Borel-Laplace resummation and a wavelet-based method for the incompressible Navier–Stokes equation

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Outline

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1. Introduction and motivation
2. Navier-Stokes resolution with time series decomposition
3. Divergence free wavelet and Helmholtz-Hodge decomposition

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2. Navier-Stokes resolution with time series decomposition
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4. Divergence-free wavelet-based time series resummation method
5. Preliminary numerical results.

Introduction and motivation

Turbulent flow simulation:

- Parameters depend on Reynolds number: $Re^{\frac{9}{4}}$ in 3d and $Re^{\frac{6}{4}}$ in 2d.
- Large CPU ressources and computation times.

Numerical methods:

- Spectral methods: fast, efficient, effective $\neq$ physical boundary conditions.
- Finite element method, finite volume method or finite difference method: complex geometries $\neq$ expensive in time and CPU memory.
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- $\longrightarrow$ **Competitive methods in Lagrangian formulation.**

Objective: High-order method in both time and space, with lowest numerical cost.

Incompressible Navier Stokes equations

$$
\begin{cases}
\partial_t \mathbf{v} - \nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = \mathbf{f}, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^d, \quad t > 0, \\
\nabla \cdot \mathbf{v} = 0, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^d, \\
+ \mathit{Boundary} \quad \mathit{conditions}.
\end{cases}
\tag{1}
$$

where $\mathbf{v}$ is the fluid velocity field, p the pressure and ν the kinematic viscosity.

Boundary conditions:

Periodic, Free-slip $\rightarrow \mathbf{v} \cdot \mathbf{n} = 0$, No-slip $\rightarrow \mathbf{v}|_{\partial\Omega} = 0$, Dirichlet $\rightarrow \mathbf{v}|_{\partial\Omega} = g$.

$\longrightarrow$ The unknowns are the velocity $\mathbf{v}$ and the pressure p.

Navier-Stokes resolution with time series decomposition

The unknowns $\mathbf{v}$, p and the force $\mathbf{f}$ are supposed to be formal series in time t:

$$\mathbf{v}(t,\mathbf{x}) = \sum_{n=0}^{+\infty} t^n \mathbf{v}^n(\mathbf{x}), \qquad p(t,\mathbf{x}) = \sum_{n=0}^{+\infty} t^n p^n(\mathbf{x}), \qquad \mathbf{f}(t,\mathbf{x}) = \sum_{n=0}^{+\infty} t^n \mathbf{f}^n(\mathbf{x}). \quad (2)$$

Inserting these expressions into equation (1), we obtain:

$$\begin{cases} (n+1)\mathbf{v}^{n+1} + \sum_{k=0}^{n} (\mathbf{v}^k \cdot \nabla)\mathbf{v}^{n-k} + \nabla p^n = \nu \Delta \mathbf{v}^n + \mathbf{f}^n, \\ \nabla \cdot \mathbf{v}^{n+1} = 0. \end{cases} \quad (3)$$

For each n, both p^n and $\mathbf{v}^{n+1}$ must be computed.

$\longrightarrow$ The series in equation (2) do not necessarily converge.

Navier-Stokes resolution with time series decomposition

Projection method allows to decouple the computation of $\mathbf{v}^n$ from p^n:

$$\begin{cases} (n+1)\tilde{\mathbf{v}}^{n+1} + \sum_{k=0}^{n} (\mathbf{v}^k \cdot \nabla)\mathbf{v}^{n-k} = \nu \Delta \mathbf{v}^n + \mathbf{f}^n, \\ (n+1)(\mathbf{v}^{n+1} - \tilde{\mathbf{v}}^{n+1}) + \nabla p^n = 0. \end{cases} \quad (4)$$

The intermediate velocity $\tilde{\mathbf{v}}^{n+1}$ is not necessarily divergence-free and in practice, the computation of $\mathbf{v}^{n+1}$ leads to Poisson equation:

$$\frac{1}{n+1}\Delta p^n = \nabla \cdot \tilde{\mathbf{v}}^{n+1} \neq 0, \quad \nabla p^n \cdot \mathbf{n} = 0 \ \text{ and } \quad \nabla p^n \cdot \mathbf{t} = 0 \ \text{ on } \ \partial\Omega. \quad (5)$$

where $\mathbf{n}$ and $\mathbf{t}$ are the unit normal and the unit tangent to $\partial\Omega$, respectively.

$\longrightarrow$ Poisson solver can't incorporate both boundary conditions of (5).

$\longrightarrow$ **Another objective:** change the projection step in (5), using divergence-free wavelet basis, to project $\tilde{\mathbf{v}}^{n+1}$ onto:

$$\mathcal{H}_{div}(\Omega) = \{\mathbf{v} \in (L^2(\Omega))^d : \ \nabla \cdot \mathbf{v} = 0, \ \mathbf{v} \cdot \mathbf{n} = 0\}, \quad \mathcal{H}_{div,0}(\Omega) \rightarrow \mathbf{v}|_{\partial\Omega} = 0.$$

Principle of the divergence-free wavelet construction

(i) Starting with 1D regular biorthogonal MRA of $L^2(0,1)$: $(V_j^1, \tilde{V}_j^1)$.

(ii) Construct a 1D biorthogonal MRA $(V_j^0, \tilde{V}_j^0)$ linked to $(V_j^1, \tilde{V}_j^1)$ by:

$$\frac{d}{dx} V_j^1 = V_j^0 \quad \text{and} \quad \frac{d}{dx} \tilde{V}_j^0 = \tilde{V}_j^1$$

(iii) Construct the divergence-free multiresolution analysis by:

$$\mathbf{V}_j^{div} = \mathbf{curl}[V_j^1 \otimes V_j^1]$$

$$\longrightarrow \mathbf{div}[\mathbf{curl}(\mathbf{u})] = 0 \quad \text{and} \quad \mathbf{curl}[\mathbf{grad}(\mathbf{u})] = 0$$

Biorthogonal wavelet basis (BMRA) of $L^2(\mathbb{R})$.

- $\cdots V_0 \subset \cdots \subset V_j \subset V_{j+1} \cdots \subset L^2(\mathbb{R}), \quad \cap V_j = \{0\}$
- $\forall\, j,\ V_j = \text{span} < \varphi_{j,k} = 2^{\frac{j}{2}}\varphi(2^j x - k),\ k \in \mathbb{Z} > \quad \varphi$ scaling function).

$\rightarrow$ There is another sequence $\tilde{V}_j$ with the same structure.

- $\cdots \tilde{V}_0 \subset \cdots \subset \tilde{V}_j \subset \tilde{V}_{j+1} \cdots \subset L^2(\mathbb{R}), \quad \cap \tilde{V}_j = \{0\}$
- $\forall\, j,\ \tilde{V}_j = \text{span} < \tilde{\varphi}_{j,k} = 2^{\frac{j}{2}}\tilde{\varphi}(2^j x - k),\ k \in \mathbb{Z} > \quad (\tilde{\varphi}$ dual scaling function).

Biorthogonality means:

- $\forall\, j,\ L^2(\mathbb{R}) = V_j \oplus \tilde{V}_j^{\perp} \qquad \Leftrightarrow \qquad \langle \varphi_{j,k}, \tilde{\varphi}_{j,k'} \rangle = \delta_{k,k'}$

Biorthogonal wavelet basis

Biorthogonal wavelet basis:

- Basis for the detail spaces: $W_j = V_{j+1} \cap \tilde{V}_j^{\perp}$ and $\tilde{W}_j = \tilde{V}_{j+1} \cap V_j^{\perp}$.
- $W_j = \text{span} < \psi_{j,k} = 2^{\frac{j}{2}} \psi(2^j x - k),\ k \in \mathbb{Z} >$ $\qquad$ (ψ wavelet generator)

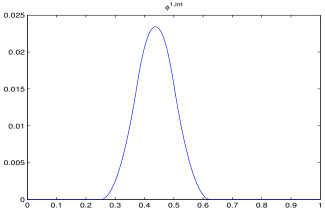
$\longrightarrow$ The space $\tilde{W}_j$ has the same structure (with $\tilde{\psi}$ dual wavelet generator)

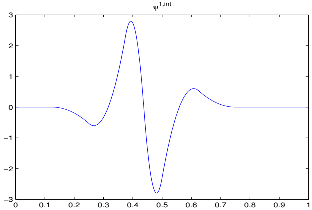
Finite dimensional spaces on $[0,1]$:

$$\#V_j = \#\tilde{V}_j = I_j < +\infty \quad \text{and} \quad \#W_j = \#\tilde{W}_j = 2^j$$

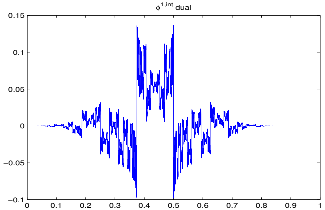
Biorthogonal B-Spline wavelets (3 vanishing moments)

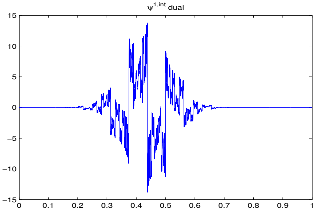
Primal scaling function (left) and associated wavelet (right):





Dual scaling function (left) and associated wavelet (right):





Wavelet basis satisfying boundary conditions

- Start with $(V_j^1, \tilde{V}_j^1)$ regular MRA of $L^2(0,1)$ associated to $(\varphi^1, \tilde{\varphi}^1)$.
- Polynomial reproduction at boundaries 0 and 1:

$$0 \leq \ell \leq r-1, \quad \frac{2^{j/2}(2^j x)^\ell}{\ell!} = \Phi_{j,\ell}^{1,\flat}(x) + \sum_{k=k_0}^{2^j-k_1} p_\ell^1(k)\, \varphi_{j,k}^1(x) + \Phi_{j,\ell}^{1,\sharp}(1-x)$$

with $\varphi_{j,k}$ internal scaling functions, $\Phi_{j,\ell}^{1,\flat}$ and $\Phi_{j,\ell}^{1,\sharp}$ the edge scaling functions.

- Similar structure for $\tilde{V}_j^1$ with $\tilde{\varphi}_{j,k}$, $\tilde{\Phi}_{j,\ell}^{1,\flat}$ and $\tilde{\Phi}_{j,\ell}^{1,\sharp}$.

Wavelet basis satisfying boundary conditions

- Homogeneous boundary conditions:

- $f^{(\lambda)}(\alpha) = 0$ for $0 \leq \lambda \leq r-1$ and $\alpha = 0$ ou $\alpha = 1$

- It suffices to remove from V_j^1, the scaling functions:

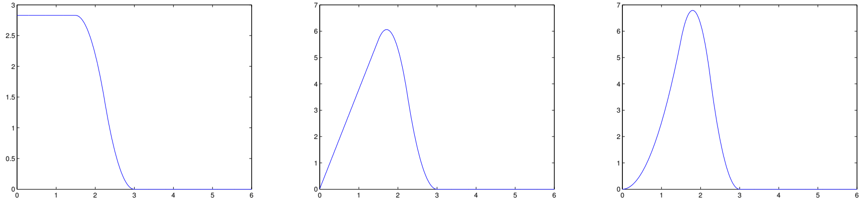
- $\{\Phi_{j,\ell}^{1,\flat}\}_{\ell=\lambda+1}$ if $\alpha = 0$ or $\{\Phi_{j,\ell}^{1,\sharp}\}_{\ell=\lambda+1}$ if $\alpha = 1$

- The dual space dimension must be adjusted $\tilde{V}_j^1$:

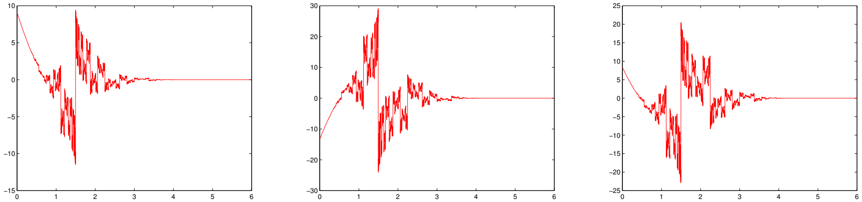
- One can proceed similarly for $\tilde{V}_j^1$.

Wavelet basis satisfying boundary conditions

Edge 0 **scaling function of** V_j^1**: B-Spline** 3.3

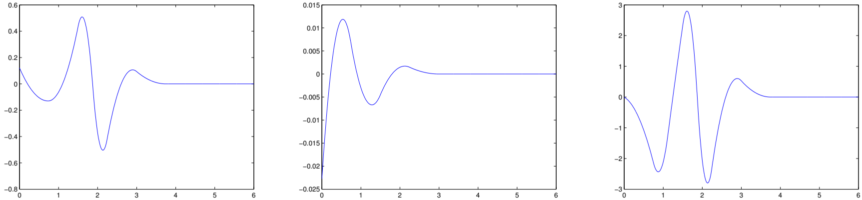


Edge 0 **scaling function of** $\tilde{V}_j^1$**: B-Spline** 3.3

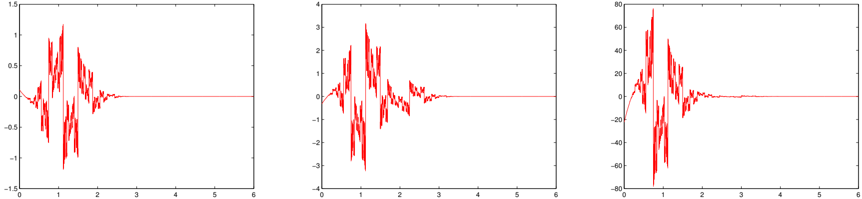


Wavelet basis satisfying boundary conditions

Edge 0 **wavelets of** W_j^1**: B-Spline** 3.3

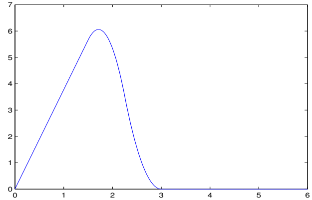


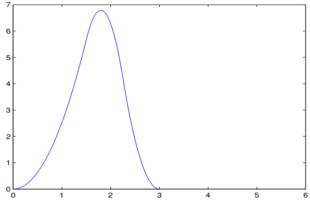
Edge 0 **wavelets of** $\tilde{W}_j^1$**: B-Spline** 3.3



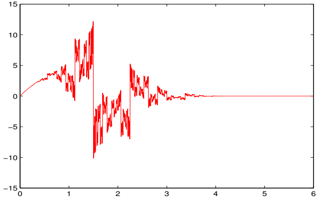
Wavelet basis satisfying boundary conditions

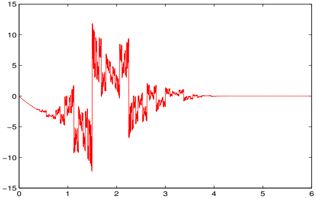
Edge 0 **scaling functions with Dirichlet: B-Spline** 3.3





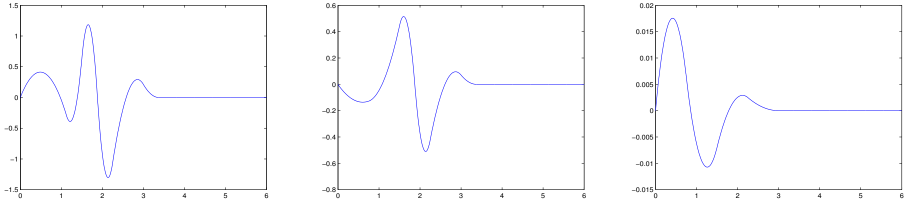
Edge 0 **dual scaling functions with Dirichlet: B-Spline** 3.3



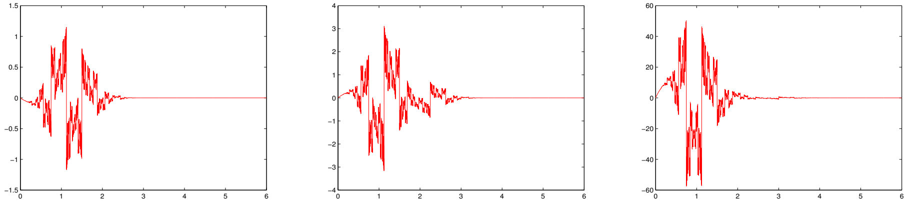


Wavelet basis satisfying boundary conditions

Edge 0 **wavelets with Dirichlet: B-Spline** 3.3



Edge 0 **dual wavelets with Dirichlet: B-Spline** 3.3



Wavelet basis for $\mathcal{H}_{div}(\Omega)$

- **Divergence-free Multiresolution Analysis** $\mathbb{V}_j^{div}$:

$$\mathbb{V}_j^{div} = \text{span} < \Phi_{j,\mathbf{k}}^{div};\ \mathbf{k} \in \mathbb{Z}^2 >, \quad j \in \mathbb{Z}$$

- **Divergence-free scaling functions:**

$$\Phi_{j,\mathbf{k}}^{div} = \mathbf{curl}[\varphi_{j,k_1}^1 \otimes \varphi_{j,k_2}^1] = \left| \begin{array}{l} \varphi_{j,k_1}^1 \otimes (\varphi_{j,k_2}^1)' \\ -(\varphi_{j,k_1}^1)' \otimes \varphi_{j,k_2}^1 \end{array} \right.$$

(As: $\frac{d}{dx} V_j^1 = V_j^0$)

$$\longrightarrow \ \mathbb{V}_j^{div} = (V_j^1 \otimes V_j^0) \times (V_j^0 \otimes V_j^1) \cap \mathcal{H}_{div}(\Omega) \ \longrightarrow \ \textbf{FWT!}$$

- **Anisotropic divergence-free wavelets:**

$$\Psi_{\mathbf{j},\mathbf{k}}^{div} = \mathbf{curl}[\psi_{j_1,k_1}^1 \otimes \psi_{j_2,k_2}^1] = \left| \begin{array}{l} 2^{j_2} \psi_{j_1,k_1}^1 \otimes \psi_{j_2,k_2}^0 \\ -2^{j_1} \psi_{j_1,k_1}^0 \otimes \psi_{j_2,k_2}^1 \end{array} \right. \qquad (\psi_{j,k}^0 = 2^{-j} (\psi_{j,k}^1)')$$

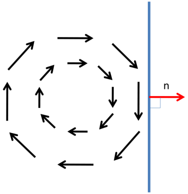
$\mathbf{j} = (j_1, j_2),\ \mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$.

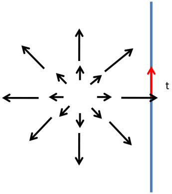
Wavelet basis for $\mathcal{H}_{div}(\Omega)$

Boundary conditions for $\mathcal{H}_{div}(\Omega)$

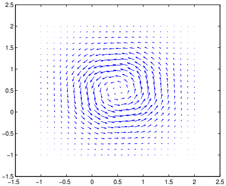
- Free-slip: $V_j^d = V_j^1 \cap H_0^1(0,1) \longrightarrow \Phi_{j,\mathbf{k}}^{div} \cdot \vec{\mathbf{n}} = 0$

- No-slip: $V_j^d = V_j^1 \cap H_0^2(0,1) \longrightarrow \Phi_{j,\mathbf{k}}^{div}|_\Gamma = 0$

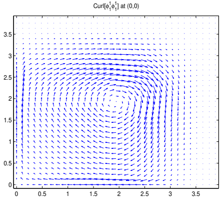




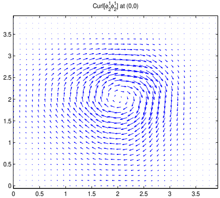
Example of free-slip generators



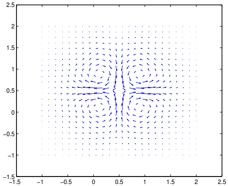
(a) Div-free scaling int



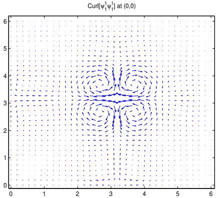
(b) Div-free scaling edge



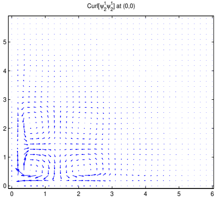
(c) Div-free scaling edge



(d) Div-free wavelet int



(e) Div-free wavelet edge



(f) Div-free wavelet edge

$\Phi^{div}_{j,\mathbf{k}} \cdot \vec{\mathbf{n}} = 0$ and $\Psi^{div}_{j,\mathbf{k}} \cdot \vec{\mathbf{n}} = 0$

Helmholtz-Hodge decomposition

Practical computation:

$$\mathbf{u} = \mathbf{u}_{div} + \mathbf{u}_{curl} + \mathbf{u}_{har} = \mathbf{u}_{div} + \nabla q.$$

Then:

$$\langle \mathbf{u}, \Psi^{div}_{\mathbf{j},\mathbf{k}} \rangle = \langle \mathbf{u}_{div}, \Psi^{div}_{\mathbf{j},\mathbf{k}} \rangle$$

Searching $\mathbf{u}_{div}$ in the divergence-free form:

$$\mathbf{u}_{div} = \sum_{\mathbf{j},\mathbf{k}} d^{div}_{\mathbf{j},\mathbf{k}} \ \Psi^{div}_{\mathbf{j},\mathbf{k}}$$

leads to:

$$(d^{div}_{\mathbf{j},\mathbf{k}}) = \mathbb{M}^{-1}_{div}(\langle \mathbf{u}, \Psi^{div}_{\mathbf{j},\mathbf{k}} \rangle)$$

$\mathbb{M}_{div}$: Gram matrix of the basis $\{\Psi^{div}_{\mathbf{j},\mathbf{k}}\}$.

$\longrightarrow$ In the periodic case, one can use the Fourier basis!

Divergence-free wavelet time series resummation method

Helmholtz-Hodge decomposition

$$\tilde{\mathbf{v}}^{n+1} = \mathbf{v}^{n+1} + \nabla \xi^{n+1}, \quad \text{with} \quad \tilde{\mathbf{v}}^{n+1} \in (H_0^1(\Omega))^d.$$

- **Prediction step:**

$$\tilde{\mathbf{v}}^{n+1} + \sum_{k=0}^{n} (\mathbf{v}^k \cdot \nabla)\mathbf{v}^{n-k} = \nu \Delta \mathbf{v}^n + \mathbf{f}^n, \qquad \tilde{\mathbf{v}}^{n+1} = 0, \quad \text{on} \quad \partial\Omega. \qquad (6)$$

- **Correction step:**

$$\mathbf{v}^{n+1} = \mathbb{P}^{div,0}(\tilde{\mathbf{v}}^{n+1}). \qquad (7)$$

⟶ Consistent splitting and resummation is used to achieve convergence of:

$$\mathbf{v}(t, \mathbf{x}) = \sum_{n=0}^{+\infty} t^n \mathbf{v}^n(\mathbf{x}).$$

Borel-Laplace resummation

Let us assume that $\mathbf{v}^n$ is Gevrey, ie, there is $C, A > 0$ such that:

$$\forall n \geq 0, \quad \|\mathbf{v}^n\| \leq CA^n n!. \tag{8}$$

The resummation consists first to calculate this Borel transform:

$$\mathcal{B}\mathbf{v}(\xi) = \sum_{l=0}^{\infty} B^l \xi^l \qquad \text{with} \qquad B^l = \frac{\mathbf{v}^{l+1}}{l!}, \qquad l \geq 0. \tag{9}$$

This serie converges at the origin and it is then analytically extended to a function $P(\xi)$. Finally, one applies the Laplace transform to $P(\xi)$ as $1/t$, which is the formal inverse of the Borel transform. This gives the following representation of $\mathbf{v}(t)$:

$$\mathcal{S}\mathbf{v}(t) = \mathbf{v}^0 + \int_0^{+\infty} P(\xi) e^{-\xi/t} d\xi.$$

Borel-Laplace resummation

$v(t) = \sum_{k=0}^{+\infty} \mathbf{v}^k t^k$	$\sim$	$\mathcal{S}\mathbf{v}(t) = \mathbf{v}^0 + \int_0^{+\infty} P(\xi) e^{-\xi/t} d\xi$
Borel $\downarrow$		$\uparrow$ Laplace
$\mathcal{B}\mathbf{v}(\xi) = \sum_{k=0}^{+\infty} B^k \xi^k$	$\xrightarrow{\text{Prolongation}}$	$P(\xi)$

Table: Borel-Laplace summation

In practice, the coefficients $\mathbf{v}^n$ are only known up to a finite order N, then we have an approximation to order N of $\mathcal{S}\mathbf{v}(t)$ valid for $t \in [0, \delta t]$, for a certain $\delta t > 0$.

Vortex Merging-2D

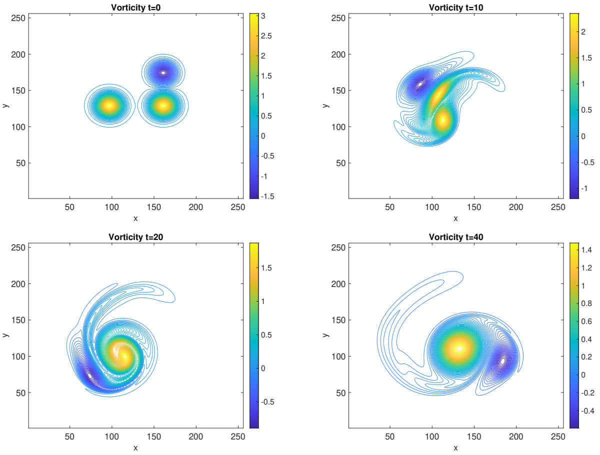


Figure: Vorticity contour, for $\nu = 1.8182 \times 10^{-5}$ and $j = 8$.

Vortex tubes collision-3D

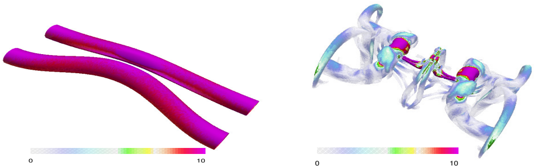


Figure: Vorticity snapshot: Q-criterion for $j = 7$.

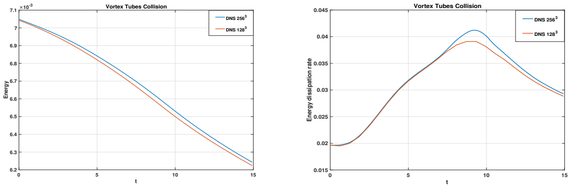


Figure: Evolution of the energy and its dissipation rate, $\nu = 4.5473 \times 10^{-5}$.

Thank you!