Evolution stochastique d'un endommagement de type Barenblatt : quelques résultats théoriques et numériques.

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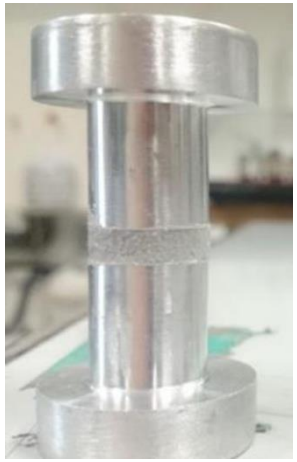
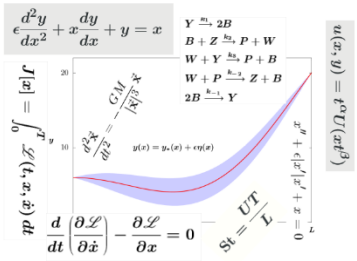


Keyword: damage in interfaces



Mathematical Models

Numerics

Experiments

Plan of the presentation

- Motivations
- A general methodology for interfaces

Modelling

- Damage
- Stochastics
- Conclusions

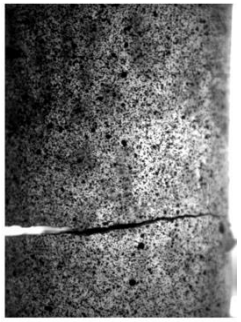
MOTIVATION

Bonding problem

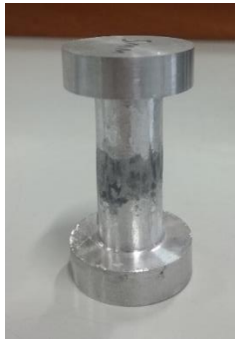
Bonding of secondary structures
(berthing structures, external ladders, internal suspended platforms, …)

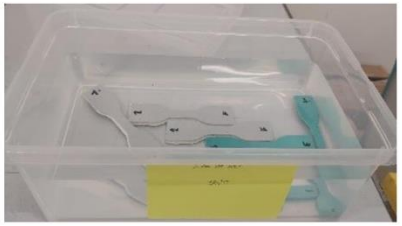
Ultimate strengh
Damage
Ageing

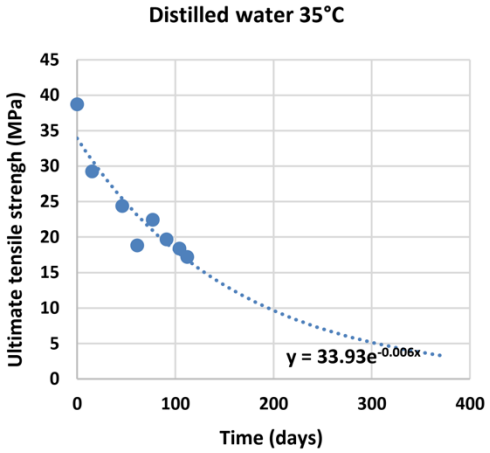
DIC



X ray

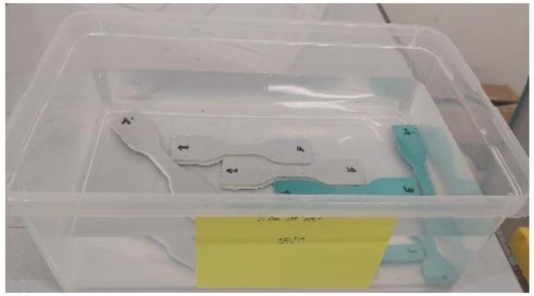


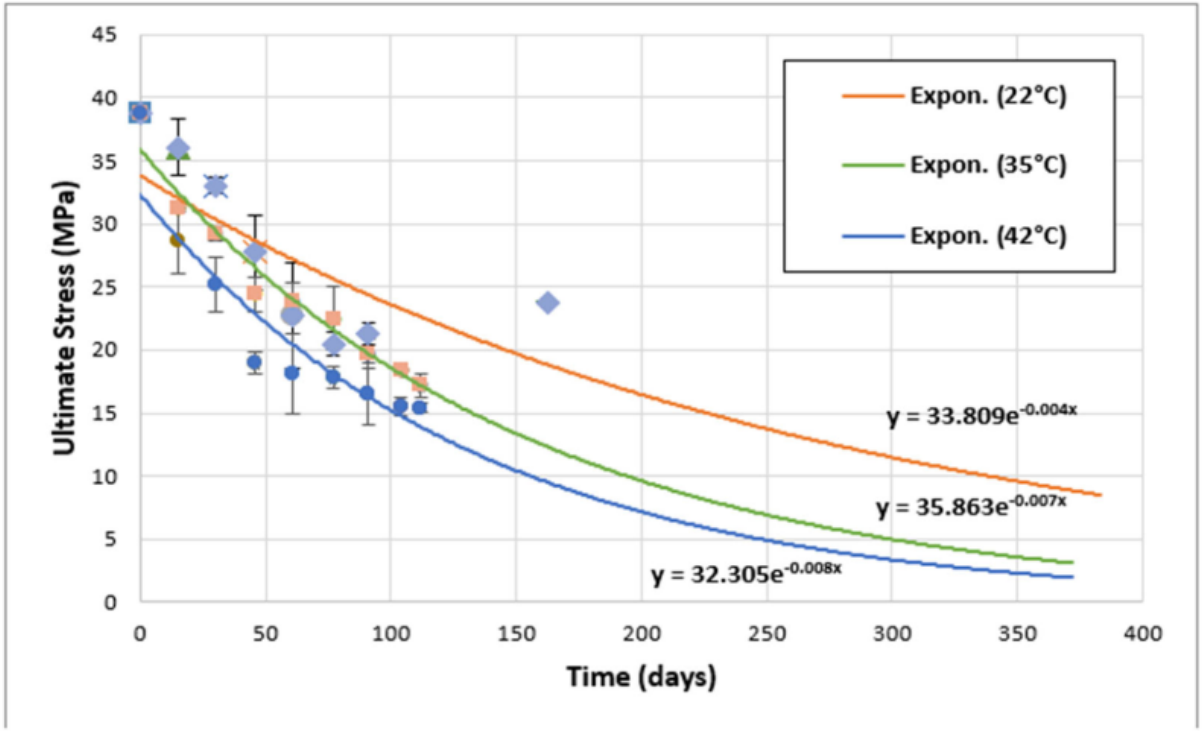




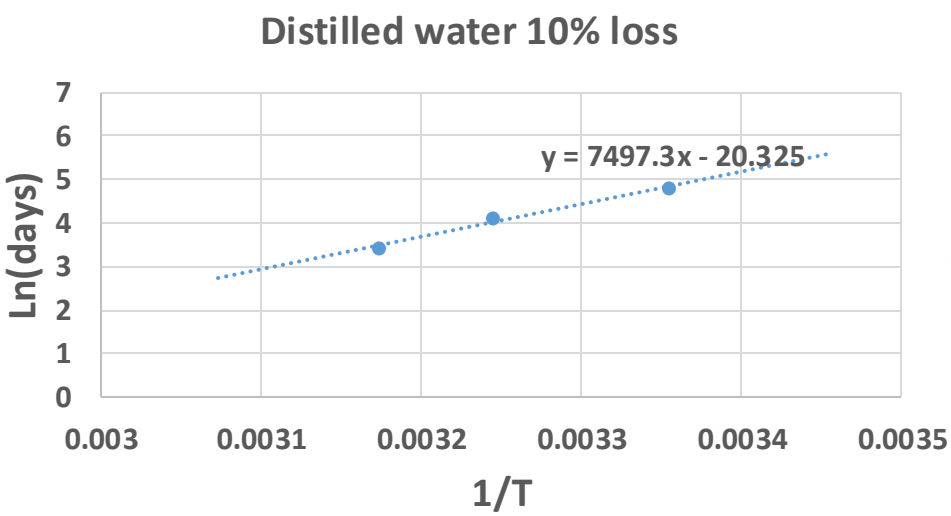
Development of experimental tools

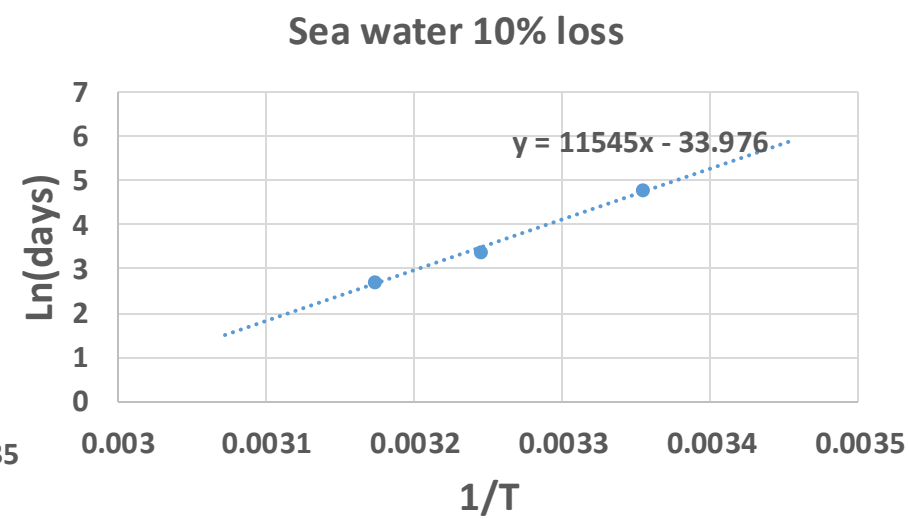
Sikadur 30 distilled water



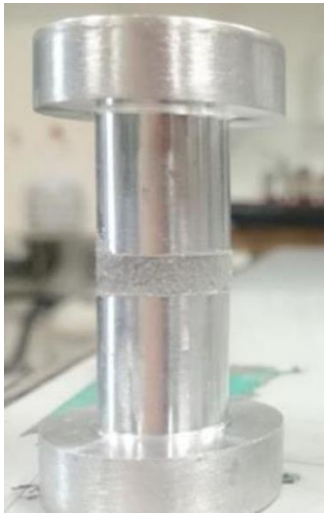


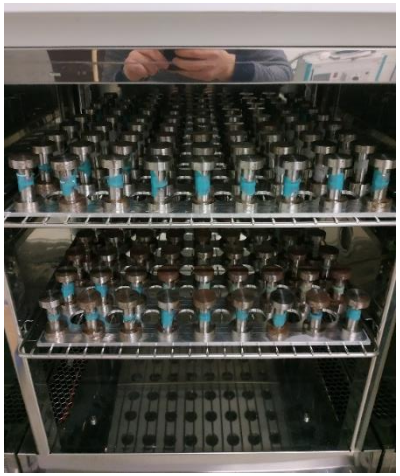
Loctite


Distilled water 10% loss
y = 7497.3x - 20.325
Ln(days)
1/T
0
1
2
3
4
5
6
7
0.003
0.0031
0.0032
0.0033
0.0034
0.0035


Sea water 10% loss
y = 11545x - 33.976
Ln(days)
1/T
0
1
2
3
4
5
6
7
0.003
0.0031
0.0032
0.0033
0.0034
0.0035

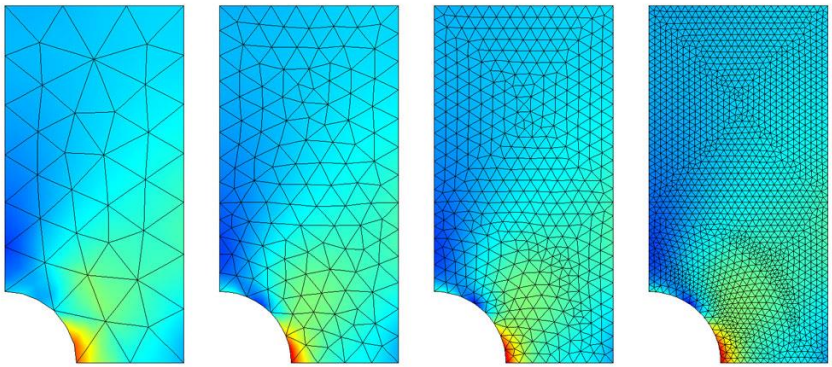
An evidence: small thickness
(and a small parameter)

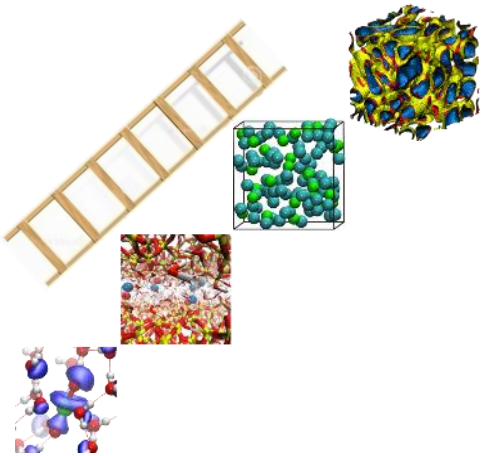




At least 2 possibilities

- *Adapted numerical methods*



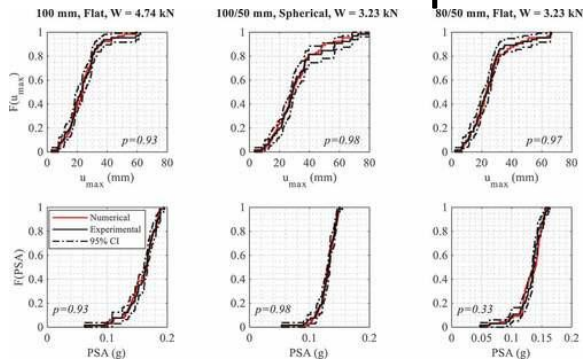


- « Simplified » models

Imperfect Interface: Jumps
- In the displacements
- and/or In the stress vector

(At least) two families of simplified models

- Phenomenological

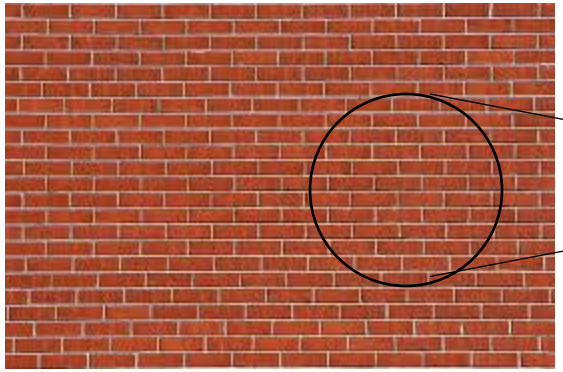


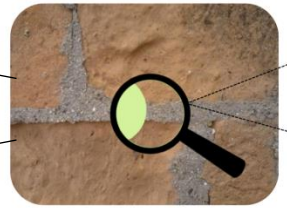
- Deductive (micromechanics)

Thin Layers (Soft ?)

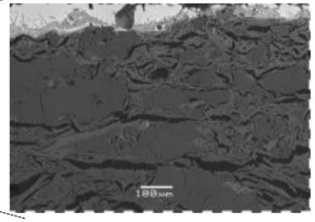
At least one small parameter (multiscale) : <u>thickness</u>

Macro scale



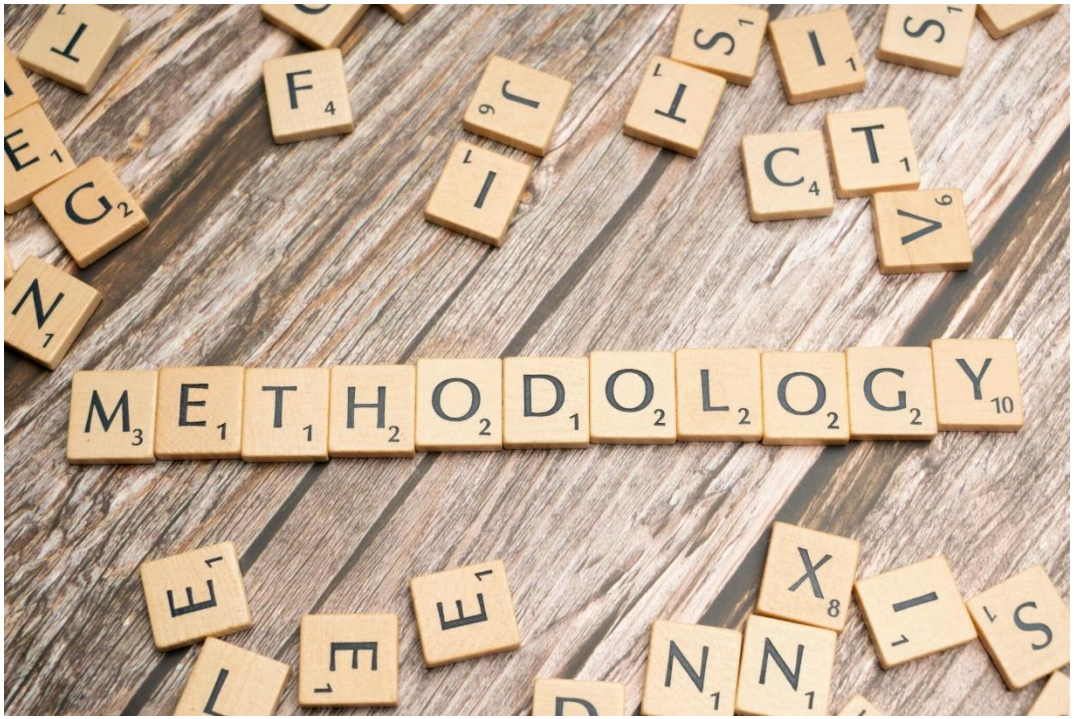


Micro scale



Behavior, cracks, roughness, couplings, temperature,etc.
At local scale

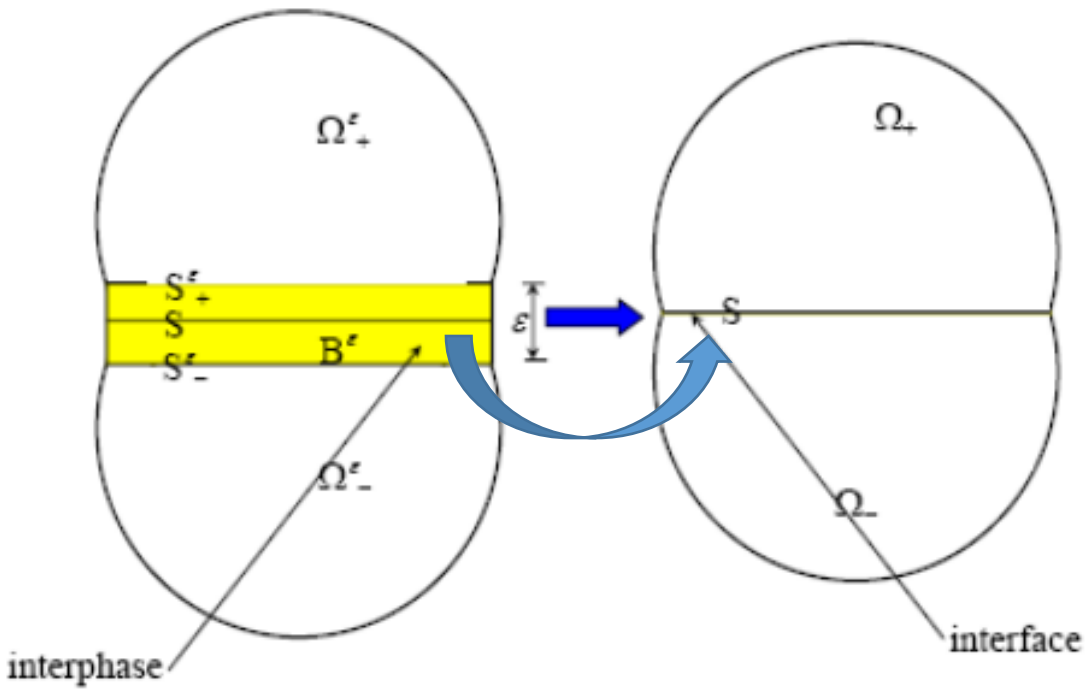
A methodology for interfaces modelling



Proposed general methodology

INTERPHASE

INTERFACE



Asymptotic theory

$f^\varepsilon(\sigma,S,P, e, E, \ldots)=0$

$f^0(\sigma n,Sn,Pn, [u],\ldots)=0$

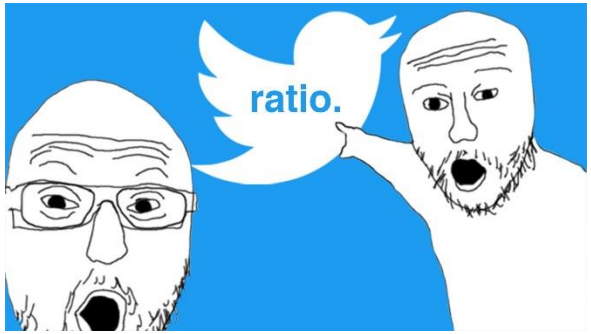
How best to preserve the mechanical characteristics of the interphase?

4 RECALL

1st Various kinds of interfaces (a tentative of classification)

- Importance of the ratios

adhesive/substrates stiffnesses versus thickness ε^p



- p = 1 Soft (Epoxy/Steel)
- p = 0 Hard/stiff (Araldite/Plexiglass)
- p = -1 Rigid (Welding)

Caillerie 1980

2nd Convergence results in energy (elastostatics)

Soft interface (Licht-Michaille 96&97, PhD Zaittouni) (using Γ-convergence)

$a^m \sim e(u)^q$

(interphase energy density)

$$\sigma n = \bar{b}([u] \otimes_s n)n$$

$$\bar{b} = lim\ a^m/\varepsilon^{q-1}$$

Generalized spring/Imperfect interface

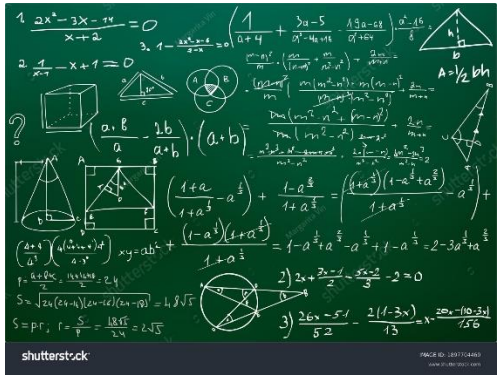
Hard interface (Lebon-Rizzoni 2010)

$$[u^0] = 0\ ,\quad [\sigma^0 n] = 0$$

Perfect interface

$$\frac{u^\varepsilon - u^0}{\varepsilon} \rightharpoonup u^1 \quad in\ L^2(\Omega)\ (\text{weak}),$$

$$\frac{\sigma^\varepsilon - \sigma^0}{\varepsilon} \rightharpoonup \sigma^1 \quad in\ D'(\Omega)\ (\text{weak}).$$



$[u^1]$ given by order 0

$[\sigma^1]$ given by order 0

Imperfect Interface

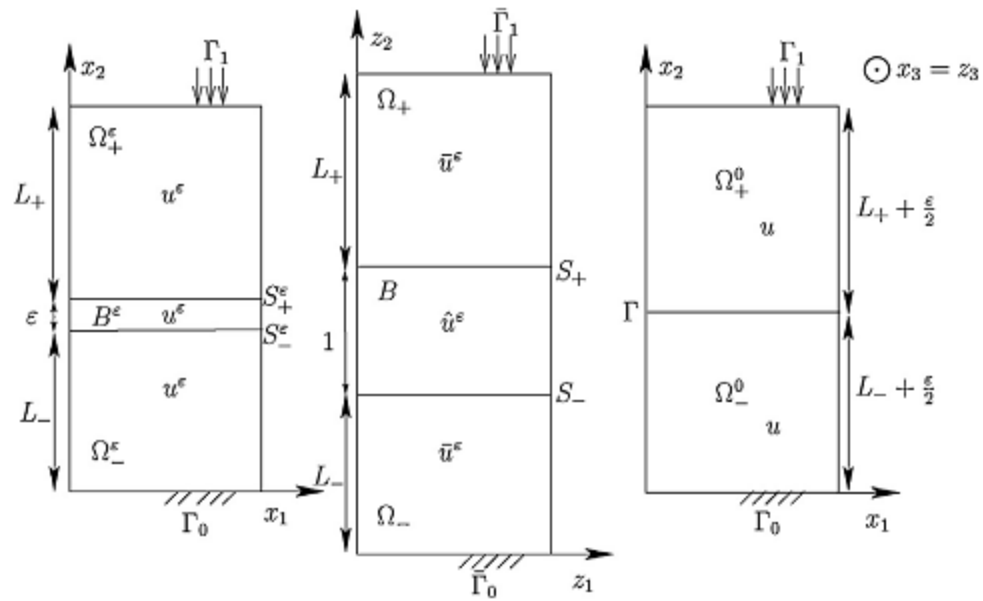
Non local model (tangential derivatives)

In addition (formal) *Formal* equivalence between the soft interface law and the hard interface law

3rd Recall « FORMAL » ASYMPTOTIC EXPANSIONS

5 STEPS TO...

Expansion
Rescaling
Analysis
Matching
Model

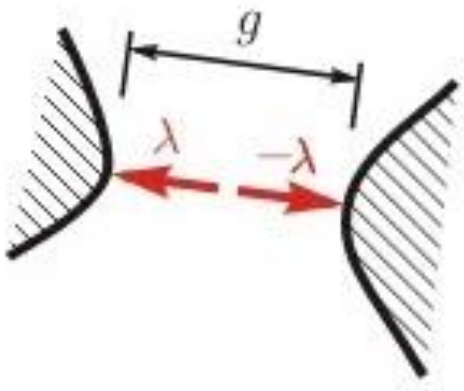


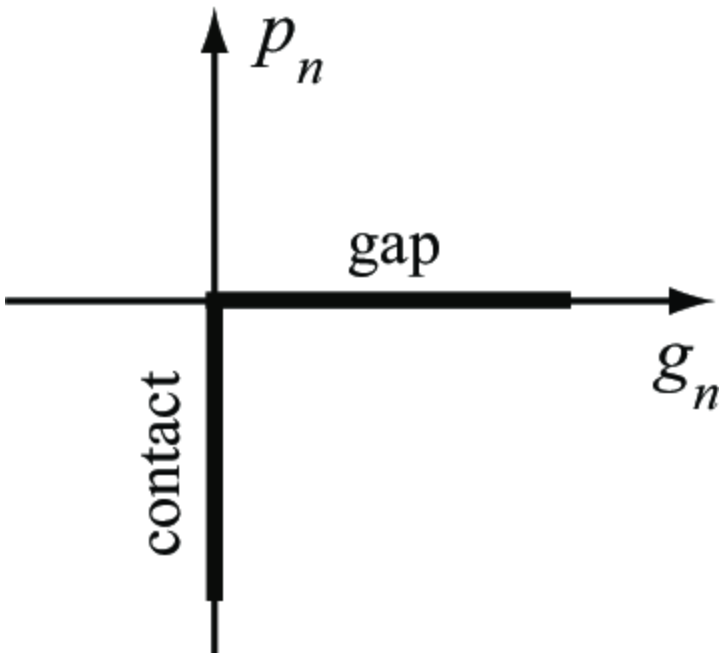
$$\begin{cases} u^\varepsilon = u^0 + \varepsilon u^1 + o(\varepsilon) \\ \sigma^\varepsilon = \sigma^0 + \varepsilon \sigma^1 + o(\varepsilon) \end{cases}$$

In the following, n = e_3

Sanchez-Palencia, Ciarlet, …

4th UNILATERAL CONTACT





Bonetti et al. 2017, Lebon et al. 2022

Two regimes (piecewise material)

Mielke et al., 2012

Thickness ε

2 regimes

- Traction soft material $K^\varepsilon = \varepsilon K$

$$\text{if } e^s(u^\varepsilon) \geq 0$$ (Spheric)

- Compression hard material $K^\varepsilon = K$
  (in fact not necessary for shear)

$$\text{if } e^s(u^\varepsilon) \leq 0$$

K: εb or B

Soft

$$\sigma_{ij}^\varepsilon = \begin{cases} \varepsilon b_{ijhk}(l) e_{hk}(u^\varepsilon) & \text{if } e^s(u^\varepsilon) \geq 0 \\ e^s(u^\varepsilon) B_{ijhk} \delta_{hk} + \varepsilon b_{ijhk}(l) e^d_{hk}(u^\varepsilon) & \text{if } e^s(u^\varepsilon) \leq 0 \end{cases}$$

Hard

Asymptotic study (soft case)

Expansion of the strain tensor spheric at first order → $u^0_{3,3}$ → $[u^0_3]$

$$b_{i3j3} = K^{33}_{ij}$$

Constitutive equation in the adhesive → $\sigma^0 e_3 = K^{33}[u^0] \quad if\, [u^0_3] \geq 0$

→ $[u^0_3] = 0 \quad if\, [u^0_3] \leq 0$

$$\boxed{\sigma^0 e_3 = \bar{K}[u^0] + \tau^0_{33}\, e_3}$$

Remark

Isotropy → $[u_3^0] \geq 0, \quad \tau_{33}^0 [u_3^0] = 0,$ $\tau_{33}^0 \leq 0$

Due to expansion at second order (quasi)Signorini

(spheric + constitutive equation)

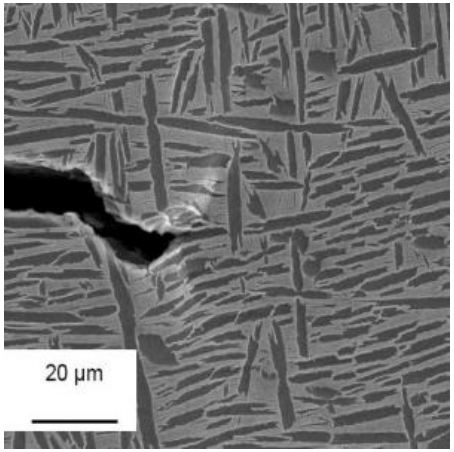
Anisotropy → Adhesion ?

(Curnier et al.) Why ? (Lebon-Rizzoni 2022)

Plan of the presentation

- Motivations
- A general methodology for interfaces Modelling
- Damage
- Stochastics
- Conclusions

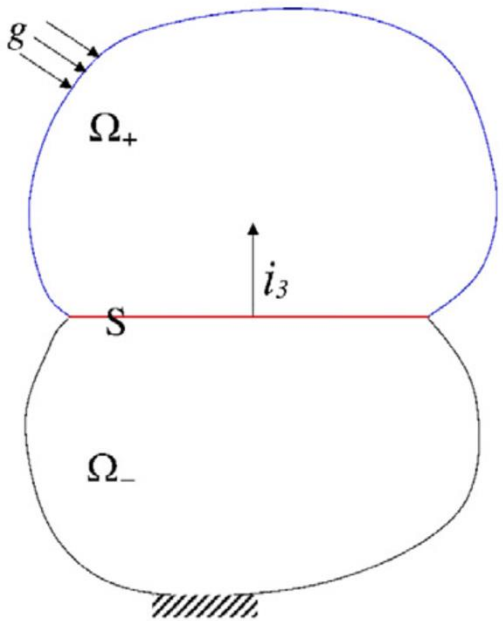
DAMAGE

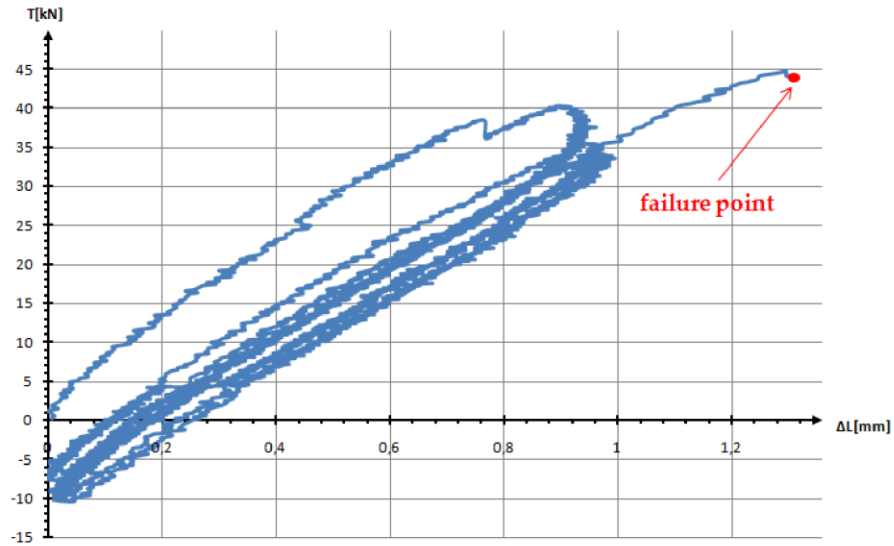


Bonetti et al. 2017, Mailto et al., 2019, Raffa et al. 2021,2022,2023

Asymptotic modeling of interfaces
How to introduce damage?

Soft Interface $K^{\varepsilon} = \varepsilon K$





Cyclic experiments Orefice et al. 2016

$$\sigma e_3 = \bar{K}[u]$$

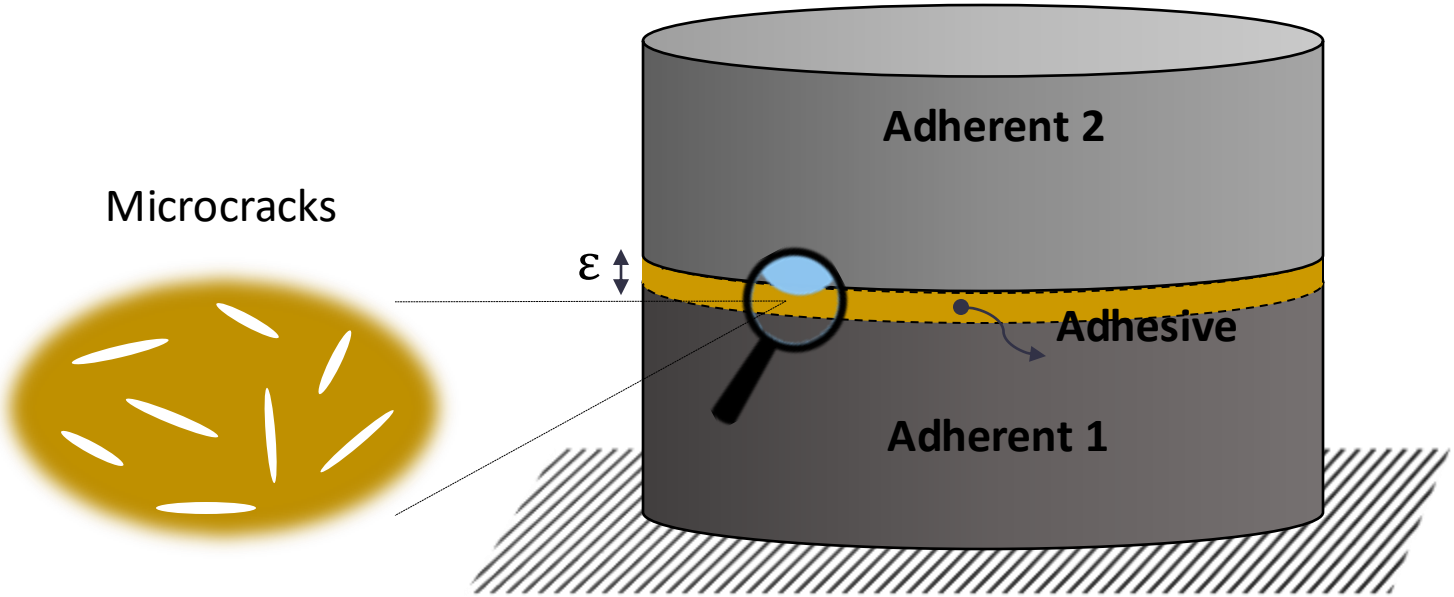
Spring model

Including UC $\sigma e_3 = \bar{K}[u] + s_{33} e_3$

K = K(R) ?

Asymptotic modeling of interfaces
How to introduce damage?

Proposed methodology

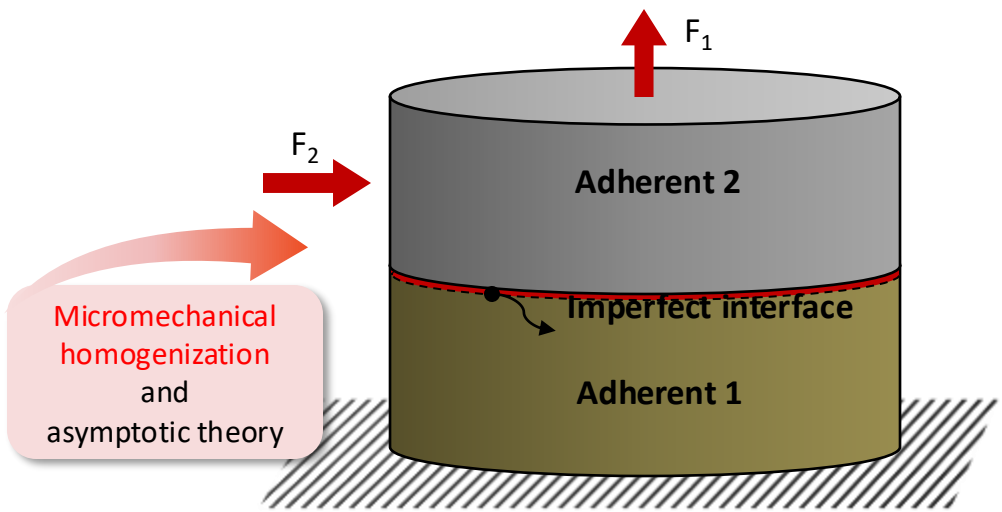


Kachanov, Sevastianov, Welemane, …

$K = K(R)$

Asymptotic modeling of interfaces
How to introduce damage?

Proposed methodology



Imperfect interface with evolutive damage $\dot{R}$

$$[\![\mathbf{u}]\!] \neq 0$$

Asymptotic modeling of interfaces
How to introduce damage?

Proposed methodology (in the interphase)

Free Energy

$$\psi(e(u),R) = \underbrace{\frac{1}{2}\mathbb{K}(R)e(u):e(u)}_{\text{Elasticity with damage}} + \overbrace{\omega(R)}^{\text{Threshold}} + \overbrace{\frac{\alpha}{p}|\nabla R|^p}^{\text{Non local}} + \overbrace{\boxed{I_{[0,1]}(R)}}^{0 \leq R \leq 1}$$

$$I_A(x) = 0 \; if \; x \in A$$

$$I_A(x) = +\infty \; if \; x \notin A$$

Constitutive equation

$$\sigma = \psi_{,e(u)} = \mathbb{K}(R)e(u)$$

Asymptotic modeling of interfaces
How to introduce damage?

Proposed methodology (in the interphase)

Dissipation potential

$$\phi(\dot{R}) = \overbrace{\frac{1}{n+1}\eta(R)\dot{R}^{n+1}}^{\text{Rate dependent}} + \overbrace{I_{[0,+\infty[}(\dot{R})}^{\text{Irreversibility}}$$

Evolution law Difficulty: 2 subdifferentials

$$0 \in \partial I_{[0,1]}(R) + \partial I_{[O,+\infty]}(\dot{R}) + \frac{1}{2}K_{,R}(R)e(u):e(u) + \omega_{,R}(R) + \alpha\,\Delta_p R + \eta(R)\dot{R}^n$$

$$\eta(R)\dot{R}^n = -(\omega_{,R}(R) + \frac{1}{2}\mathbb{K}_{,R}(R)e(u)e(u) + \alpha\Delta_p R)_-; \;\; R(0) = R_0$$

Asymptotic modeling of interfaces
How to introduce damage?

Proposed methodology (in the interface, asymptotic expansions)

$$u^\varepsilon \approx u^0 + \varepsilon u^1$$
$$\sigma^\varepsilon \approx \sigma^0 + \varepsilon \sigma^1$$

Hard interface $\mathbb{K}^\varepsilon = \mathbb{K}$

- Zero-order interface law:

$$[[\mathbf{u}^0]] = \mathbf{0}$$
$$[[\sigma^0\, \mathbf{i}_3]] = \mathbf{0}$$

- One-order interface law:

$$[[\mathbf{u}^1]] = (\mathbf{K}^{33})^{-1}\left(\sigma^0 \mathbf{i}_3 - \mathbf{K}^{\alpha 3}\mathbf{u}^0_{,\alpha}\right) - \langle\langle \mathbf{u}^0_{,3}\rangle\rangle$$
$$[[\sigma^1\, \mathbf{i}_3]] = \left(-\mathbf{K}^{\beta\alpha}\mathbf{u}^0_{,\beta} - \mathbf{K}^{3\alpha}(\mathbf{K}^{33})^{-1}\left(\sigma^0\mathbf{i}_3 - \mathbf{K}^{\beta 3}\mathbf{u}^0_{,\beta}\right)\right)_{,\alpha}$$
$$-\langle\langle \sigma^0_{,3}\mathbf{i}_3\rangle\rangle$$

Asymptotic modeling of interfaces
How to introduce damage?

Proposed methodology (interface)

Hard interface $\mathbb{K}^{\varepsilon} = \mathbb{K}$

$$\eta(R)\dot{R}^n = -\left\{ \omega(R) + \frac{1}{2} K^{\varepsilon}_{,R}(R) \begin{pmatrix} \langle u^{\varepsilon}_{,1} \rangle \\ \langle \hat{u}^{\varepsilon}_{,2} \rangle \\ [\hat{u}^{\varepsilon}] \end{pmatrix} \cdot \begin{pmatrix} \langle \hat{u}^{\varepsilon}_{,1} \rangle \\ \langle \hat{u}^{\varepsilon}_{,2} \rangle \\ [\hat{u}^{\varepsilon}] \end{pmatrix} + \alpha \Delta_p^2 R \right\}_-$$

$$u^{\varepsilon} \approx u^0 + \varepsilon u^1$$
$$\sigma^{\varepsilon} \approx \sigma^0 + \varepsilon \sigma^1$$

$$K^{\varepsilon} = \begin{pmatrix} \varepsilon K^{11} & \varepsilon K^{12} & K^{13} \\ \varepsilon K^{12} & \varepsilon K^{22} & K^{23} \\ K^{13} & K^{23} & \frac{1}{\varepsilon} K^{33} \end{pmatrix}$$

Asymptotic modeling of interfaces
How to introduce damage?

Proposed methodology (in the interface, asymptotic expansions)

$$u^\varepsilon \approx u^0$$
$$\sigma^\varepsilon \approx \sigma^0$$

Soft interface $\mathbb{K}^\varepsilon = \varepsilon \mathbb{K}$

$$\boldsymbol{\sigma}^0\, \mathbf{i}_3 = \mathbb{K}[[\mathbf{u}^0]]$$
$$[[\boldsymbol{\sigma}^0\, \mathbf{i}_3]] = \mathbf{0}$$

Asymptotic modeling of interfaces
How to introduce damage?

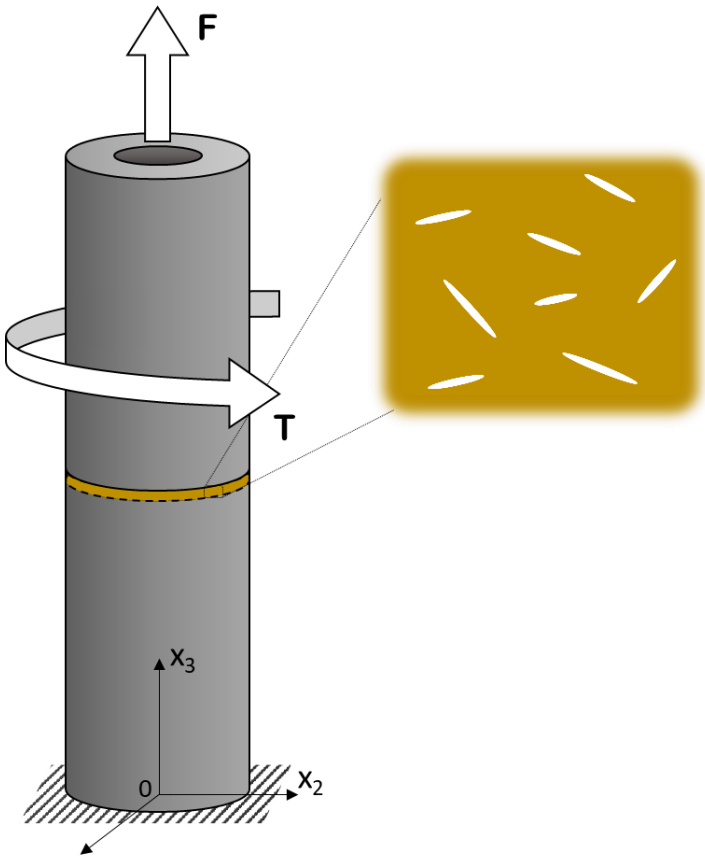
Proposed methodology (interface)

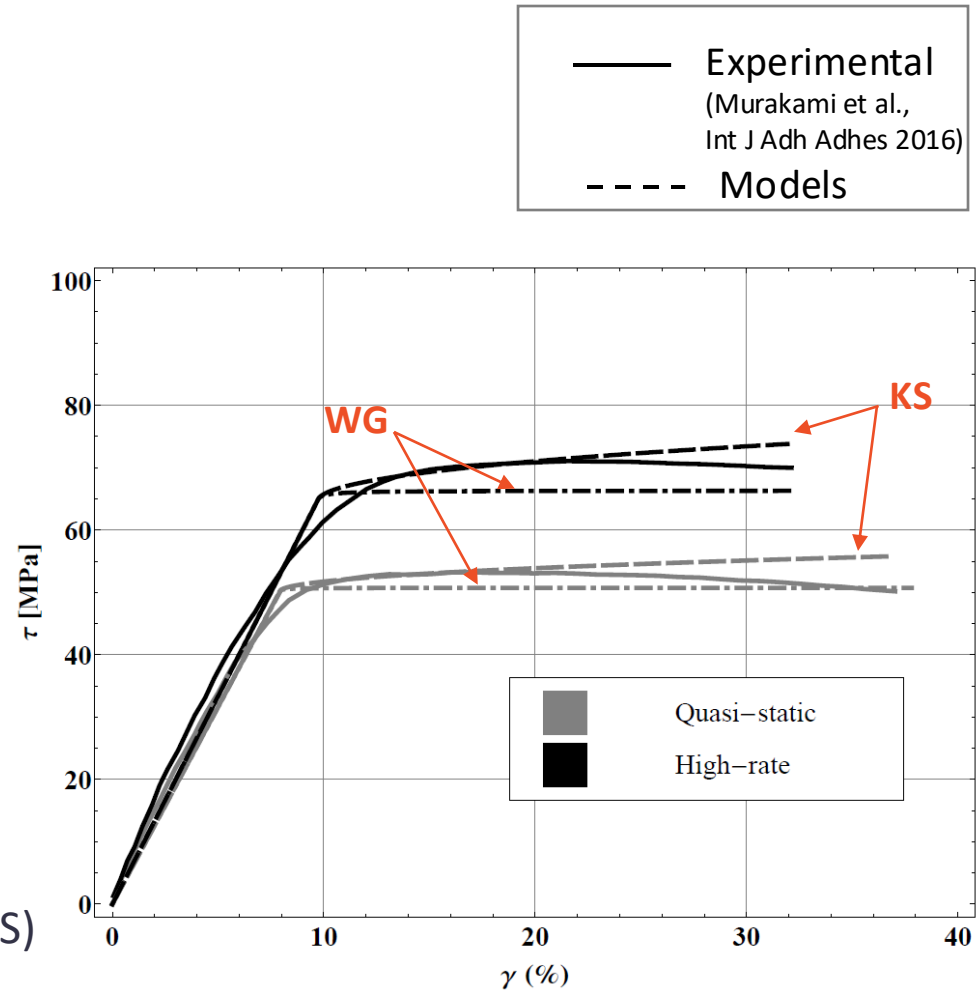
Soft interface $\mathbb{K}^{\varepsilon} = \varepsilon \mathbb{K}$

$$u^{\varepsilon} \approx u^{0}$$
$$\sigma^{\varepsilon} \approx \sigma^{0}$$

$$\eta(R)\dot{R}^{n} = -\left\{ \omega(R) + \frac{1}{2} K_{,R}(R)[[u^{\varepsilon}]][[u^{\varepsilon}]] + \alpha \Delta_{p}^{2} R \right\}_{-}$$

Numerical example (Hard, n=1, p=1)



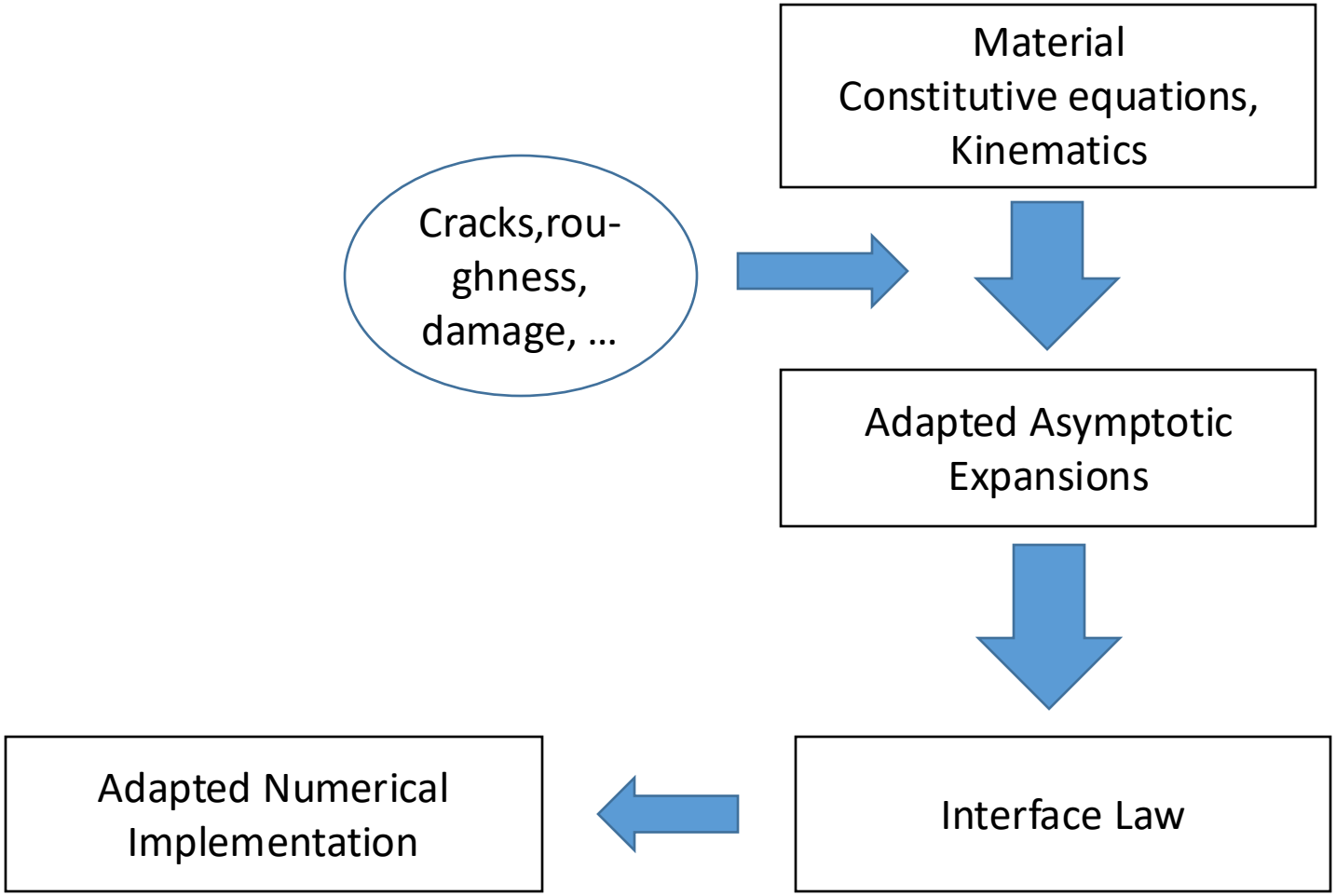


stress-based (Kachanov-Sevostianov, KS) and strain-based (Welemane-Goidescu, WG).

$$K = \frac{K_0}{1 + CR} \qquad K = (1 - CR)$$

Raffa M. L. et al., Technologies, 2021
Rizzoni R. et al., KEM, 2022

PARTIAL CONCLUSION



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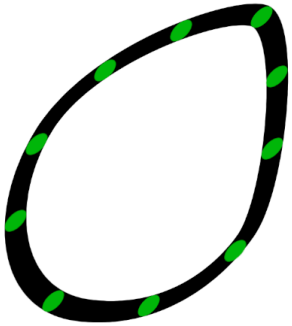
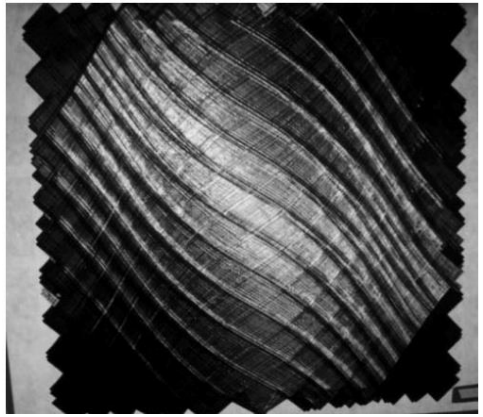
Stochastics

Lengths and orientation of cracks



Thickness



Stiffness

Threshold, viscosity, etc.

Stochastics

Itô Integral

We define a partition $0 = t_0 < t_1 < \cdots < t_k = T$ of $[0, T]$

We define $\sum_{i=0}^{k-1} h_{t_i} \left(B_{t_{i+1}} - B_{t_i}\right)$

B is Brownian motion and h is a stochastic process

$$\int_0^T h_t dB_t$$

is the root mean square limit when the step of the partition tends to zero

Stochastics

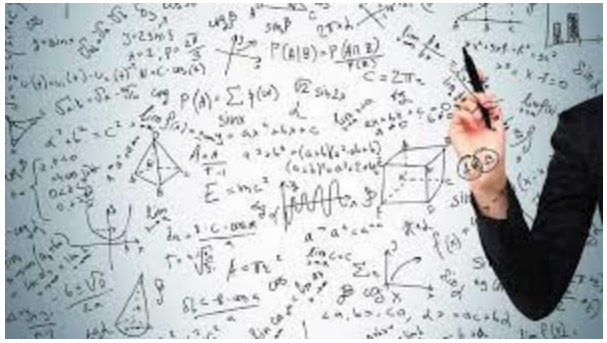
Ito Integral $\partial_t R$ → $\partial_t \left(R - \int_0^t h(R)dW\right)$

$$\phi(\dot{R}) = \frac{1}{n+1}\eta(R)\dot{R}^{n+1} + I_{[0,+\infty[}(\dot{R})$$

Two possibilities

Itô integral

In or out Indicator function ?



Mathematical results (in the volume)

(où R devient χ)

On an « Allen-Cahn » type problem with double constraints :

$$\begin{cases} \eta\dot{\chi} - \alpha\Delta\chi + \partial I_{[0,1]}(\chi) + \partial I_{[0,+\infty)}(\dot{\chi}) & \ni & k(\chi) + h & \text{in } (0,T)\times\Lambda, \\ \chi(t=0,x) & = & \chi_0(x) & \omega\in\Omega, x\in\Lambda, \\ \nabla\chi\cdot n & = & 0 & \text{on } (0,T)\times\partial\Lambda. \end{cases}$$

Finding $0 \leqslant \chi \leqslant 1$ satisfying

2 subdifferentials → 1 subdifferential

$$\begin{cases} \partial_t\Big(\chi - \int_0^{\cdot} g(\chi(s))dW(s)\Big) - \Delta\chi + \partial I_{[0,1]}(\chi) & \ni & k(\chi) + h & \text{in } \Omega\times(0,T)\times\Lambda, \\ \chi(\omega, t=0,x) & = & \chi_0(x) & \omega\in\Omega, x\in\Lambda, \\ \nabla\chi\cdot n & = & 0 & \text{on } \Omega\times(0,T)\times\partial\Lambda. \end{cases}$$

- $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, $\Lambda \subset \mathbb{R}^d$, $d \geqslant 1$ and $T > 0$.
- $g : \mathbb{R} \to \mathbb{R}$ a Lipschitz-continuous function with $\mathrm{supp}(g) \subset (0,1)$.
- $k : \mathbb{R} \to [0, +\infty[$ a Lipschitz-continuous function with $k(0) = 0$.
- h a random external force of damage belonging to $L^2(\Omega\times(0,T)\times\Lambda)$.
- $\chi_0 : \Omega\times\Lambda \to \mathbb{R}$ the initial condition, $0 \leqslant \chi_0 \leqslant 1$ and $\chi_0 \in L^2(\Omega; H^1(\Lambda))$.

Using Lagrange multipliers

Finding a pair (χ, ψ) such that $0 \leqslant \chi \leqslant 1$, $\psi \in \partial I_{[0,1]}(\chi)$ and

$$(\mathcal{P}) \begin{cases} \partial_t\Big(\chi - \int_0^{\cdot} g(\chi(s))dW(s)\Big) - \Delta\chi + \psi &= k(\chi) + h \quad \text{in } \Omega \times (0,T) \times \Lambda, \\ \chi(\omega, t=0, x) &= \chi_0(x) \quad \omega \in \Omega, x \in \Lambda, \\ \nabla\chi \cdot n &= 0 \quad \text{on } \Omega \times (0,T) \times \partial\Lambda. \end{cases}$$

Definition of a solution

Any pair (χ, ψ) belonging to $\left(L^2_{\mathcal{P}_T}\left((0,T)\times\Omega; L^2(\Lambda)\right)\right)^2 \subset \left(L^2\left((0,T)\times\Omega; L^2(\Lambda)\right)\right)^2$ with

1. $\chi \in L^2_{\mathcal{P}_T}\left((0,T)\times\Omega; H^1(\Lambda)\right) \cap L^2\left(\Omega; \mathscr{C}\left(0,T; L^2(\Lambda)\right)\right)$
2. $\partial_t\left(\chi - \int_0^{\cdot} g(\chi(s))dW(s)\right), \Delta\chi \in L^2(\Omega; L^2(0,T; L^2(\Lambda)))$

is a solution to Problem $(\mathcal{P})$ if a.e in $(0,T)\times\Lambda$ and $\mathbb{P}$-a.s in Ω,

$$0 \leq \chi \leq 1 \text{ and } \psi \in \partial I_{[0,1]}(\chi)$$

and if for all $t \in [0,T]$

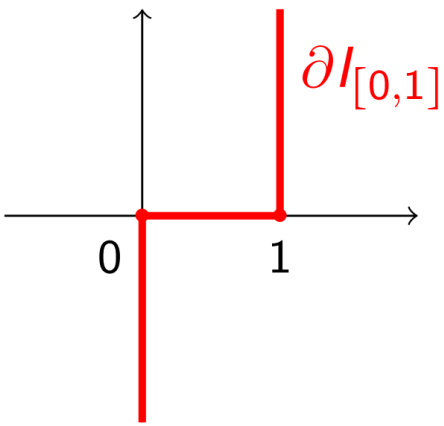
$$\chi(t) = \chi_0 + \int_0^t \left(\Delta\chi(s) - \psi(s) + k(\chi(s)) + h(s)\right)ds + \int_0^t g(\chi(s))dW(s),$$

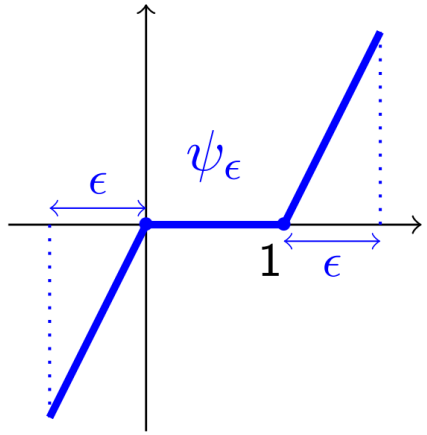
in $L^2(\Lambda)$ and $\mathbb{P}$-a.s in Ω, where Δ denotes the Laplace operator on $H^1(\Lambda)$ associated with the formal Neumann boundary conditions.

Theorem ● [Bauzet-Bonetti-Bonfanti-Lebon-Vallet, Math. Meth. Appl. Sci., 2017]

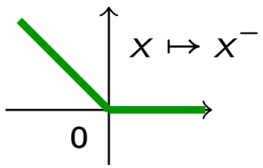
There exists a unique pair (χ, ψ) solution of Problem $(\mathcal{P})$.

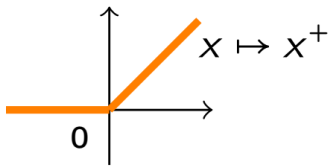
Moreau-Yosida approximation





$$\psi_\epsilon(x) = -\frac{x^-}{\epsilon} + \frac{(x-1)^+}{\epsilon}.$$





Intermediate problem ● [BBBLV17]

For a fixed $\epsilon > 0$, find χ_ϵ satisfying the following Problem $(\mathcal{P}_\epsilon)$

$$\begin{cases} \partial_t\Big(\chi_\epsilon - \int_0^\cdot g(\chi_\epsilon(s))dW(s)\Big) - \Delta\chi_\epsilon + \psi_\epsilon(\chi_\epsilon) = k(\chi_\epsilon) + h & \text{in } \Omega\times(0,T)\times\Lambda, \\ \chi_\epsilon(\omega, x, t=0) = \chi_0(x) & \omega\in\Omega, x\in\Lambda, \\ \nabla\chi_\epsilon\cdot n = 0 & \text{on } \Omega\times(0,T)\times\partial\Lambda. \end{cases}$$

Coupled problems (Bauzet et al. 2021)

Additive (1.1)

$$\begin{cases} \partial_t \vartheta + \partial_t \Big(\chi - \int_0^{\cdot} h dw\Big) - \Delta \vartheta &= 0 \text{ in } (0,T) \times D \times \Omega, \\ \tilde{\alpha}\left(\partial_t (\chi - \int_0^{\cdot} h dw)\right) - \Delta \chi &= \vartheta \text{ in } (0,T) \times D \times \Omega, \\ \nabla \chi . \mathbf{n} = \nabla \vartheta . \mathbf{n} &= 0 \text{ on } (0,T) \times \partial D \times \Omega, \\ \chi(0,.) = \chi_0 \quad \text{and} & \vartheta(0,.) = \vartheta_0, \end{cases}$$

Multiplicative (1.2)

$$\begin{cases} \partial_t \vartheta + \partial_t \Big(\chi - \int_0^{\cdot} \mathscr{H}(\chi) dw\Big) - \Delta \vartheta &= 0 \text{ in } (0,T) \times D \times \Omega, \\ \tilde{\alpha}\left(\partial_t (\chi - \int_0^{\cdot} \mathscr{H}(\chi) dw)\right) - \Delta \chi &= \vartheta \text{ in } (0,T) \times D \times \Omega, \\ \nabla \chi . \mathbf{n} = \nabla \vartheta . \mathbf{n} &= 0 \text{ on } (0,T) \times \partial D \times \Omega, \\ \chi(0,.) = \chi_0 \quad \text{and} & \vartheta(0,.) = \vartheta_0, \end{cases}$$

We assume the following assumptions:

H_1: $h \in \mathcal{N}_w^2(0,T,H^1(D))^\dagger$.

H_2: $\tilde{\alpha} = I_d + \alpha$ where $I_d : \mathbb{R} \to \mathbb{R}$ is the identity function and $\alpha : \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous, coercive and non-decreasing function such that $\alpha(0) = 0$.

H_3: $\chi_0, \vartheta_0 \in H^1(D)$.

H_4: $\mathscr{H} : H^1(D) \to H^1(D)$ is a Lipschitz-continuous mapping with Lipschitz constant $C_{\mathscr{H}} > 0$.

Theorem 1.4. *Under assumptions H_1 to H_3, there exists a unique pair (ϑ, χ) solution of System (1.1) in the sense of Definition 1.1.*

Theorem 1.6. *Under Assumptions H_2 to H_4, there exists a unique pair (ϑ, χ) solution of Problem (1.2) in the sense of Definition 1.2.*

Definition 1.1. Any pair of predictable processes $(\vartheta, \chi) \in \left(\mathcal{N}_w^2(0, T, H^1(D))\right)^2$ such that $\vartheta \in L^2\big(\Omega, H^1(Q)\big)$ and $\partial_t(\chi - \int_0^\cdot hdw) \in L^2(\Omega \times Q)$, is a solution of System (1.1) if t-almost everywhere in $(0, T)$, P-almost surely in Ω, the following variational formulations hold: for any $v \in H^1(D)$,

$$\int_D \partial_t \vartheta v dx + \int_D \partial_t(\chi - \int_0^\cdot hdw) v dx + \int_D \nabla\vartheta.\nabla v dx = 0 \tag{1.3}$$

$$\int_D \tilde{\alpha}\Big(\partial_t(\chi - \int_0^\cdot hdw)\Big) v dx + \int_D \nabla\chi.\nabla v dx = \int_D \vartheta v dx, \tag{1.4}$$

with $\chi(0, .) = \chi_0 \in H^1(D)$ and $\vartheta(0, .) = \vartheta_0 \in H^1(D)$.

Definition 1.2. Any pair of predictable processes $(\vartheta, \chi) \in \left(\mathcal{N}_w^2(0, T, H^1(D))\right)^2$ such that $\vartheta \in L^2\big(\Omega, H^1(Q)\big)$ and $\partial_t(\chi - \int_0^\cdot \mathscr{H}(\chi)dw) \in L^2(\Omega \times Q)$, is a solution of System (1.2) if t-almost everywhere in $(0, T)$, P-almost surely in Ω, the following variational formulations hold: for any $v \in H^1(D)$,

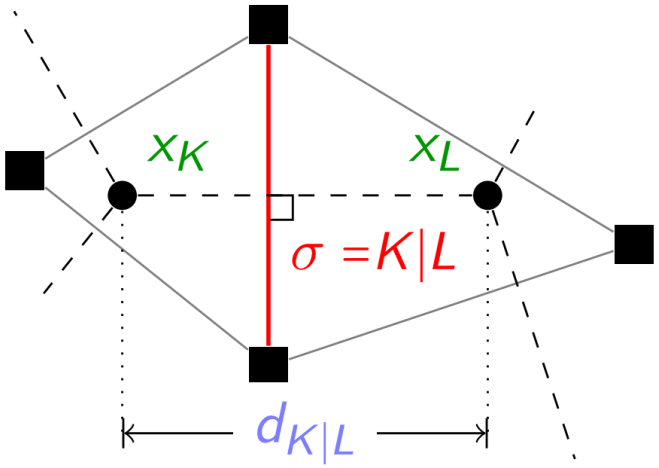
$$\int_D \partial_t \vartheta v dx + \int_D \partial_t(\chi - \int_0^\cdot \mathscr{H}(\chi)dw) v dx + \int_D \nabla\vartheta.\nabla v dx = 0$$

$$\int_D \tilde{\alpha}\Big(\partial_t(\chi - \int_0^\cdot \mathscr{H}(\chi)dw)\Big) v dx + \int_D \nabla\chi.\nabla v dx = \int_D \vartheta v dx,$$

with $\chi(0, .) = \chi_0 \in H^1(D)$ and $\vartheta(0, .) = \vartheta_0 \in H^1(D)$.

The considered mesh on Λ

Let $\mathcal{T}$ be an admissible finite-volume mesh of $\Lambda \subset \mathbb{R}^2$ consisting of open, polygonal and convex subsets $K \in \mathcal{T}$ called control volumes such that $\overline{\Lambda} = \bigcup_{K \in \mathcal{T}} \overline{K}$.



- $\forall K, L \in \mathcal{T}$, if $K \neq L$ then $K \cap L = \varnothing$ and $\overline{K} \cap \overline{L}$ is either empty, or a common vertex or an edge.
- $\sigma = K|L$ is the interface between two neighboring control volumes $K, L \in \mathcal{T}$, called *edge*.
- To each $K \in \mathcal{T}$ we associate a point $x_K \in \overline{K}$, called *center*.
- For two neighboring control volumes $K, L \in \mathcal{T}$, $\overrightarrow{x_K x_L} \perp K|L$.

$$\partial_t\Big(\chi_\epsilon - \int_0^{\cdot} g(\chi_\epsilon(s))dW(s)\Big) - \Delta\chi_\epsilon + \psi_\epsilon(\chi_\epsilon) = k(\chi_\epsilon) + h \quad \text{in } \Omega \times (0,T) \times \Lambda$$

On a semi-implicit TPFA scheme [Bauzet-Nabet-Vallet-Zimmermann]

$$\begin{cases} \chi^0_{\epsilon,K} = \frac{1}{|K|}\int_K \chi_0(x)\,dx, \quad \forall K \in \mathcal{T} \\ \frac{|K|}{\Delta t}(\chi^{n+1}_{\epsilon,K} - \chi^n_{\epsilon,K}) - \frac{|K|}{\Delta t} g(\chi^n_{\epsilon,K})(W^{n+1} - W^n) + \sum_{\sigma = K|L \in \mathcal{E}_{int} \cap \mathcal{E}_K} \frac{|\sigma|}{d_{K|L}}(\chi^{n+1}_{\epsilon,K} - \chi^{n+1}_{\epsilon,L}) \\ + |K|\psi_\epsilon(\chi^{n+1}_{\epsilon,K}) = |K| k(\chi^{n+1}_{\epsilon,K}) + |K| \int_{n\Delta t}^{(n+1)\Delta t} \int_K h(s,x)\,dx\,ds. \end{cases}$$

Well posedness of the scheme

For any $\epsilon > 0$, any $n \in \{0, \dots, N-1\}$ and any given $\mathcal{F}_{t_n}$-measurable random vector $(\chi^n_{\epsilon,K})_{K\in\mathcal{T}}$, there exists a unique $\mathcal{F}_{t_{n+1}}$-measurable random vector $(\chi^{n+1}_{\epsilon,K})_{K\in\mathcal{T}}$ satisfying the semi-implicit « ϵ/TPFA » scheme.

Aim of the study [Bauzet-Nabet-Vallet-Zimmermann]

Proving that the sequence $(\chi_{\epsilon,\mathcal{T},\Delta t})_{\epsilon,\mathcal{T},\Delta t}$ converges towards the unique solution of Problem $(\mathcal{P})$ as ϵ, size$(\mathcal{T})$ and Δt tend simultaneously to 0.

Prove the convergence of a semi-implicit « ϵ/TPFA » scheme towards the unique pair (χ, ψ) satisfying $0 \leqslant \chi \leqslant 1$, $\psi \in \partial I_{[0,1]}(\chi)$ and

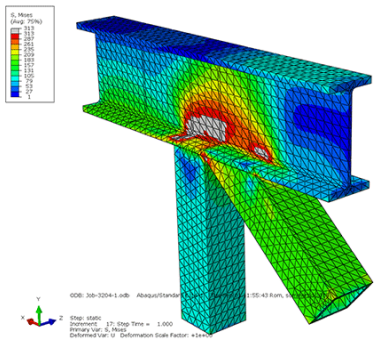
$$(\mathcal{P})\left\{\begin{array}{rcll} \partial_t\Big(\chi - \int_0^{\cdot} g(\chi(s))dW(s)\Big) - \Delta\chi + \psi & = & k(\chi) + h & \text{in } \Omega\times(0,T)\times\Lambda, \\ \chi(\omega, t=0, x) & = & \chi_0(x) & \omega\in\Omega, x\in\Lambda, \\ \nabla\chi\cdot n & = & 0 & \text{on } \Omega\times(0,T)\times\partial\Lambda. \end{array}\right.$$

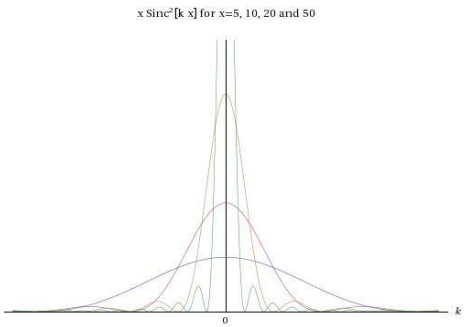
What remains to be done

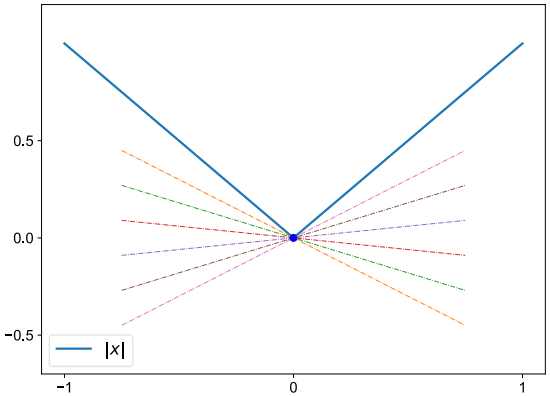
$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + B_x = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + B_y = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + B_z = 0$$







- Coupling with mechanics
- Finite elements
- Thickness tends to zero
- Two subdifferentials

Numerical procedure

$$dR + h(R)dW = f(t, R)dt$$

1. The Euler-Maruyama method [**KP92**]

$$R_{n+1} \simeq R_n + f(t_n, R_n)\delta t - h(R_n)dW_{n+1}$$

2. The Milstein method [**M75**]

$$R_{n+1} \simeq R_n + f(t_n, R_n)\delta t - h(R_n)dW_{n+1} + \frac{1}{2}h(R_n)h'(R_n)(dW_{n+1}^2 - \delta t)$$

Estimation of the Expectation

We estimate the Expectation using the Central Limit Theorem.

The Algorithm is the following

1. Let X be a random variable, simulate $(X_1, \ ..., X_N)$ a sample drawn along the law of X;
2. Compute estimators of the expectation and the variance

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N} X_i \qquad\qquad \sigma_N^2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \mu_N)^2$$

3. Then, μ_N is an estimation of the expectation with an interval of confidence $I_{\alpha,N}$ and a level of confidence α given by:

$$I_{\alpha,N} = \left[\mu_N - c_\alpha \frac{\sigma_n}{\sqrt{N}}, \mu_N + c_\alpha \frac{\sigma_n}{\sqrt{N}}\right],$$

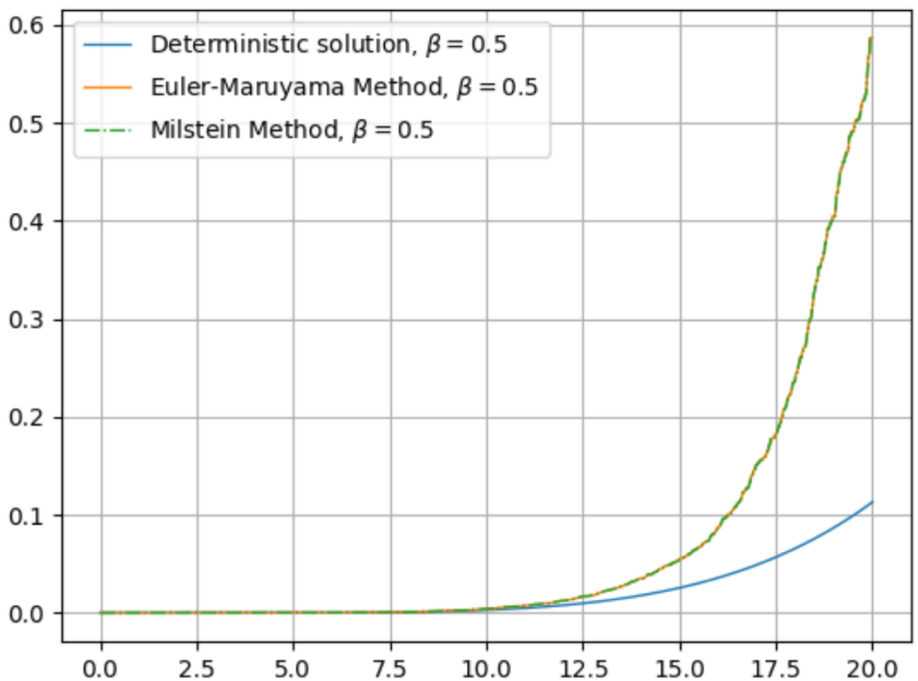
and for $\alpha = 95\%$, one has $c_\alpha \simeq 1.96$.

Example: non increasing

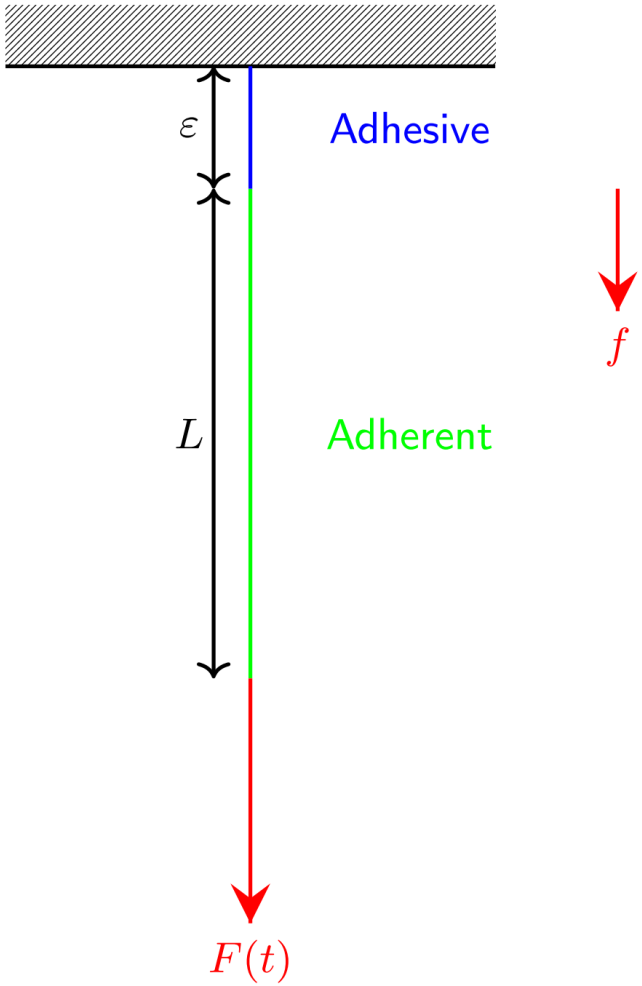
$$R_{n+1} \simeq R_n + \text{Max}\Big(0; f(t_n, R_n)\delta t - h(R_n)dW_{n+1}\Big)$$

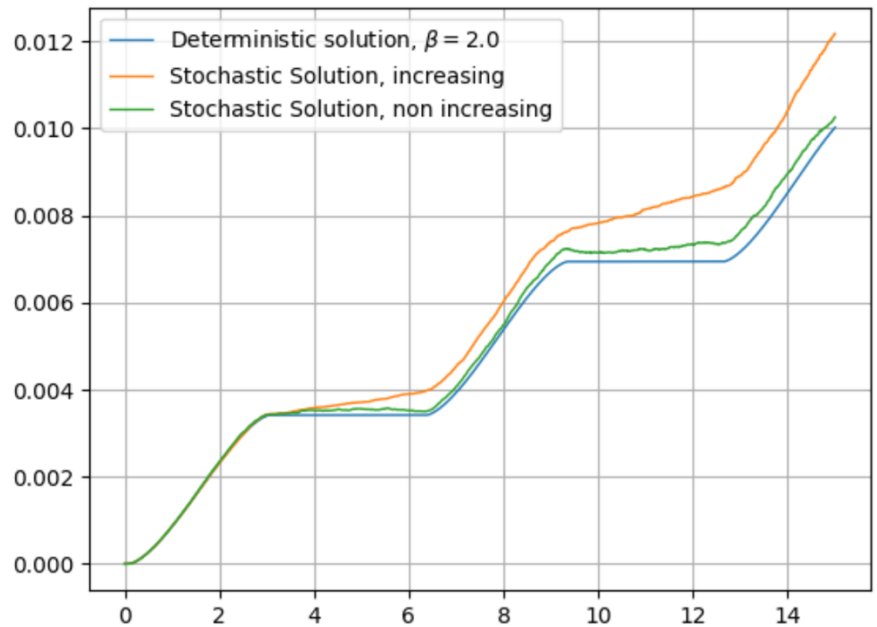
$$R_{n+1} \simeq R_n + \text{Max}\Big(0; f(t_n, R_n)\delta t - h(R_n)dW_{n+1} + \frac{1}{2}h(R_n)h'(R_n)(dW_{n+1}^2 - \delta t)\Big)$$

β = n



- $\bar{\omega} = 0.06$ Pa; $\eta(R) = \bar{\eta} = 3.6 \cdot 10^2$ Pa.
- $\varepsilon = 10^{-2}$cm., $L = 1$cm.;
- Young's modulus of the adherent: $E = 1$Pa;
- Young's modulus of the undamaged adhesive : $E_0 = 1$Pa;
- Normalized gravity: $f = 1$N, External force: $F(t) = 1 + 0.1\sin(t)$N.





$$\bar{\rho} = \rho + \int_0^T h_t(\rho) dB_t$$

$$\eta(\rho)\dot{\bar{\rho}}^r = -\left(a_{,\rho}(\rho) + \frac{1}{2}\mathbb{C}_{,\rho}(\rho)e(u)e(u) + b\Delta_p\rho \right) H(\dot{\rho}),$$

$$\rho(0) = \rho_0,$$

Two models

$$\eta(\rho)\dot{\bar{\rho}}^r = -\left\{ a_{,\rho}(\rho) + \frac{1}{2}\mathbb{C}_{,\rho}(\rho)e(u)e(u) + b\Delta_p\rho \right\}_- ,$$

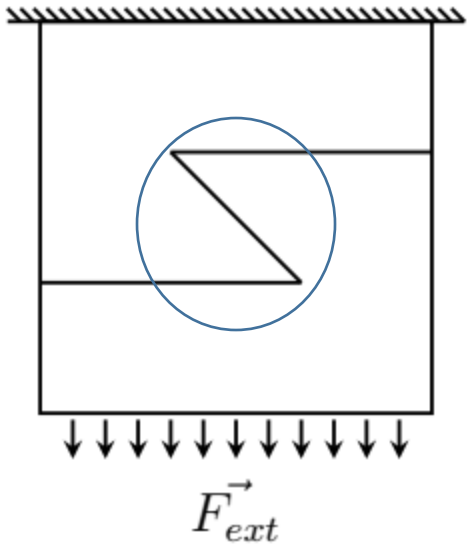
$$\rho(0) = \rho_0,$$

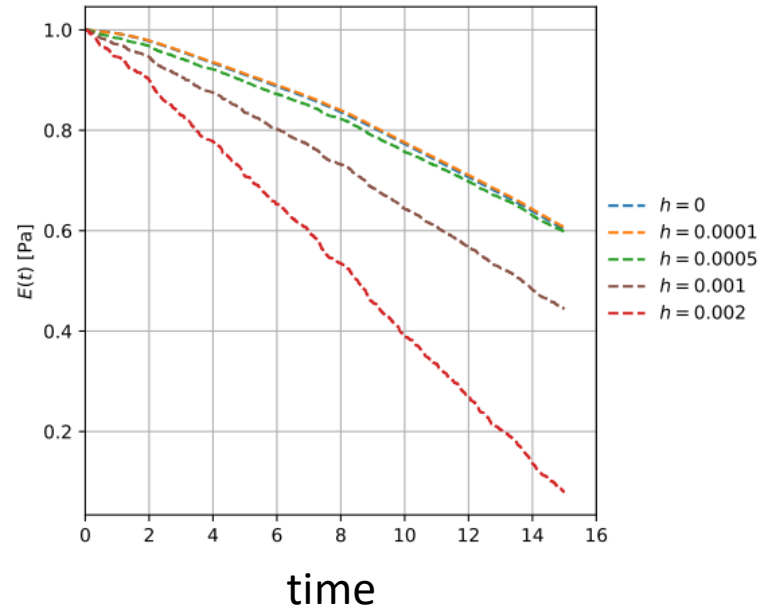
Thickness → 0

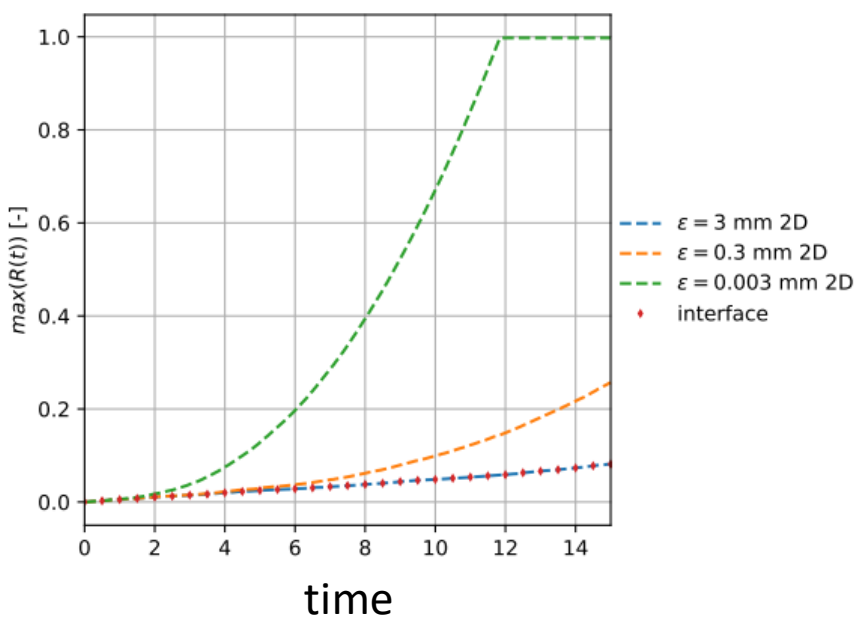
$$\tilde{\eta}(\rho)\dot{\bar{\rho}}^r = -\left(\tilde{a}_{,\rho}(\rho) + \frac{1}{2}\mathbf{C}^{\varepsilon}_{,\rho}(\rho) \begin{pmatrix} \langle\langle u^{\varepsilon}_{,1}\rangle\rangle \\ \langle\langle u^{\varepsilon}_{,2}\rangle\rangle \\ [[u^{\varepsilon}]] + \varepsilon\langle\langle\langle\langle u^{\varepsilon}_{,3}\rangle\rangle \end{pmatrix} . \begin{pmatrix} \langle\langle u^{\varepsilon}_{,1}\rangle\rangle \\ \langle\langle u^{\varepsilon}_{,2}\rangle\rangle \\ [[u^{\varepsilon}]] + \varepsilon\langle\langle\langle\langle u^{\varepsilon}_{,3}\rangle\rangle \end{pmatrix} + \tilde{b}\Delta_p^2\,\rho \right) H(\dot{\bar{\rho}})$$

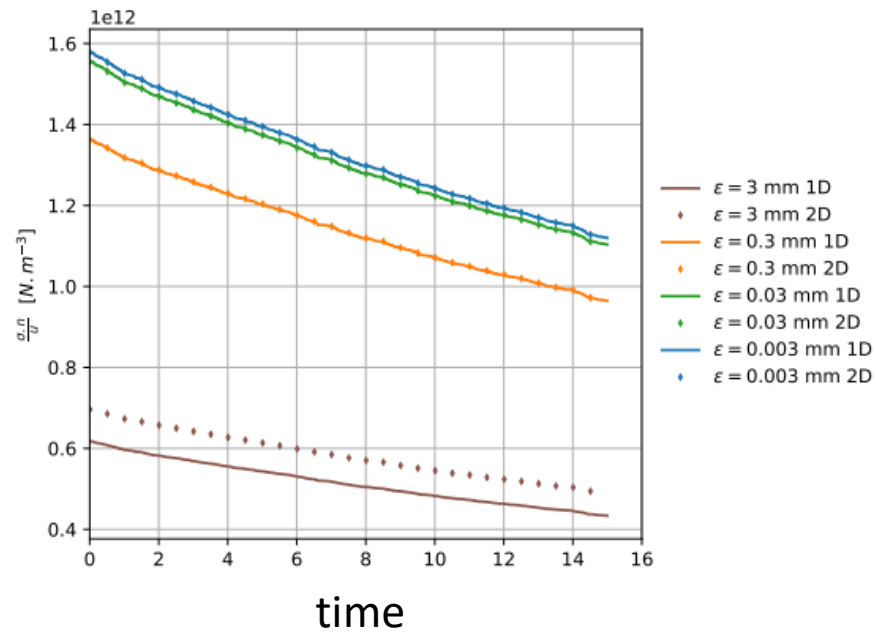
$$\tilde{\eta}(\rho)\dot{\bar{\rho}}^r = -\left(\tilde{a}_{,\rho}(\rho) + \frac{1}{2}\mathbf{C}^{\varepsilon}_{,\rho}(\rho) \begin{pmatrix} \langle\langle u^{\varepsilon}_{,1}\rangle\rangle \\ \langle\langle u^{\varepsilon}_{,2}\rangle\rangle \\ [[u^{\varepsilon}]] + \varepsilon\langle\langle\langle\langle u^{\varepsilon}_{,3}\rangle\rangle \end{pmatrix} . \begin{pmatrix} \langle\langle u^{\varepsilon}_{,1}\rangle\rangle \\ \langle\langle u^{\varepsilon}_{,2}\rangle\rangle \\ [[u^{\varepsilon}]] + \varepsilon\langle\langle\langle\langle u^{\varepsilon}_{,3}\rangle\rangle \end{pmatrix} + \tilde{b}\Delta_p^2\,\rho \right)_-$$

Fubini Stochastic Theorem


$\vec{F_{ext}}$


E(t) [Pa]
time
h = 0
h = 0.0001
h = 0.0005
h = 0.001
h = 0.002


max(R(t)) [-]
time
ε = 3 mm 2D
ε = 0.3 mm 2D
ε = 0.003 mm 2D
interface


1e12
$\frac{\alpha.n}{v}$ [N.m^{-3}]
time
ε = 3 mm 1D
ε = 3 mm 2D
ε = 0.3 mm 1D
ε = 0.3 mm 2D
ε = 0.03 mm 1D
ε = 0.03 mm 2D
ε = 0.003 mm 1D
ε = 0.003 mm 2D

Conclusion

- A lot of phenomenological/asymptotic models
- Damage + unilateral conditions
- Numerical implementation
- Elasto(plastic)-statics plane/curved soft/stiff interfaces, temperature + dynamics + finite strain + smooth roughness + visco elasticity + …

In progress

- A lot of conjectures
- Experiments: ageing
- Non local p-laplacian
- Stochastics

Merci pour votre attention !