Déferlement d'un Paquet d'Ondes de Vorticité dans un Vortex en Rotation Rapide

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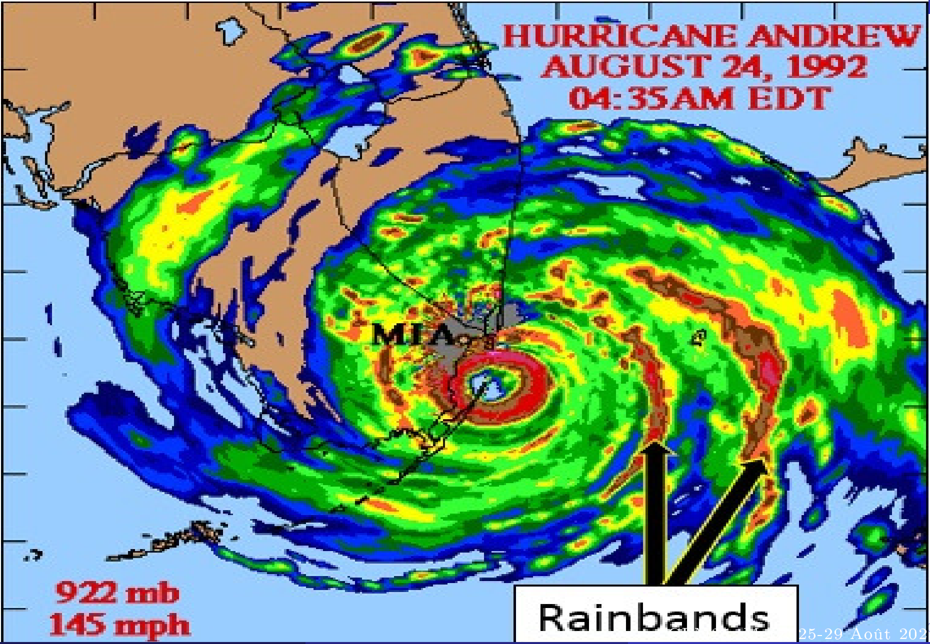
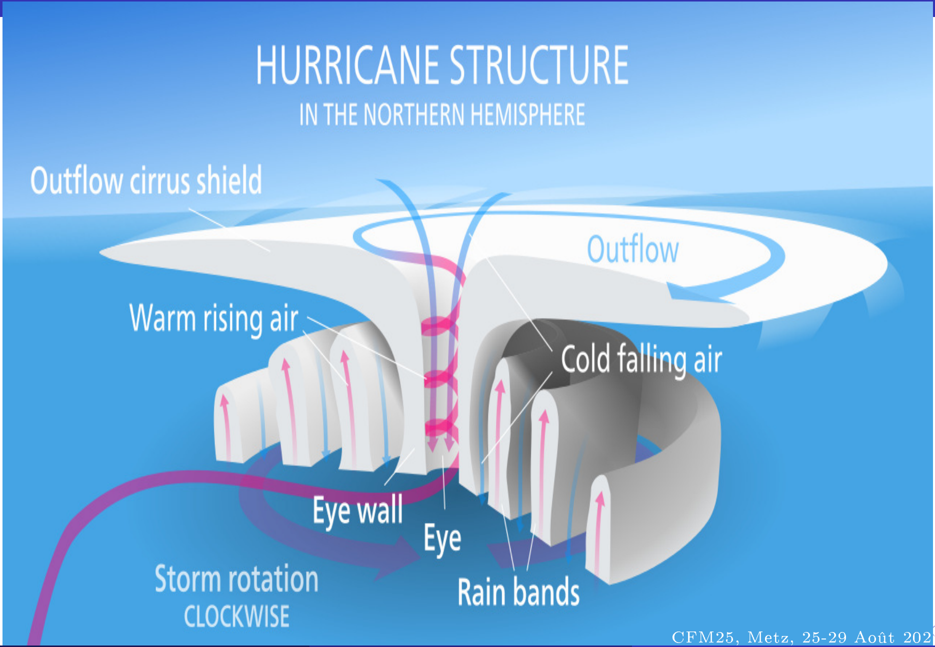
Motivations



Motivations



Motivations

• Vorticity and inertio-gravity waves propagate in geophysical vortices: tornadoes, mesocyclones, tropical cyclones. Vorticity waves have a stronger activity (Chen and Yau, 2001).

• Vortices only interact with vorticity waves, the relevant modes are continuous (Moller and Montgomery, 1999), (Chen, Brunet and Yau, 2003).

• The interaction of vortices with continuous vortical modes creates Nonlinear Critical Layers. The absorption of such waves at the critical radius leads to a wind acceleration around the emission radius and a deceleration outside the critical radius (Montgomery and Kallenbach, 1997).

• The inner spiral rainbands are outward propagating vorticity waves sheared by the differential rotation. The outer spiral bands are quasi-stationnary convective bands and are observed near the critical radius (Wang, 2002).

Modeling

stable, columnar and axisymmetric dry vortex

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{R_o} \mathbf{z} \times \mathbf{u} = \mathbf{f}_i - \nabla \Gamma + \frac{1}{R_e} \Delta \mathbf{u} + \mathbf{F}$$

$$\frac{\partial r u_r}{\partial r} + \partial_\theta u_\theta + r \partial_z u_z = 0$$

$\mathbf{u}$ is the 3D velocity (u_r, u_θ, u_z), Γ the modified pressure,

$\mathbf{f}_i$ inertial forces and $\mathbf{F}$ a body force,

$R_e \gg 1, \quad R_o > 1$

Assumption: the existence of an asymptotic quasi-steady regime after the initial strong Wave/Vortex Interaction.

Formulation

We consider the weak interaction of
a basic rotational velocity field $\overline{V}_0(\eta)$ with
a $O(\epsilon)$ free amplitude-modulated continuous helical mode

$$U_r^{(0)} = U_0(\eta, \Phi, T) \sin \xi, \qquad \xi = kz + m\theta - \omega t,$$

quasi-Lagrangian strained radial coordinate η
slow modulation space and time scales

$$\Phi \equiv \epsilon^{1/2} z, \quad T \equiv \epsilon\, t, \quad 1/Re = \lambda\, \epsilon^{3/2}$$

The Nonlinear CL imposes its scaling.

Outer Flow Formulation

$$r = \eta + h(\eta, \xi, \Theta, \Phi, T), \quad h = \epsilon\, h_2 + \epsilon^{\frac{3}{2}}\, h_3 + \ldots$$

$$\eta = \eta_c = r_c, \quad \text{spiralling critical radius}$$

$$\eta = \eta_a = \eta_c + \delta_r \epsilon^{\frac{1}{2}}, \quad \text{spiralling CL symmetry axis}$$

$$\Theta = m\,\theta, \quad \text{secular azimuthal variable}$$

decomposition phase-averaged flow + disturbance

$$u_r = \overline{U}(\eta, \Theta, \Phi) + \epsilon\, U_r^{(0)}(\eta, \xi, \Phi) + \epsilon^{\frac{3}{2}}\, U_r^{(1)}(\eta, \xi, \Theta, \Phi) + \ldots ,$$

$$u_\theta = \overline{V}(\eta, \Theta, \Phi) + \epsilon\, U_\theta^{(0)}(\eta, \xi, \Phi) + \epsilon^{\frac{3}{2}}\, U_\theta^{(1)}(\eta, \xi, \Theta, \Phi) + \ldots ,$$

$$u_z = \overline{W}(\eta, \Theta, \Phi) + \epsilon\, U_z^{(0)}(\eta, \xi, \Phi) + \ldots ,$$

Mean Flow Expansion

$$\overline{U} = \epsilon\,\overline{U}_2(\eta,\Theta,\Phi) + \epsilon^{\frac{3}{2}}\,\overline{U}_3(\eta,\Theta,\Phi) + \ldots,$$

$$\overline{V} = \overline{V}_0(\eta) + \epsilon^{\frac{1}{2}}\,\overline{V}_1(\eta,\Theta,\Phi) + \epsilon\,\overline{V}_2(\eta,\Theta,\Phi) + \ldots,$$

$$\overline{W} = \epsilon^{\frac{1}{2}}\,\overline{W}_1(\eta,\Theta,\Phi) + \epsilon\,\overline{W}_2(\eta,\Theta,\Phi) + \ldots,$$

The Quasi-Steady Vortical Mode

$$D\left[\frac{S}{\eta}D[\eta U_0]\right] + \left[k^2 S\frac{\varrho_0}{\omega^D}Q_{z,0} + \frac{m}{\eta^2}(\eta D - 2)[SQ_{z,0}] - 1\right]\frac{U_0}{\omega^D} = 0$$

Howard-Gupta modal equation on the f-plane

$$\text{with} \quad \omega^D(\eta) = \omega - m\Omega_0(\eta)$$

$$S(\eta) = \frac{\eta^2}{k^2\eta^2 + m^2}\,,\ \Omega_0(\eta) = \frac{V_0(\eta)}{\eta}\,,\ \varrho_0(\eta) = 2\Omega_0(\eta) + \frac{1}{R_{o,0}},$$

$$Q_{z,0}(\eta) = \frac{1}{\eta}D[\eta\overline{V}_0(\eta)] + \frac{1}{R_{o,0}}\,, \qquad D = \frac{d}{d\eta}$$

$$\text{there exists } \eta_c,\ \omega^D(\eta_c) = 0,\ \omega \text{ real}$$

The QS Vortical Mode Asymptotic Solution

Near $\eta = \eta_c$, in the quasi-steady stage

$$U_0^{\pm}(\eta, \Phi, T) = [a^{\pm}\, \phi_a(\eta) + \phi_b(\eta)]\, \hat{U}_0(\Phi, T),$$

$$\phi_a = \eta - \eta_c + \sum_{n=2}^{\infty} a_{0,n}(\eta - \eta_c)^n\,, \quad + = \text{outer side}, \quad - = \text{inner side},$$

$$\phi_b = 1 + \sum_{n=2}^{\infty} b_{0,n}(\eta - \eta_c)^n + b_0 \phi_a(\eta) \ln \frac{\eta - \eta_c}{\eta_0},$$

$$Q_z(\eta_a) = O(\epsilon^{\frac{1}{2}}), \quad Q_\theta(\eta_a) = -D[\overline{W}](\eta_a) = O(\epsilon^{\frac{1}{2}}),$$

$$\zeta = \eta_c \frac{D[Q_{z,0}]}{\varrho_0}(\eta_a) = O(1), \quad \varpi = k\eta_c/m = O(1), \text{ phase tilt}$$

The QS Vortical Mode: Singular Degree/Interaction Intensity

equivalent Richardson number $J_0(\eta_a) = \dfrac{k^2 \eta^2 \varrho_{0,a} Q_{z,0,a}}{m^2 (Q_{z,0,a} - \varrho_{0,a})^2}$

basic vorticity erosion in the CL $Q_{z,0}(\eta_a) \to 0$ as t $\to \infty$

$$J = J_0(\eta) + \epsilon^{\frac{1}{2}} J_1(\eta) + \epsilon \, J_2(\eta) + \dots$$

$J_0(\eta_a) \to 0$ in the unsteady interaction

$$J_1(\eta_a) = \varpi^2 Q_{z,1}(\eta_a, \Phi, T) + \varpi Q_{\theta,1}(\eta_a, \Phi, T)$$

asymmetry $\delta_r \to J_1(\eta_a) \neq 0$ weak quasi-steady interaction

$\delta_r = 0 \to J_1(\eta_a) = 0$ very weak interaction

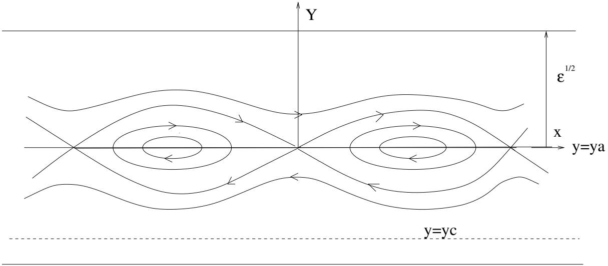
Inner Flow

The inner radial variable Z characterizing a streamline is

$$r - \eta_a = \epsilon^{\frac{1}{2}}\rho + \epsilon H_2(\Theta, \Phi) + \epsilon^{\frac{3}{2}} H_3(\rho, \Theta, \Phi) + \dots,$$

$$Z = \frac{1}{2}\rho^2 + A\cos X, \quad A(\Phi, T) = \left|\frac{\hat{U}_0(\Phi, T)}{m\varrho_0(\eta_a)\eta_c}\right|, \; X = \xi + \frac{\pi}{2}[1 + \mathrm{sign}(m\hat{U}_0)]$$

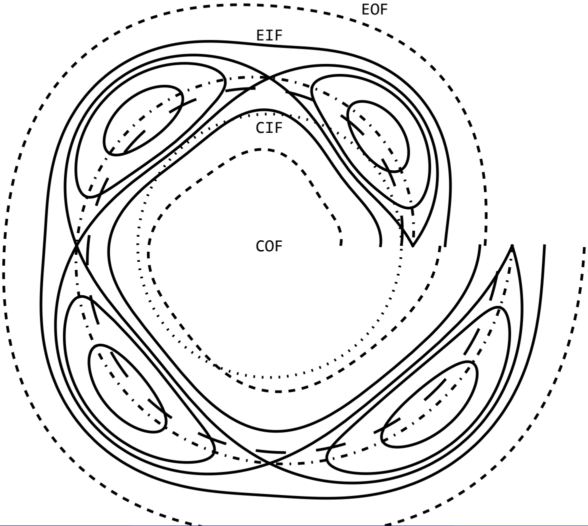
elongated and spiralling Kelvin cat's eye



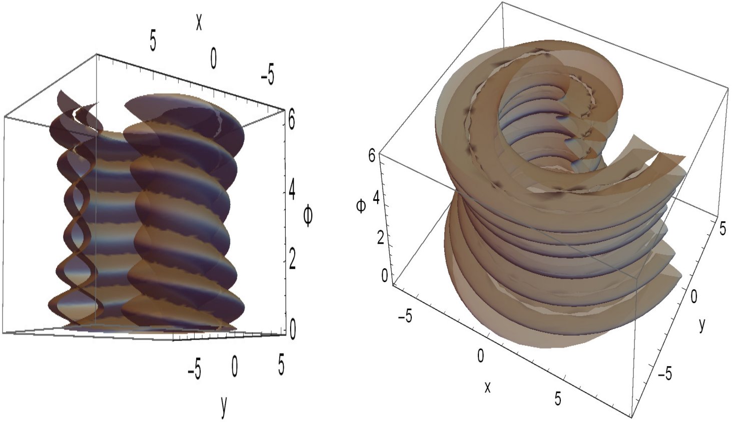
$\delta_r = 0$: leading-order axial vorticity inside the cat's eyes $= f(\Phi, T)$ extended Prandl-Batchelor theorem (Caillol, 2022),

$\delta_r \neq 0$: axial vorticity function of the wave phase ξ.

A few streamline envelopes at a height Φ, spiralling around the CL axis (dotdashed line), critical radius (long-dashed line), $r = \eta_c = r_c$ (dotted line), $m = 4$. The CL is bounded between the 2 dashed lines.

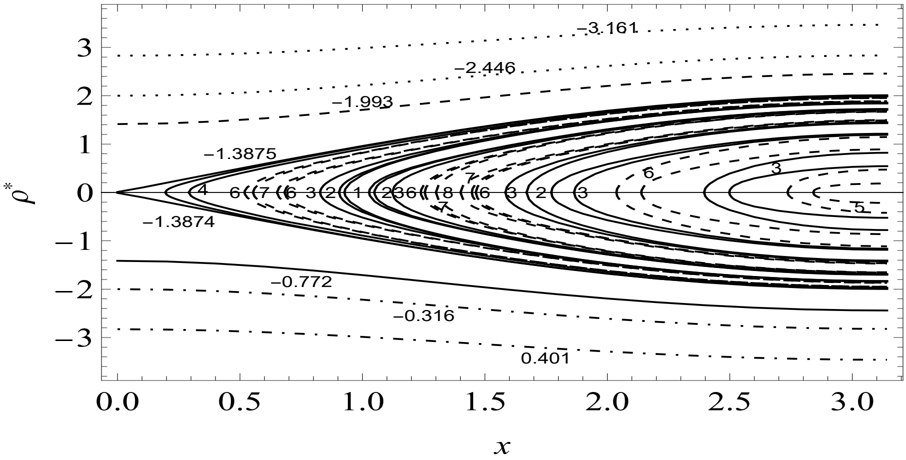


3D Helical and Spiralling CL



Separatrices $Z = A$ as functions of x, y and Φ, $m = 2$, $\delta_r = 0$, $\epsilon = 0.1$, $\varpi = 2$, $\eta_c = 2$, $\overline{U}_{2,a}/(m\eta_c\varrho_{0,a}) = 1$, $\overline{V}_{0,a}/(\eta_c\varrho_{0,a}) = 0.5$, $A(\Phi) = [1 + \cos(\Phi)/2]^2$.

Axial Iso-Vorticity Lines within the CL



$J_1^* = J_{1,a}/(|\zeta|\varpi^2 A^{1/2}) = 1.2\ 10^{-2}$ cat's eye minimum vorticity $q_z = -2.03$ between contours 8, maximum vorticity $q_z = -0.777$ between contours 1. Vorticity rescaled by $\varrho_{0,a}$.

Retrograde Waves, Outer Flow Distortions

A vorticity wave propagating with a phase angular speed less the vortex rotation at $\eta = \eta_a$ is characterized by a bounded mean axial velocity

$$\varpi \overline{W}_1(\eta_a) < \varrho_0(\eta_a)\delta_r,$$

a retrograde wave (outward energy propagation) is favored by $\delta_r > 0$.

$$[\overline{V}_1(\eta_a)]_-^+ = [\overline{W}_1(\eta_a)]_-^+ = [\overline{U}_2(\eta_a)]_-^+ = 0, \qquad [q]_-^+ = q^+ - q^-$$

$$[Q_{\theta,1}(\eta_a)]_-^+ = -\varpi[Q_{z,1}(\eta_a)]_-^+ \quad \rightarrow \quad \text{helical flow},$$

Outer Flow Distortions

$$[a]_-^+ = \zeta(1+\varpi^2)\,\mathcal{C}_a(J_1^*), \quad \zeta = O(1) \text{ mean vorticity radial gradient}$$

$$[Q_{z,1}(\eta_a)]_-^+ = -\zeta\Big(2\mathcal{C} + \mathcal{C}_q(J_1^*)\Big)A^{\frac{1}{2}}(\Phi, T)\varrho_0(\eta_a), \qquad \mathcal{C} \simeq 1.3788$$

$$[\overline{V}_2(\eta_a)]_-^+ = \zeta\,\mathcal{C}_v(J_1^*)\,A(\Phi, T)r_c\varrho_0(\eta_a),$$

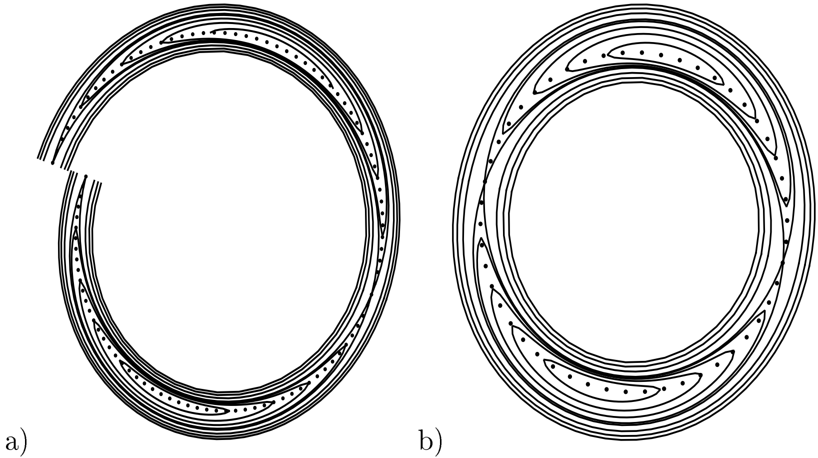
$\mathcal{C}_a$, $\mathcal{C}_q$ and $\mathcal{C}_v$ linear functions of the leading-order wave-induced inner spiralling axial vorticity (non zero as $\delta_r \neq 0$) integrated inside the whole CL and nonlinear functions of $J_1^* = J_{1,a}/(|\zeta|\varpi^2 A^{1/2})$, implicit functions of $\delta_r/A^{\frac{1}{2}}(\Phi, T)$

CL-induced mean flow/wave packet coupling (Caillol, 2025)

$$[\overline{W}_2(\eta_a)]_-^+ = \zeta\, \mathcal{C}_w(J_1^*)\, A(\Phi, T) r_c \varrho_0(\eta_a),$$

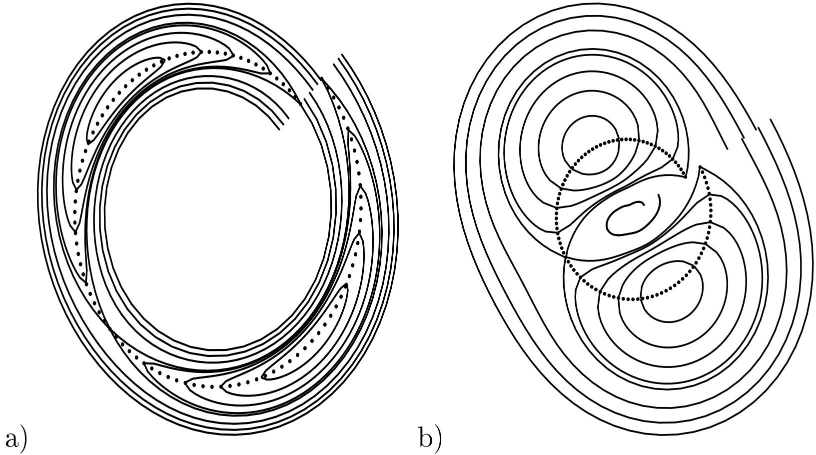
$\mathcal{C}_w$ linear function of the leading-order inner spiralling axial vorticity and axial velocity, nonlinear function of J_1^* and implicit function of $\delta_r / A^{\frac{1}{2}}(\Phi, T)$

Horizontal Sections of a Tripole at the QS stage beginning



$m = 2$, $\epsilon = 2.5\ 10^{-4}$, $t = 0$, $\delta_r = 0$: a) $A = 0.757$, $\overline{U}_{2,a}/(m\eta_c\varrho_{0,a}) = 42.8$, b) $A = 1.13$, $\overline{U}_{2,a}/(m\eta_c\varrho_{0,a}) = 0$. The CL axis $\eta = \eta_c$ is in dotted line (Caillol, 2022).

Horizontal Sections of a Tripole at the Breaking Onset



$m = 2$, $\epsilon = 2.5\ 10^{-4}$, $t \simeq 1.2\ h$, $\delta_r = 0$: a) $A = 1.86$, $\overline{U}_{2,a}/(m\eta_c \varrho_{0,a}) = 49.5$, b) $A = 4.72$, $\overline{U}_{2,a}/(m\eta_c \varrho_{0,a}) = 26.8$. The CL axis $\eta = \eta_c$ is in dotted line (Caillol, 2022).

Conclusion

• Analytical description of the quasi-steady interaction of an amplitude-modulated continuous shear mode with a rotating flow through a nonlinear asymmetric critical layer.

• The mean radial velocity breaks CL symmetries in the radial direction. Critical-layer patterns are elongated spiralling Kelvin cat's eyes.

• The asymmetry δ_r enhances the CL flow distortions, creates a phase jump $[a]_-^+$, a second-order velocity jump $([\overline{V}_{2,a}]_-^+, [\overline{W}_{2,a}]_-^+)$.

• The asymmetry δ_r enhances the modal singularity, $J_a = O(\epsilon^{\frac{1}{2}})$ and subsequently the wave/vortex interaction.

• The Prandtl-Batchelor theorem is no longer valid: the leading-order axial vorticity within the cat´s eye is not uniform (Caillol, 2025).

• The vortex erosion from the spiralling helical CL into the neighboring spiralling diffusion boundary layers partly explains the observed inner spiral bands.

• Going work: study of the coupled system: the low-order mean flow equations + A evolution equation, effect of the asymmetry on the breaking. Relationship between $J_{1,a}$ and δ_r.

• Future work with a $O(\epsilon^{-1/4})$ envelope vertical extent, dispersive effects should appear (Caillol and Grimshaw, 2007, 2008, 2012). Possible coherent vortical structures.

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