Geometric Reformulation and Integration of Hypoelasticity

Dedicaced to P. Rougée

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Introduction

- In the 1950s, Truesdell introduced hypoelasticity as an extension of elasticity, expressed in differential rather than integral form.
- In the 1960s, Bernstein try to generalze using the Zaremba-Jaumann derivative.
- He also provided a necessary condition for an hypoelasticity law to derive from an elastic law.

We propose:

- a geometric reformulation of hypoelasticity on the manifold of Riemannian metrics.
- a new integrability condition similar to the Saint-Venant compatibility condition for the displacement problem.
- an integrator calculating the elasticity law from an integrable hypoelasticity law.
- an application to Oldroyd's isotropic hypoelasticity laws.

Outline

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1. A geometric reformulation of hypoelasticity
2. A differential complex for hypoelasticity

Elasticity and hypoelasticity laws

- Infinitesimal elasticity law: linear relation between stress and strain.
  On deformed media Ω:
  $$\boldsymbol{\tau} = \mathbf{E} : \boldsymbol{\varepsilon},$$
  where $\boldsymbol{\tau} = \frac{\boldsymbol{\sigma}}{\rho}$ Kirchhoff stress 2-tensor, $\boldsymbol{\varepsilon}$ strain 2-tensor and $\mathbf{E}$ elasticity contravariant 4-tensor.

- Elasticity law: functionnal relation between stress and strain.
  On reference configuration Ω_0:
  $$\mathbf{S} = f(\mathbf{C}),$$
  where $\mathbf{S}$ Piola-Kirchhoff 2 stress 2-tensor and $\mathbf{C}$ right Green-Cauchy 2-tensor.

- Hypoelasticity law: linear relation between derivatives of stress and strain.
  On deformed configuration Ω:
  $$\overset{\circ}{\boldsymbol{\tau}} = \mathbf{h} : \mathbf{d} = \mathbf{d} : \mathbf{h},$$
  where $\overset{\circ}{\boldsymbol{\tau}}$ objective derivative, $\mathbf{d} := \frac{1}{2}\mathcal{L}_{\boldsymbol{u}}\mathbf{q}$ strain rate, $\mathbf{q}$ Euclidean metric and $\boldsymbol{u}$ Eulerian velocity.

The contravariant 4-tensor $\mathbf{h} = (h^{ijkl})$ is called *hypoelsaticity tensor*. It has symmetries of $\mathbf{E}$.

Example: Oldroyd's isotropic hypoelasticity law

- on deformed configuration Ω

$$\overset{\triangle}{\boldsymbol{\tau}} = 2\mu\mathbf{q}^{-1}\mathbf{d}\mathbf{q}^{-1} + \lambda\,\mathrm{tr}(\mathbf{q}^{-1}\mathbf{d})\mathbf{q}^{-1}$$

- on reference configuration Ω_0

$$\partial_t\mathbf{S} = \mu\mathbf{C}^{-1}\partial_t\mathbf{C}\mathbf{C}^{-1} + \frac{\lambda}{2}\,\mathrm{tr}(\mathbf{C}^{-1}\partial_t\mathbf{C})\mathbf{C}^{-1}$$

where $\overset{\triangle}{\boldsymbol{\tau}} = \partial_t\boldsymbol{\tau} + \mathcal{L}_{\boldsymbol{u}}\boldsymbol{\tau}$ is the Oldroyd derivative and λ, μ the Lamé coefficients.

Question: can we integrate this law into an elasticity law $\mathbf{S} = f(\mathbf{C})$?

$$\partial_t\mathbf{S} = -\mu\partial_t\mathbf{C}^{-1} + \frac{\lambda}{2}\,\boxed{\cdots ??? \cdots}$$

It seems that there is no primitive for the term in $\frac{\lambda}{2}$!

$\Longrightarrow$ **We will show that this term is non-integrable**

Configurations of a deformable solid

Configuration of a deformable solid:

$$p : \begin{cases} \mathcal{B} & \xrightarrow{\text{embedding } C^\infty} & \Omega \subset \mathbb{R}^3 \\ \mathbf{X} & \longmapsto & \mathbf{x} \end{cases}$$

Deformation gradient:

$$Tp := \mathbf{F} : T\mathcal{B} \longrightarrow T\Omega$$

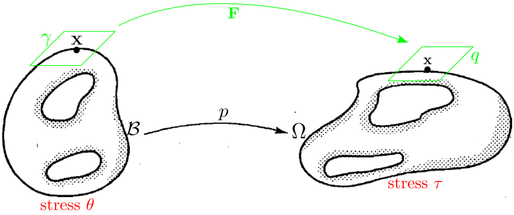


Figure: *J.E. Marsden and J.R. Hughes 1984*

Metrics:

- Euclidean $\mathbf{q}$ on Ω
- $\boldsymbol{\gamma} = p^*\mathbf{q}$ on $\mathcal{B}$
- (Cauchy right $\mathbf{C} = \phi^*\mathbf{q}$ on Ω_0)

Stresses:

- Kirchhoff $\boldsymbol{\tau}$ on Ω
- Rougée $\boldsymbol{\theta} = p^*\boldsymbol{\tau}$ on $\mathcal{B}$
- (Piola-Kirchhoff 2 $\mathbf{S} = \phi^*\mathbf{q}$ on Ω_0)

The manifold of Riemannian metrics

- To each embedding p corresponds a Riemannian metric $\boldsymbol{\gamma} = p^{*}\mathbf{q}$ on the body $\mathcal{B}$.
- The set of Riemannian metrics on $\mathcal{B}$, denoted $\mathrm{Met}(\mathcal{B})$, is a convex open set in the vector space of second-order covariant tensor fields on $\mathcal{B}$.
- It is a differential manifold (of infinite dimension).
- A point $\boldsymbol{\gamma} \in \mathrm{Met}(\mathcal{B})$ is a Riemannian metric on the body $\mathcal{B}$
- Its tangent space $T_{\boldsymbol{\gamma}}\mathrm{Met}(\mathcal{B})$ corresponds to small strains $\boldsymbol{\varepsilon}$ around the reference state $\boldsymbol{\gamma}$.
- Its cotangent space $T^{\star}_{\boldsymbol{\gamma}}\mathrm{Met}(\mathcal{B})$ corresponds to virtual powers $\mathcal{P}$ of stresses (internal forces).

Geometric formulation of elasticity according to Rougée

- Rougée's metric on the manifold of Riemannian metrics $\mathrm{Met}(\mathcal{B})$:

$$G_{\boldsymbol{\gamma}}(\delta\gamma_1, \delta\gamma_2) := \int_{\mathcal{B}} \operatorname{tr}\left(\boldsymbol{\gamma}^{-1}\delta\gamma_1\,\boldsymbol{\gamma}^{-1}\delta\gamma_2\right)\,\mu, \quad \delta\gamma_i \in T_{\boldsymbol{\gamma}}\mathrm{Met}(\mathcal{B})$$

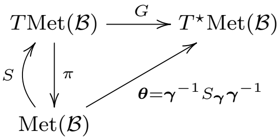
- This metric induces an injective (but not surjective) linear application

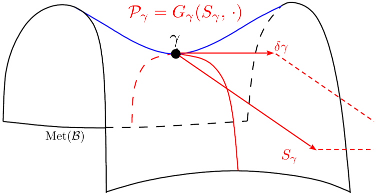
$$T_{\boldsymbol{\gamma}}\mathrm{Met}(\mathcal{B}) \to T^{\star}_{\boldsymbol{\gamma}}\mathrm{Met}(\mathcal{B}), \qquad S_{\boldsymbol{\gamma}} \mapsto \mathcal{P}_{\boldsymbol{\gamma}} = G_{\boldsymbol{\gamma}}(S_{\boldsymbol{\gamma}}, \cdot\,)$$

- The image of this application corresponds to distributions with density

$$\mathcal{P}_{\boldsymbol{\gamma}}(\delta\gamma) = \int_{\mathcal{B}} (\boldsymbol{\theta} : \delta\gamma)\mu, \qquad \boldsymbol{\theta} = \boldsymbol{\gamma}^{-1} S_{\boldsymbol{\gamma}} \boldsymbol{\gamma}^{-1}.$$

- An elastic constitutive law is then interpreted as a vector field $S_{\boldsymbol{\gamma}}$ on $Met(\mathcal{B})$.





Geometric formulation of hypoelasticity

Theorem (Rougée 1991): Given a path of embeddings p_t with $\boldsymbol{\gamma}(t) = p_t^* \mathbf{q}$, we have

$$\partial_t \boldsymbol{\gamma} = 2p^* \mathbf{d}.$$

Theorem (Kolev and Desmorat): All known objective derivatives correspond to covariant derivatives ∇_t on $T\mathrm{Met}(\mathcal{B})$ by the magic formula

$$p^* \overset{\circ}{\boldsymbol{\tau}} = \nabla_t \boldsymbol{\theta}.$$

- On the deformed configuration Ω: $$\overset{\circ}{\boldsymbol{\tau}} : \boldsymbol{\varepsilon} = \mathbf{d} : \mathbf{h} : \boldsymbol{\varepsilon}.$$
- By pullback on the body $\mathcal{B}$: $$\underbrace{p^* \overset{\circ}{\boldsymbol{\tau}}}_{=\nabla_t \boldsymbol{\theta}} : \underbrace{p^* \boldsymbol{\varepsilon}}_{:=\delta\boldsymbol{\gamma}} = \underbrace{p^* \mathbf{d}}_{=\frac{1}{2}\partial_t \boldsymbol{\gamma}} : \underbrace{p^* \mathbf{h}}_{:=\mathbf{H}} : \underbrace{p^* \boldsymbol{\varepsilon}}_{:=\delta\boldsymbol{\gamma}}.$$
- Integrating on the body $\mathcal{B}$: $$\int_{\mathcal{B}} (\nabla_t \boldsymbol{\theta} : \delta\boldsymbol{\gamma})\, \boldsymbol{\mu} = \frac{1}{2} \int_{\mathcal{B}} (\partial_t \boldsymbol{\gamma} : \mathbf{H} : \delta\boldsymbol{\gamma})\, \boldsymbol{\mu}.$$

Proposition (Reformulation of hypoelasticity on the manifold of Riemannian metrics)

An hypoelasticity law recasts on $\mathrm{Met}(\mathcal{B})$ *as:*

$$\nabla_t \mathcal{P}(\delta\boldsymbol{\gamma}) = \mathcal{H}(\partial_t \boldsymbol{\gamma}, \delta\boldsymbol{\gamma}),$$

where ∇ *is a covariant derivative on* $\mathrm{Met}(\mathcal{B})$ *and* $\mathcal{H}$ *a symmetric covariant* 2*-tensor on* $\mathrm{Met}(\mathcal{B})$.

Example: Oldroyd's isotropic hypoelasticity law

$$\overset{\triangle}{\boldsymbol{\tau}} = 2\mu \mathbf{q}^{-1}\mathbf{d}\mathbf{q}^{-1} + \lambda\,\mathrm{tr}(\mathbf{q}^{-1}\mathbf{d})\mathbf{q}^{-1}$$

$$\Big\Downarrow \quad :\boldsymbol{\varepsilon}$$

$$\overset{\triangle}{\boldsymbol{\tau}} : \boldsymbol{\varepsilon} = 2\mu\,\mathrm{tr}(\mathbf{q}^{-1}\mathbf{d}\mathbf{q}^{-1}\boldsymbol{\varepsilon}) + \lambda\,\mathrm{tr}(\mathbf{q}^{-1}\mathbf{d})\,\mathrm{tr}(\mathbf{q}^{-1}\boldsymbol{\varepsilon})$$

$$\Big\Downarrow \quad :p^*$$

$$\underbrace{\partial_t \boldsymbol{\theta}}_{=\nabla_t^0 \boldsymbol{\theta}} : \delta\boldsymbol{\gamma} = \mu\,\mathrm{tr}\left(\boldsymbol{\gamma}^{-1}\partial_t\boldsymbol{\gamma}\boldsymbol{\gamma}^{-1}\delta\boldsymbol{\gamma}\right) + \frac{\lambda}{2}\,\mathrm{tr}(\boldsymbol{\gamma}^{-1}\partial_t\boldsymbol{\gamma})\,\mathrm{tr}(\boldsymbol{\gamma}^{-1}\delta\boldsymbol{\gamma})$$

$$\Big\Downarrow \quad :\int$$

$$\underbrace{\int_{\mathcal{B}} \left(\nabla_t^0\boldsymbol{\theta} : \delta\boldsymbol{\gamma}\right)\boldsymbol{\mu}}_{=\nabla_t^0\mathcal{P}(\delta\boldsymbol{\gamma})} = \underbrace{\int_{\mathcal{B}} \left[\mu\,\mathrm{tr}\left(\boldsymbol{\gamma}^{-1}\partial_t\boldsymbol{\gamma}\boldsymbol{\gamma}^{-1}\delta\boldsymbol{\gamma}\right) + \frac{\lambda}{2}\,\mathrm{tr}(\boldsymbol{\gamma}^{-1}\partial_t\boldsymbol{\gamma})\,\mathrm{tr}(\boldsymbol{\gamma}^{-1}\delta\boldsymbol{\gamma})\right]\boldsymbol{\mu}}_{=\mathcal{H}(\partial_t\boldsymbol{\gamma},\delta\boldsymbol{\gamma})}$$

Outline

De Rham differential complex

De Rham 1-complex

A graded sequence of spaces of differential forms:

$$\Omega^0_1(\mathbb{R}^3) \xrightarrow{\mathbf{grad}} \Omega^1_1(\mathbb{R}^3) \xrightarrow{\mathbf{rot}} \Omega^2_1(\mathbb{R}^3) \xrightarrow{\mathbf{div}} \Omega^3_1(\mathbb{R}^3) \longrightarrow \{0\}.$$

Fundamental property:

$$\begin{cases} \mathbf{rot} \circ \mathbf{grad} = 0 \\ \mathbf{div} \circ \mathbf{rot} = 0 \end{cases}$$

antisymmetric forms

$\Longrightarrow$ but elasticity has other symmetries ($\xi^\flat$, ε, $\mathbf{E}$)

A differential complex for elasticity

Elasticity 2-complex: naturally adapted to the symmetries of elasticity tensors.

- Dubois-Violette theory on $\mathbb{R}^3$:

$$\begin{array}{ccccc} \Omega_2^1(\mathbb{R}^3) & \xrightarrow{D_1} & \Omega_2^2(\mathbb{R}^3) & \xrightarrow{D_2} & \Omega_2^4(\mathbb{R}^3) \\ \boldsymbol{\xi}^\flat & \longmapsto & \boldsymbol{\varepsilon} & \longmapsto & \mathbf{W} \\ \text{(displacement)} & & \text{(strain)} & & \text{(Saint-Venant)} \end{array} \qquad D_2 D_1 = 0.$$

- Differentials operators defined by Young symmetrization:

$$(D_1 \boldsymbol{\xi}^\flat)_{ij} := \frac{1}{2}\left(\partial_i \xi_j^\flat + \partial_j \xi_i^\flat\right)$$

$$(D_2 \boldsymbol{\varepsilon})_{ijkl} := (\partial_i \partial_j \varepsilon)_{kl} - (\partial_k \partial_j \varepsilon)_{il} - (\partial_i \partial_l \varepsilon)_{kj} + (\partial_k \partial_l \varepsilon)_{ij} \qquad \text{(symmetries of } \mathbf{E}\text{)}$$

Proposition (Saint-Venant compatibility condition)

If the strain ε *is generated by a displacement* $\boldsymbol{\xi}^\flat$ *we have:*

$$\mathbf{W} := D_2 \boldsymbol{\varepsilon} = D_2 D_1 \boldsymbol{\xi}^\flat = 0.$$

In coordinates: $W_{ijkl} = (\partial_i \partial_j \varepsilon)_{kl} - (\partial_k \partial_j \varepsilon)_{il} - (\partial_i \partial_l \varepsilon)_{kj} + (\partial_k \partial_l \varepsilon)_{ij} = 0.$

A differential complex for hypoelasticity

Strategy: Build a complex on $\mathrm{Met}(\mathcal{B})$ similar to the elasticity complex on $\mathbb{R}^3$.

Elasticity complex on $\mathbb{R}^3$	**Hypoelaticity complex** on $\mathrm{Met}(\mathcal{B})$
Displacement $\boldsymbol{\xi}^\flat$ (order 1)	Virtual power $\mathcal{P}$ (order 1)
Strain $\boldsymbol{\varepsilon}$ (order 2)	Hypoelasticity $\mathcal{H}$ (order 2)
Saint-Venant $\mathbf{W}$ (order 4)	Generalized Saint-Venant $\mathcal{W}$ (order 4)

- Dubois-Violette theory on $\mathrm{Met}(\mathcal{B})$:

$$\begin{array}{ccccc} \Omega^1_2(\mathrm{Met}(\mathcal{B})) & \xrightarrow{D_1} & \Omega^2_2(\mathrm{Met}(\mathcal{B})) & \xrightarrow{D_2} & \Omega^4_2(\mathrm{Met}(\mathcal{B})) \\ \mathcal{P} & \longmapsto & \mathcal{H} & \longmapsto & \mathcal{W} \end{array} .$$

- Differentials operators defined by Young symmetrization:

$$(D_1\mathcal{P})(\partial_t,\partial_s) := \frac{1}{2}\left[\nabla_t\mathcal{P}(\partial_s) + \nabla_s\mathcal{P}(\partial_t)\right]$$

$$\begin{aligned}(D_2\mathcal{H})(\partial_t,\partial_s,\partial_u,\partial_v) := {} & (\nabla_t\nabla_s\mathcal{H})(\partial_u,\partial_v) - (\nabla_u\nabla_s\mathcal{H})(\partial_t,\partial_v) \\ & - (\nabla_t\nabla_v\mathcal{H})(\partial_u,\partial_s) + (\nabla_u\nabla_v\mathcal{H})(\partial_t,\partial_s)\end{aligned}$$ (symmetries of $\mathbf{E}$)

where ∇ a covariant derivative on $T\mathrm{Met}(\mathcal{B})$ and $\partial_t,\partial_s,\partial_u,\partial_v$ are variations of $\boldsymbol{\gamma}(s,t,u,v)$.

A necessary (and sufficient) integrability condition for an hypoelasticity law

Question: Given an hypoelasticity law $\mathcal{H}$, does there exist an elastic law $\mathcal{P}$ such that

$$\nabla_t \mathcal{P}(\delta\gamma) = \mathcal{H}(\partial_t\gamma, \delta\gamma), \qquad \delta\gamma \in T_\gamma \mathrm{Met}(\mathcal{B})?$$

Thanks to symmetry of $\mathcal{H}$, this problem recast as :

$$\begin{array}{ccccc} \Omega^1_2(\mathrm{Met}(\mathcal{B})) & \xrightarrow{D_1} & \Omega^2_2(\mathrm{Met}(\mathcal{B})) & \xrightarrow{D_2} & \Omega^4_2(\mathrm{Met}(\mathcal{B})) \\ \exists \mathcal{P}\ ? & \longmapsto & \mathcal{H} & \longmapsto & \mathcal{W} \end{array}.$$

Response

- Any ∇ derivative: $D_2 D_1 \Longrightarrow$ no simple condition because of the curvature.
- Flat ∇^0 derivative: $D_2 D_1 = 0 \Longrightarrow$ similar to Saint-Venant compatibility condition.

Theorem (Necessary Integratability Condition)

If the hypoelasticity law $\mathcal{H}$ derives from an elasticity law $\mathcal{P}$, we have

$$\mathcal{W} := D_2\mathcal{H} = D_2 D_1 \mathcal{P} = 0,$$

i.e. $\nabla^0_v \nabla^0_u \mathcal{H}(\partial_t, \partial_s) - \nabla^0_v \nabla^0_t \mathcal{H}(\partial_s, \partial_u) + \nabla^0_s \nabla^0_t \mathcal{H}(\partial_v, \partial_u) - \nabla^0_s \nabla^0_u \mathcal{H}(\partial_t, \partial_v) = 0.$

Example: Oldroyd's isotropic hypoelasticity law

Generalised hypoelasticity tensor on the manifold of Riemannian metrics $\mathrm{Met}(\mathcal{B})$:

$$\mathcal{H}(\partial_t\boldsymbol{\gamma}, \delta\boldsymbol{\gamma}) = \int_{\mathcal{B}} \left[\mu \operatorname{tr}\left(\boldsymbol{\gamma}^{-1}\partial_t\boldsymbol{\gamma}\boldsymbol{\gamma}^{-1}\delta\boldsymbol{\gamma}\right) + \frac{\lambda}{2}\operatorname{tr}(\boldsymbol{\gamma}^{-1}\partial_t\boldsymbol{\gamma})\operatorname{tr}(\boldsymbol{\gamma}^{-1}\delta\boldsymbol{\gamma})\right]\boldsymbol{\mu}$$

Question: Does this law of isotropic hypoelasticity derive from an elasticity law?

- Generalized Saint-Venant tensor:

$$\mathcal{W}(\partial_t, \partial_s, \partial_u, \partial_v) = \nabla^0_v\nabla^0_u\mathcal{H}(\partial_t, \partial_s) - \nabla^0_v\nabla^0_t\mathcal{H}(\partial_s, \partial_u) + \nabla^0_s\nabla^0_t\mathcal{H}(\partial_v, \partial_u) - \nabla^0_s\nabla^0_u\mathcal{H}(\partial_t, \partial_v)$$

- Integrability condition $\mathcal{W} = 0$:

$$\forall\partial_t, \partial_s, \partial_u, \partial_v, \int_{\mathcal{B}} \lambda\left(\langle\partial_t, \partial_v\rangle\langle\partial_u, \partial_s\rangle - \langle\partial_t, \partial_s\rangle\langle\partial_u, \partial_v\rangle\right)\boldsymbol{\mu} = 0 \Longrightarrow \boxed{\lambda = 0}$$

- By contraposition: if $\lambda \neq 0$, the law does not integrate into an elasticity law.

$$\overset{\triangle}{\boldsymbol{\tau}} : \boldsymbol{\varepsilon} = 2\underbrace{\mu\operatorname{tr}(\mathbf{q}^{-1}\mathbf{d}\mathbf{q}^{-1}\boldsymbol{\varepsilon})}_{\text{shear integrable}} + \underbrace{\lambda\operatorname{tr}(\mathbf{q}^{-1}\mathbf{d})\operatorname{tr}(\mathbf{q}^{-1}\boldsymbol{\varepsilon})}_{\text{spherical non-integrable}}.$$

Sufficient condition and calculation of the integrated elasticity law

Question: How to calculate the elasticity law $\mathcal{P}$ from the integrable hypoelasticity law $\mathcal{H}$?

- Homotopy formula in elasticity complex:

$$\boxed{D_1 K_1 \mathcal{H} + K_2 D_2 \mathcal{H} = \mathcal{H}},$$

where

$$\begin{array}{ccccc} \Omega^1_2(\mathrm{Met}(\mathcal{B})) & \underset{K_1}{\overset{D_1}{\rightleftarrows}} & \Omega^2_2(\mathrm{Met}(\mathcal{B})) & \underset{K_2}{\overset{D_2}{\rightleftarrows}} & \Omega^4_2(\mathrm{Met}(\mathcal{B})) \\ \mathcal{P} & \longmapsto & \mathcal{H} & \longmapsto & \mathcal{W} \end{array}.$$

If the hypoelasticity law $\mathcal{H}$ derives from an elasticity law $\mathcal{P}$:

$$D_2\mathcal{H} = D_2 D_1 \mathcal{P} = 0$$

Injecting into the homotopy formula:

$$\underbrace{D_1 K_1 \mathcal{H}}_{\text{integrator}} + \underbrace{K_2 D_2 \mathcal{H}}_{=0} = \mathcal{H}$$

- Calculations give:

$$K_1\mathcal{H}_{\gamma}(\delta\gamma) = 2\int_0^1 r\mathcal{H}_{r\gamma}(\delta\gamma,\gamma)\,dr - \int_0^1 dy \int_0^y \left[2\mathcal{H}_{r\gamma}(\delta\gamma,\gamma) + r\,\mathrm{d}_{r\gamma}\,\mathcal{H}(\gamma,\gamma,\delta\gamma)\right]\,dr$$

$$K_2\mathcal{H}_{\gamma}(\delta_1\gamma,\delta_2\gamma) = \int_0^1 dt \int_0^t r\mathcal{W}_{r\gamma}(\delta_1\gamma,\delta_2\gamma,\gamma,\gamma)\,dr \text{ (obstruction term)}$$

Conclusion

Sumary:

- Reformulate the displacement problem using a new elasticity complex
- Reformlate geometricaly hypoelasticity on the manifold of Riemannian metrics
- Use this infinite-dimensional formalization to build a hypoelasticity complex.
- Establish a necessary and sufficient integrability condition into an elastic law
- Explicit an integrator to recover the elasticity law
- Obtain an obstruction term measuring the degree of non-integrability

Outlook: adaptation for other non-flat objective derivatives like Zaremba–Jaumann one. Curved differential complexes?

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