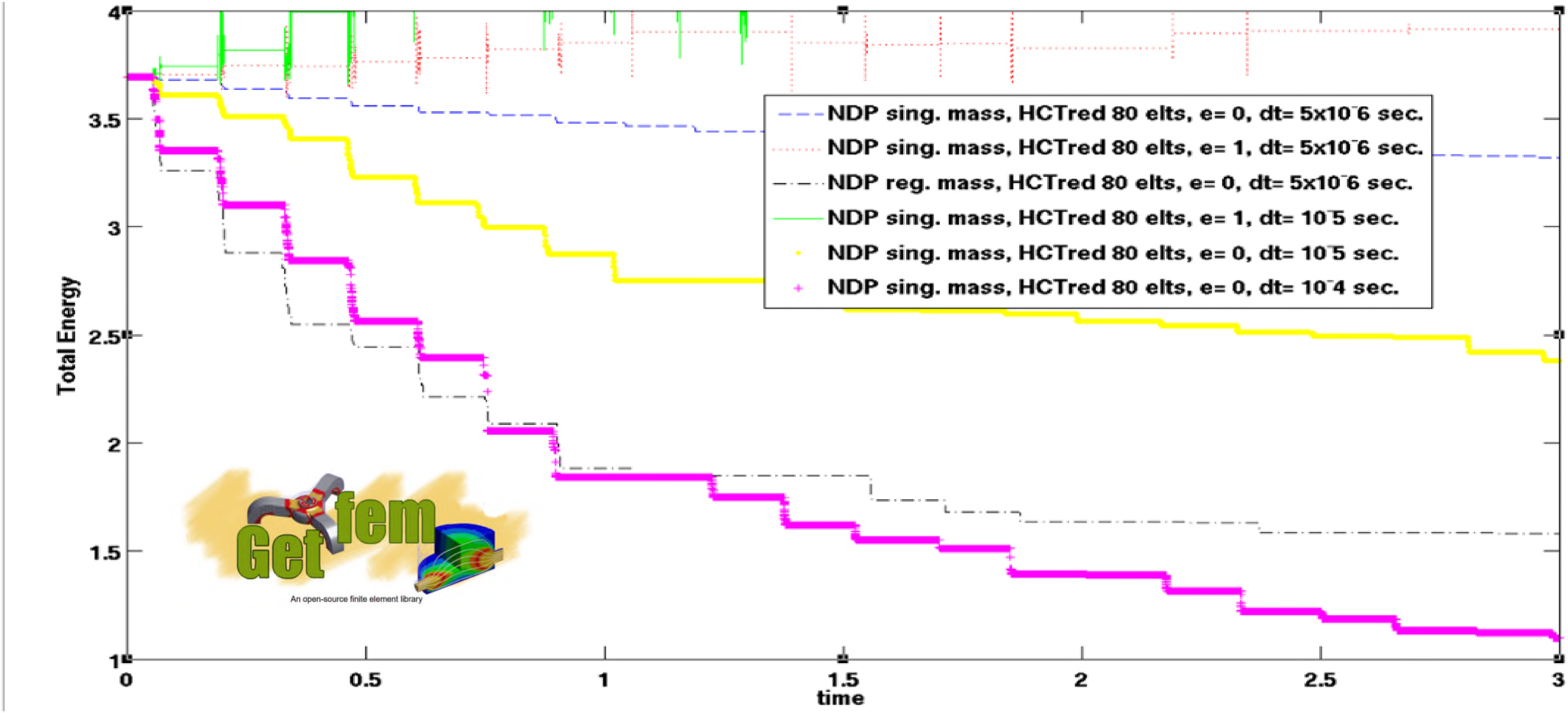


FIGURE 9. Energy for different time steps. Reduced HCT, 80 triangles. e-β-Newmark scheme. $e = 0$ or $e = 1$, regular or singular mass matrices.

From beam to elasto-dynamic 2D contact problem

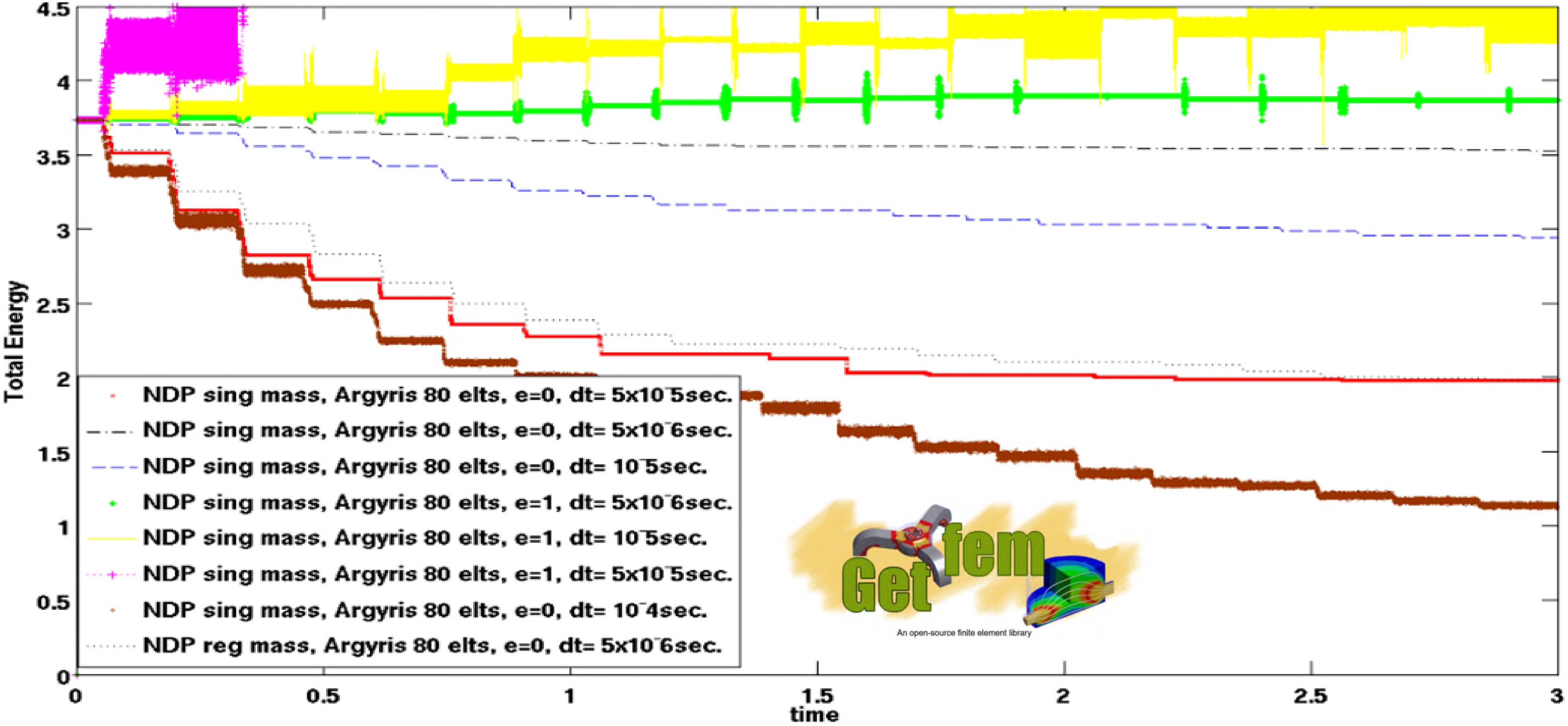


FIGURE 10. Energy for different time steps. Argyris element, 80 triangles. e-β-Newmark scheme. $e = 0$ or $e = 1$, regular or singular mass matrices.

From beam to elasto-dynamic 2D contact problem

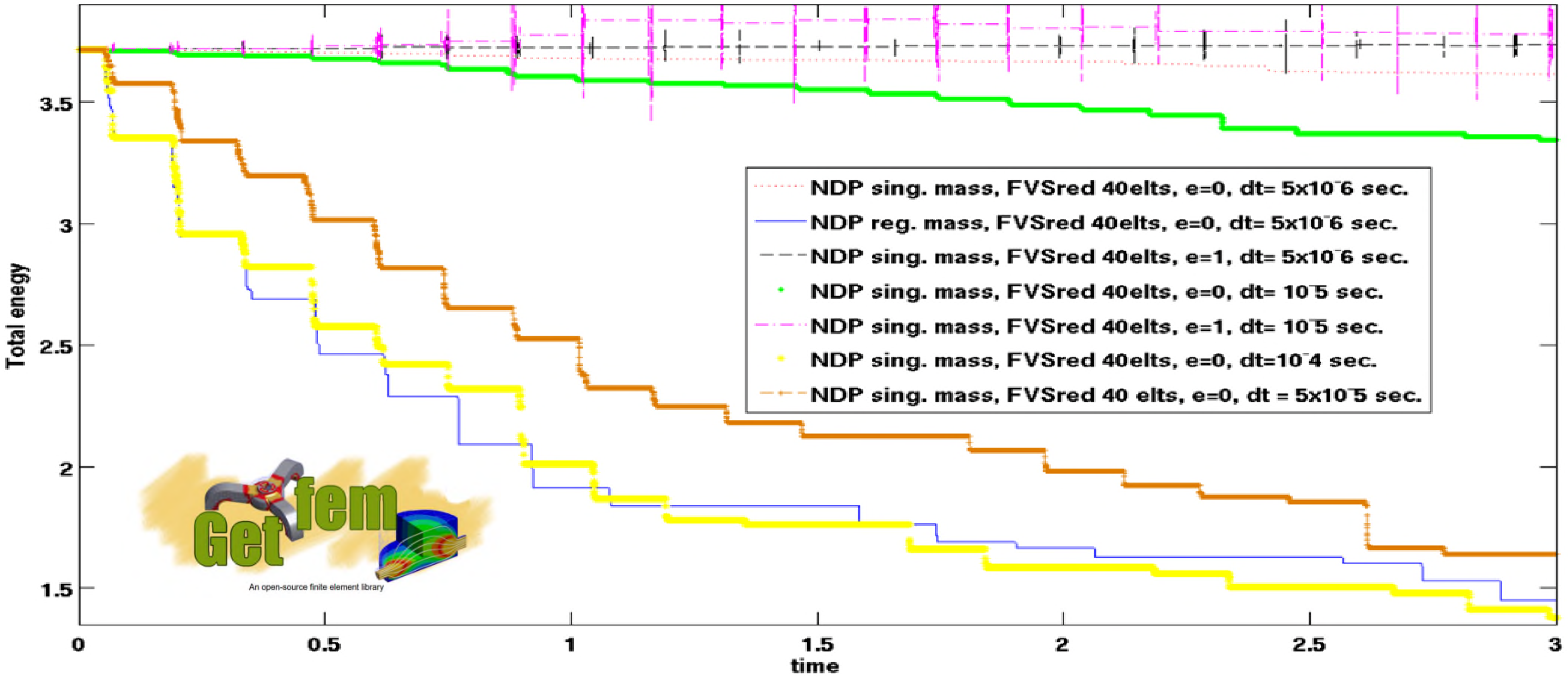


FIGURE 11. Energy for different time steps. Reduced FVS, 40 quadrangles. e-β-Newmark scheme. $e = 0$ or $e = 1$, regular or singular mass matrices.

From beam to elasto-dynamic 2D contact problem

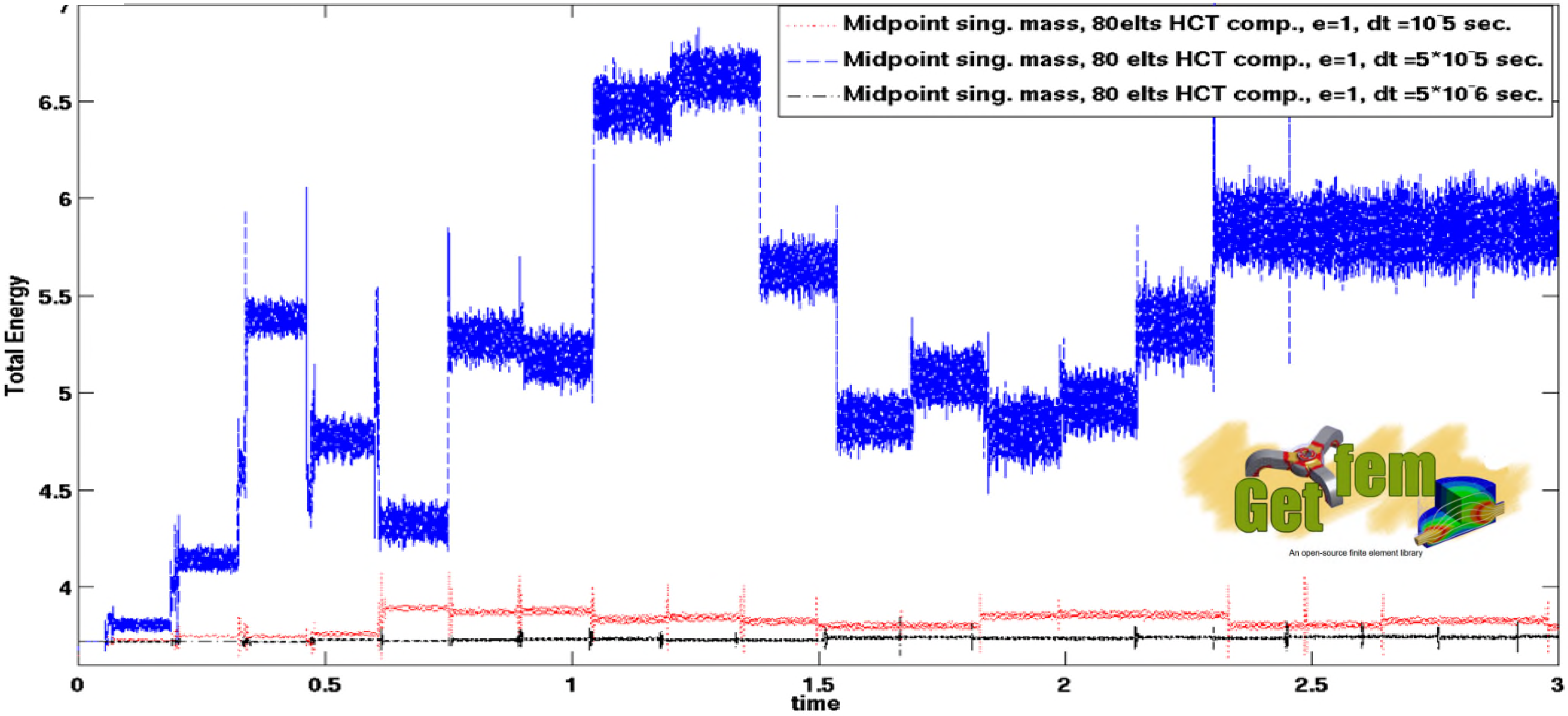
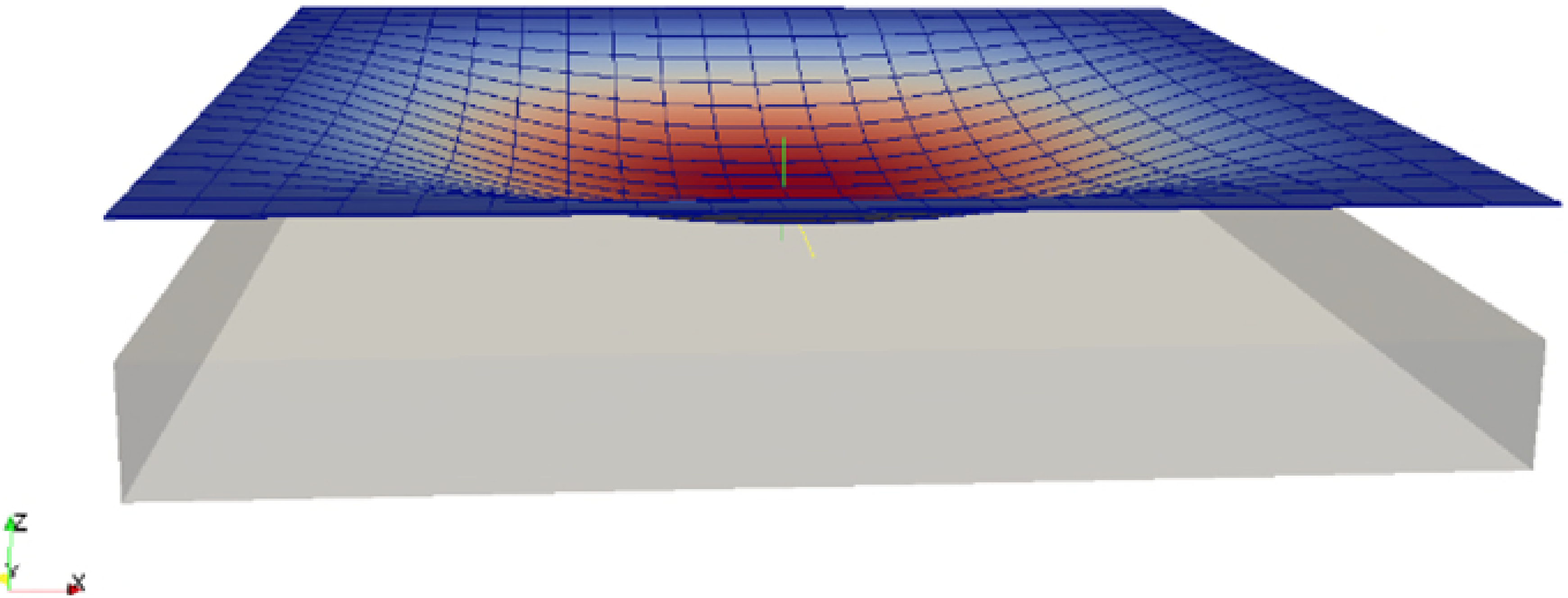


FIGURE 19. Energy for different time steps. Complete HCT, 80 triangles. Midpoint scheme, singular mass matrix.

nitsche methods for plates in contact problems ?

M. Fabre, C. Pozzolini, Y. Renard. Nitsche-based models for the unilateral contact of plates.
ESAIM: Mathematical Modelling and Numerical Analysis, 55:941--967, 2021



nitsche methods for plates

We now consider a thicker plate with $\varepsilon = 0.05$, still with a flat obstacle defined by the gap $g = 0.15$ and subjected to a very large Gaussian volume load $f^V(x) = -50000\exp(-(x_1^2 + x_2^2))$ MPa. A large uniform load has not been considered because it concentrates the deformation close to the boundaries where the plate is clamped. The reference solution is plotted in Figure 6.

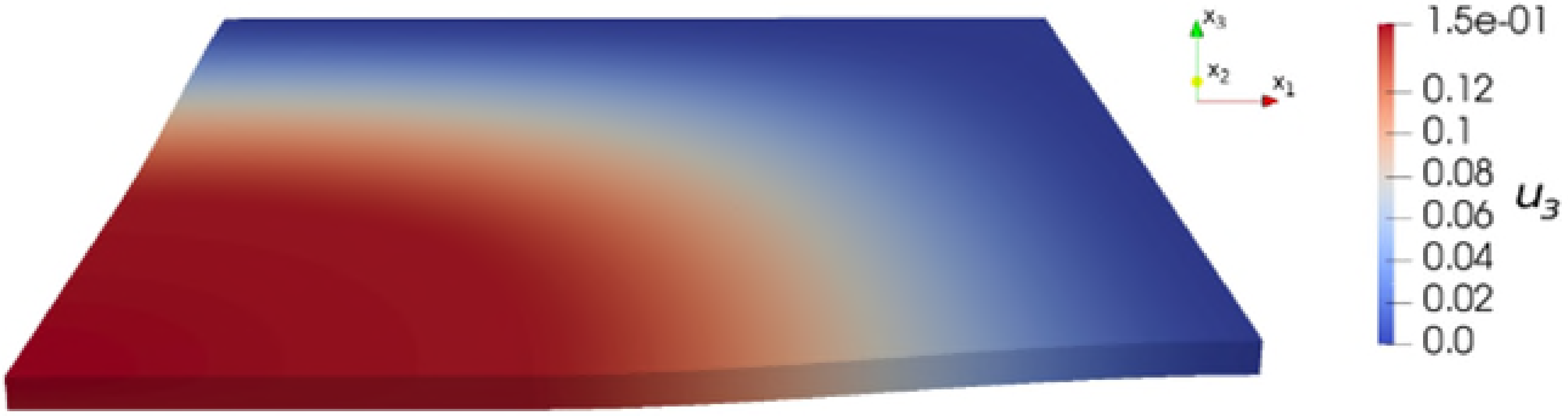
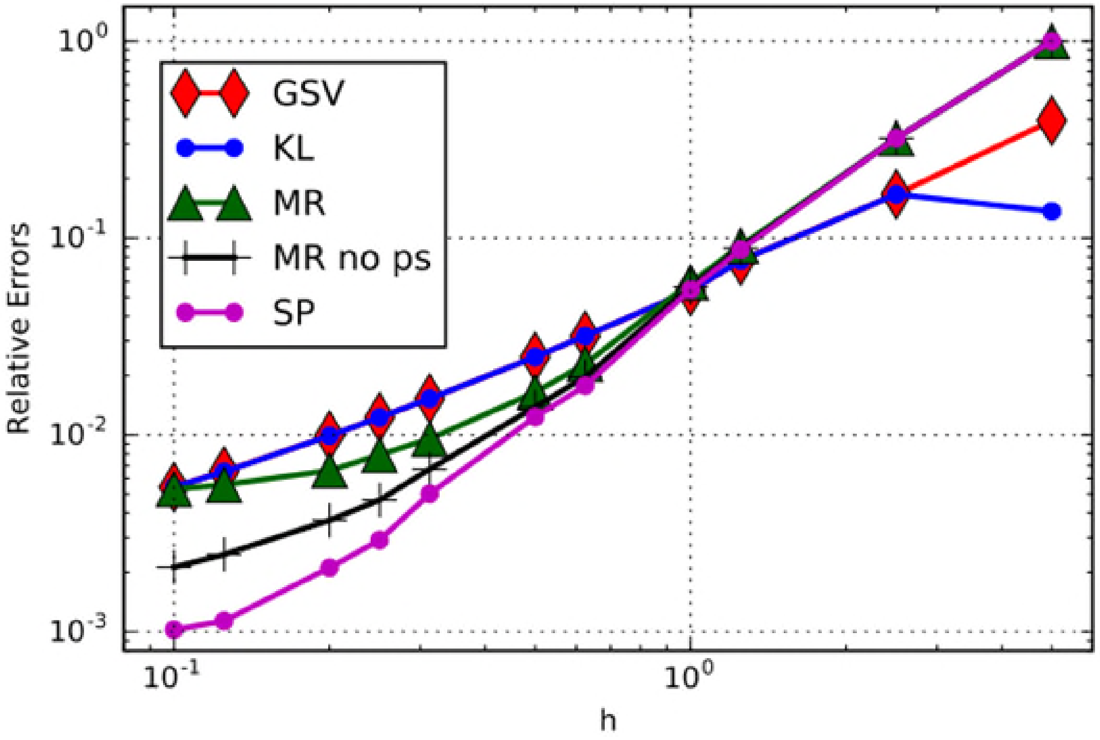


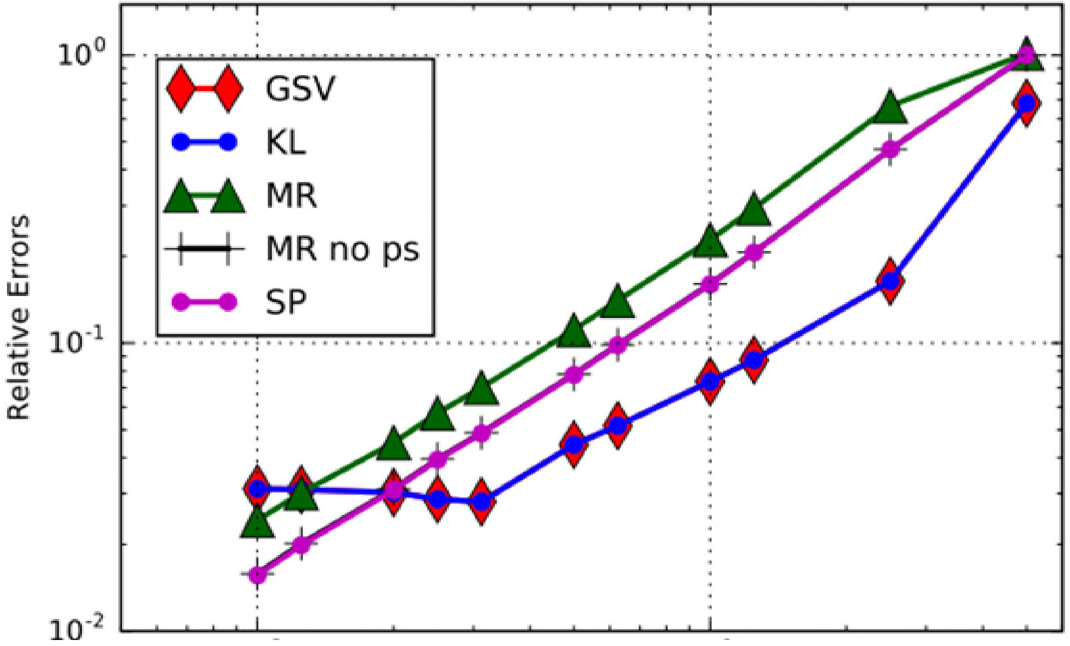
Figure 6: Thicker square plate ($\varepsilon = 0.05$m) under very large Gaussian volume load and flat obstacle: 3D reference solution with the augmented Lagrangian method and Q_2-Lagrange elements.

M. Fabre, C. Pozzolini, Y. Renard. Nitsche-based models for the unilateral contact of plates. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55:941--967, 2021

nitsche methods for thin / thick plates

Relative $L^2(\Omega^\varepsilon)$-norm of the error in displacement with respect to the mesh size

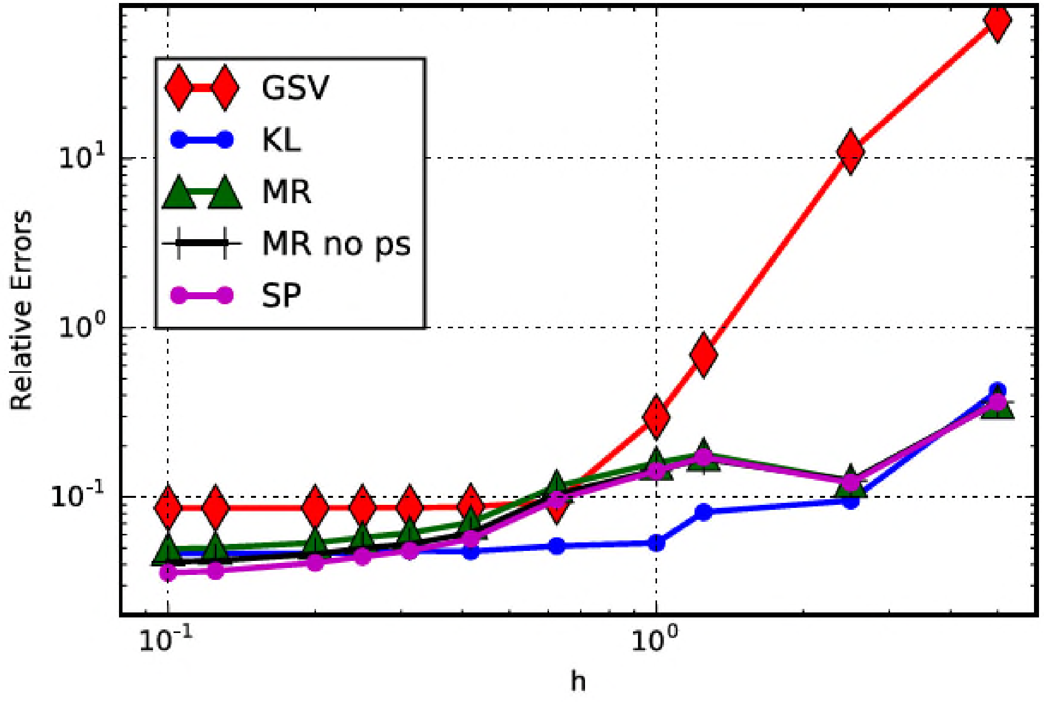




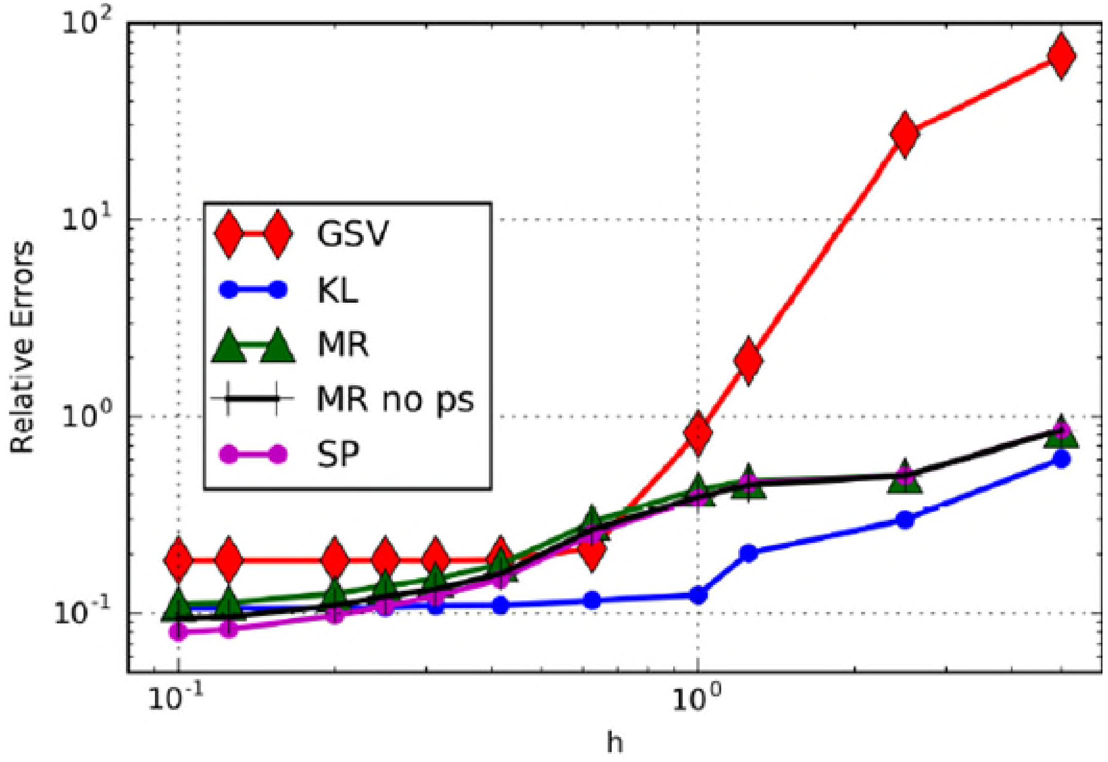
Relative $H^1(\Omega^\varepsilon)$-norm of the error in displacement with respect to the mesh size

nitsche methods for thin/thick plates

Relative $L^2(\Omega^\varepsilon)$-norm of the error in displacement with respect to the mesh size



An open-source finite element library



Relative $H^1(\Omega^\varepsilon)$-norm of the error in displacement with respect to the mesh size

framatome

Nitsche's methods

- Extended to frictionnal elastic contact problem :

. Chouly, R. Mlika, Y. Renard. An unbiased Nitsche's approximation of the frictional ontact between two elastic structures. *Numerische Matematik*, 2018.

- Extended to frictionnal elasto-plastic contact problem :

C. Pozzolini, Y. Renard, Nitsche's methods for elasto-plastic contact small deformation problems : theorical and numerical aspects CMIS 2018

- Extended to large deformation contact problem :

F. Chouly, R. Mlika, Y. Renard. An unbiased Nitsche's formulation of large deformation frictional contact and self-contact. *Comp. Meth. Appl. Mech. Eng.*, 2017.

- Extended to static elastic plate contact problem :

M. Fabre, C. Pozzolini, Y. Renard. Nitsche-based models for the unilateral contact of plates. ESAIM: Mathematical Modelling and Numerical Analysis, 2021

- Extended to dynamic contact problem :

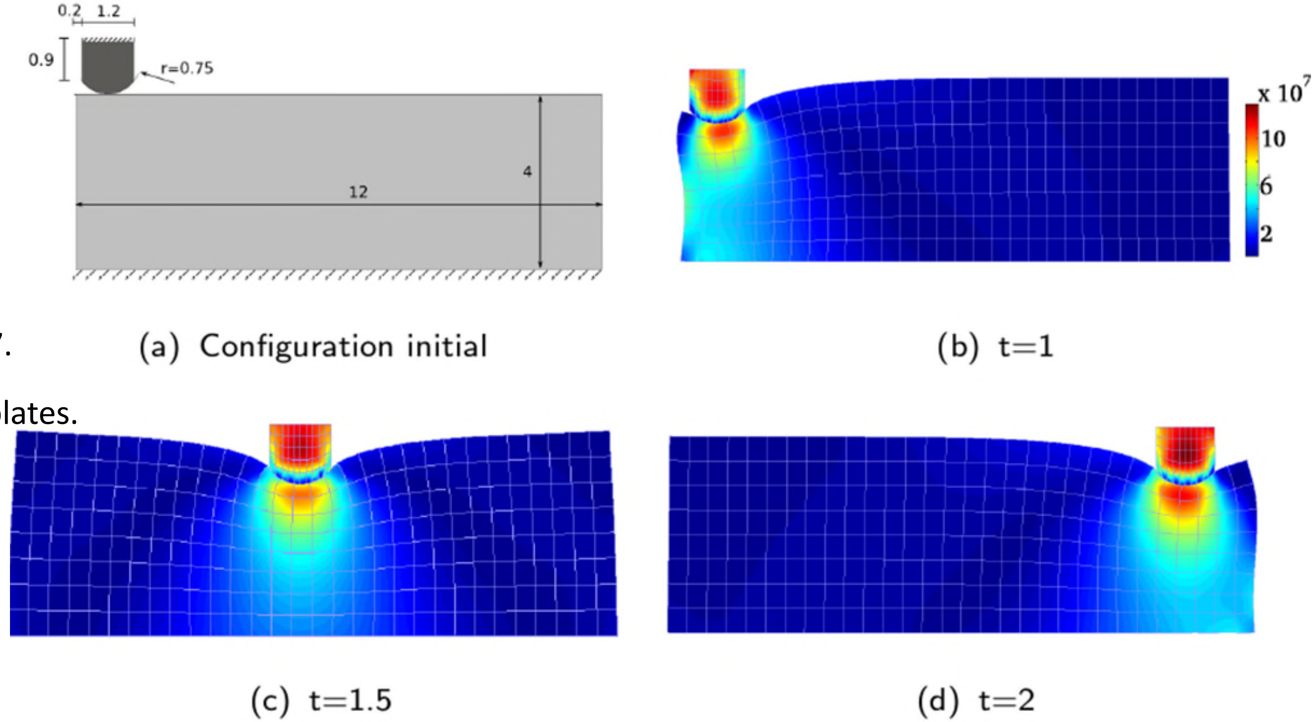
F. Chouly, P. Hild, Y. Renard. A Nitsche finite element method for dynamic contact:

1. Semi-discrete problem analysis and time-marching schemes.

ESAIM Math. Model. Numer. Anal., 2015.

F. Chouly, P. Hild, Y. Renard. A Nitsche finite element method for dynamic contact:

2. Stability of the schemes and numerical experiment. ESAIM Math. Model. Numer. Anal., 2015.



MICHELIN

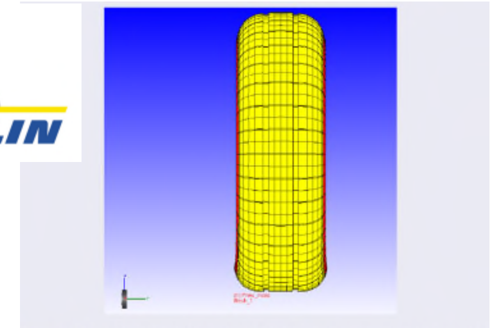
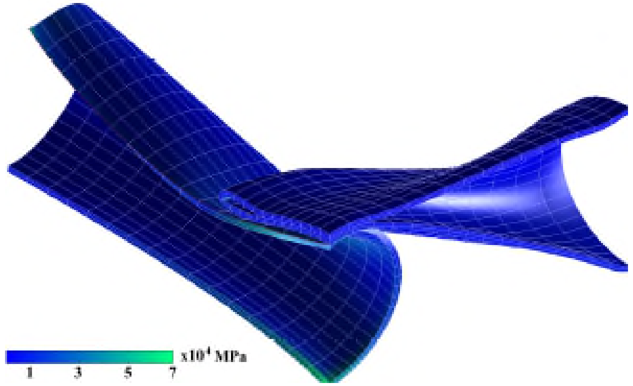


FIGURE 15 – Obtained deformation with Nitsche's method and $\gamma_N = 10^3$



GetFEM
An open-source finite element library

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Conclusions & perspectives

- Nitsche's method allows a numerical treatment of contact and friction in a simple manner since it remains a primal method, and is more robust than penalty, since it is consistent. For the Signorini problem, a rather complete numerical analysis can be carried out, to establish well-posedness and optimal convergence under assumptions on Nitsche's parameters similar to those commonly encountered for Dirichlet boundary conditions. The method can be extended to take into account various situations such as multi-body contact, large transformations and contact in elastodynamics. Most common **friction's law** such as Coulomb or Tresca can be formulated as well within this framework. Forthcoming studies may deal with numerical analysis in some situations in which results are lacking, such as contact in elastodynamics, or Coulomb's friction. The same method can be considered as well to discretize other categories of contact / friction problems, or other types of non-linear boundary conditions associated to variational inequalities, in particular for **shell and beam models**.

- As the singular mass matrix method transforms the elastodynamic inequality problem into a Lipschitz-O.D.E., for a given space step, every convergent scheme for a Lipschitz-O.D.E. will converge when Δt goes to zero, and the limit is obviously the **unique solution** of impact problem which is conservative. Consequently, the complete link between fully discretized schemes, including restitution coefficient, and initial P.D.E. has to be investigated in depth.

- The explicit method being commonly used in elastodynamic contact problems, it seemed important to complete the study that had been performed in for implicit/explicit schemes by F. Chouly, P. Hild and Y. Renard for the Nitsche's methods. Further study to obtain optimal scheme, **understanding of the effect of the mass matrix lumping/ modified/ singular methods,** particularly on the stability of the method, and of the proper choice of the Nitsche's parameter $\gamma 0$ are some perspectives for Phd's / R&D's works. Extensions to thin structures are also some interesting perspectives.

- Linear fracture mechanics does not represent one of the important physical phenomena in fatigue propagation: the concept of crack reclosing discovered by Elber. Crack propagation is a complex numerical problem where remeshing and field projection are almost inevitable. New methods implicitly modeling discontinuities such as the XFEM method have been developed. One proposes to use it to model the **reclosing of crack**. It is then necessary to take into account the plasticity and the contact along the crack. A new base of plastic enrichment coupled with a formulation of Nitsche of the contact adapted to the extended finite elements would allow it by integrating the management of the history to treat this delicate and always open problem, would be another ways of exploitation of the potentialities of the Nitsche method.

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- SCHORSCH Matthieu focus on static contact models for Timoshenko beams with pinching and "ovalization", then will seek to extend to plates first, then to shells (still in elasticity). The idea is to have a fairly complete theoretical work for static thin structures in contact.
We will try to deal with plasticity in a parallel work with Franz Chouly.
Framatome have shown great interest in shell models with.

- KHADDARI Ayman focus on the dynamic study of frictionless impacts of Euler-B beams, as he has been doing for the past few months. Once we have a validated model in enriched Timoshenko, he will be able to seek to apply his study of bifurcations to it. He should add a spring at the end of the beam and redo the bifurcation diagrams to see the effect of a rigid or elastic obstacle. We would like to highlight the role of the singular mass method compared to the regular mass on impact accumulation and momentum.

- ADIOUANE Mustpaha is currently working on identifying non-smooth differential equations (impact and friction) using artificial intelligence methods. Extracting governing equations from measurement data is a major challenge in tribology. While the sparse identification of nonlinear dynamics (SINDy) algorithm works well for smooth dynamics, it presents difficulties with non-smooth systems. We propose a new method adapted to these dynamics.

- CHAUSSADE Pierre will focus on the dynamic study with friction of Euler-B beams. Once we have a validated model in enriched Timoshenko, he will be able to seek to apply his study of bifurcations to it. The aim of the work will be to produce a continuation method inspired by the work of Yves Renard on the quasi-static friction contact problem to the dynamic case.

- GROS Julien will soon begin his thesis in a collaboration between INSA Lyon- FRAMATOME and Ansys, with the aim of proposing an extension of the harmonic balance method to the case of frictional contact.

Thank you