Computing dispersive PDEs normal forms via decorated trees

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Nonlinear Schrödinger equation

We recall the following cubic nonlinear Schrödinger equation (NLS) on the one dimensional torus $\mathbb{T}$:

$$\begin{cases} i\partial_t u + \partial_x^2 u = |u|^2 u \\ u|_{t=0} = u_0, \end{cases} \qquad (x,t) \in \mathbb{T} \times \mathbb{R}.$$

Duhamel's formula

$$u(t) = e^{it\Delta} u_0 + e^{it\Delta}\left(-i \int_0^t e^{-is\Delta} \left(|u(s)|^2 u(s)\right) ds\right)$$

With the change of variable $v(t) = e^{-it\Delta} u(t)$, one has

$$\partial_t v = -ie^{-it\Delta} \left(|e^{it\Delta} v|^2 e^{it\Delta} v\right)$$

In Fourier space

One has

$$\partial_t v_k = -\sum_{k=-k_1+k_2+k_3} ie^{itk^2}(e^{itk_1^2}\overline{v}_{k_1})(e^{-itk_2^2}v_{k_2})(e^{-itk_3^2}v_{k_3})$$
$$= -i\sum_{k=-k_1+k_2+k_3} e^{it\Phi(\bar{k})}\overline{v}_{k_1}v_{k_2}v_{k_3}.$$

with $\Phi(\bar{k}) := k^2 + k_1^2 - k_2^2 - k_3^2$. Decomposition into resonant and non-resonant part:

$$\partial_t v_k = -i\sum_{\substack{k=-k_1+k_2+k_3\\ k_1\neq k_2,k_3}} e^{i\Phi(\bar{k})t}\bar{v}_{k_1}v_{k_2}v_{k_3} - 2i\sum_{k_1\in\mathbb{Z}}\bar{v}_{k_1}v_{k_1}v_k + i|v_k|^2 v_k$$
$$=: \mathcal{N}^{(1)}(v)(k) + \mathcal{R}^{(1)}(v)(k).$$

Integration by parts

We decompose $\mathcal{N}^{(1)}$ into

$$\mathcal{N}^{(1)} = \mathcal{N}_1^{(1)} + \mathcal{N}_2^{(1)},$$

where $\mathcal{N}_1^{(1)}$ is the restriction of $\mathcal{N}^{(1)}$ onto $\Phi(\bar{k}) \leq N$.

$$\begin{aligned}
\mathcal{N}_2^{(1)}(v)(k) &= \sum_{A_1(k)^c} \partial_t \left(\frac{e^{i\Phi(\bar{k})t}}{\Phi(\bar{k})} \right) \bar{v}_{k_1} v_{k_2} v_{k_3} \\
&= \sum_{A_1(k)^c} \partial_t \left[\frac{e^{i\Phi(\bar{k})t}}{\Phi(\bar{k})} \bar{v}_{k_1} v_{k_2} v_{k_3} \right] - \sum_{A_1(k)^c} \frac{e^{i\Phi(\bar{k})t}}{\Phi(\bar{k})} \partial_t \left(\bar{v}_{k_1} v_{k_2} v_{k_3} \right) \\
&=: \partial_t \mathcal{N}_0^{(2)}(v)(k) + \widetilde{\mathcal{N}}^{(2)}(v)(k).
\end{aligned}$$

Substitution via solutions of NLS

One replaces $\partial_t \bar{v}_{k_1}, \partial_t v_{k_2}, \partial_t v_{k_3}$ by

$$\partial_t v_k = \mathcal{N}^{(1)}(v)(k) + \mathcal{R}^{(1)}(v)(k)$$

to get

$$\begin{aligned}
&\widetilde{\mathcal{N}}^{(2)}(v)(k) = \mathcal{N}^{(2)}(v)(k) + \mathcal{R}^{(2)}(v)(k) \\
&= -\sum_{A_1(k)^c} \frac{e^{i\Phi(\bar{k})t}}{\Phi(\bar{k})} \Big(\overline{\mathcal{N}^{(1)}(v)(k_1)} v_{k_2} v_{k_3} + \bar{v}_{k_1} \mathcal{N}^{(1)}(v)(k_2) v_{k_3} \\
&+ \bar{v}_{k_1} v_{k_2} \mathcal{N}^{(1)}(v)(k_3) + \overline{\mathcal{R}^{(1)}(v)(k_1)} v_{k_2} v_{k_3} \\
&+ \bar{v}_{k_1} \mathcal{R}^{(1)}(v)(k_2) v_{k_3} + \bar{v}_{k_1} v_{k_2} \mathcal{R}^{(1)}(v)(k_3) \Big)
\end{aligned}$$

Normal form decomposition

With the previous computations, one writes up the decomposition:

$$v(t) = v(0) + \mathcal{N}_0^{(2)}(v)(t) - \mathcal{N}_0^{(2)}(v)(0)$$
$$+ \int_0^t \left\{ \mathcal{N}_1^{(1)}(v)(t') + \sum_{j=1}^{2} \mathcal{R}^{(j)}(v)(t') \right\} dt' + \int_0^t \mathcal{N}^{(2)}(v)(t')dt'.$$

Many applications:

- Unconditional well-posedness.
- Study of quasi-invariant Gaussian measures.
- Stochastic context: three dimensional Zakharov system.

Duhamel's formulation

Mild solution given by Duhamel's formula for NLS:

$$u(t) = e^{it\Delta}v + e^{it\Delta}\left(-i \int_0^t e^{-is\Delta}\left(|u(s)|^2 u(s)\right) ds\right)$$

In Fourier space, one gets

$$u_k(t) = e^{-itk^2} v_k - \sum_{k=-k_1+k_2+k_3} ie^{-itk^2} \int_0^t e^{isk^2}\overline{u}_{k_1}(s)u_{k_2}(s)u_{k_3}(s)ds$$

where $e^{i\tau\Delta}$ is sent to $e^{-i\tau k^2}$ in Fourier space.

Iterating Duhamel and decorated trees

One wants to replace $u_{k_j}(s)$ for $j \in \{1,2,3\}$ by

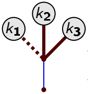
$$u_{k_j}(s) = e^{-isk_j^2} v_{k_j} + \mathcal{O}(s).$$

We obtain

$$u_k(t) = e^{-itk^2} v_k - \sum_{k=-k_1+k_2+k_3} i e^{-itk^2} \int_0^t e^{isk^2} (e^{isk_1^2} \overline{v}_{k_1})(e^{-isk_2^2} v_{k_2})(e^{-isk_3^2} v_{k_3}) ds + \mathcal{O}(t^2),$$

and define a map Π : Decorated trees $\to$ Oscillatory integrals

$(\Pi$ 

$)(t) = e^{-itk^2}$, $(\Pi$ 

$)(t) = \int_0^t e^{is(k^2+k_1^2-k_2^2-k_3^2)} ds.$

B-series type expansion

Fourier coefficient U_k^r up to order r (error t^{r+1}) :

$$U_k^r(t, v) = \sum_{T \in \mathcal{T}_0^{\leq r,k}} \frac{\Upsilon(T)(v)}{S(T)} (\Pi T)(t)$$

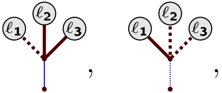
- $(\Pi T)(t)$ are oscillatory integrals.
- $\mathcal{T}_0^{\leq r,k}$: decorated trees up to order r.
- $\Upsilon(T)$: elementary differentials.
- $S(T)$: symmetry factor.

One has

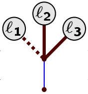
$$u_k(t) - U_k^r(t, v) = \mathcal{O}(t^{r+1}).$$

Decorated trees and words

Let A be the alphabet with letters:



The associated phase is given by

$\mathscr{F}\Big($  $\Big) = (-\ell_1 + \ell_2 + \ell_3)^2 + \ell_1^2 - \ell_2^2 - \ell_3^2.$

Then, one defines the character

$$\tilde{\Psi}\left(T_k \cdots T_1\right)(t) = \frac{e^{i \sum_{j=1}^{k} \mathscr{F}(T_j) t}}{\prod_{m=1}^{k} \sum_{j=1}^{m} \mathscr{F}(T_j)}, \quad \tilde{\Psi}\left(w \sqcup\!\sqcup \tilde{w}\right) = \tilde{\Psi}\left(w\right) \tilde{\Psi}\left(\tilde{w}\right).$$

Shuffle Hopf algebra

Let A be an alphabet and $T(A)$ the words on A. The shuffle product is defined for $a, b \in A$ and $u, v \in T(A)$

$$\varepsilon \sqcup\!\sqcup v = v \sqcup\!\sqcup \varepsilon = v, \quad (au \sqcup\!\sqcup bv) = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v).$$

Given a smooth path $t \mapsto X_t^a$ indexed by $a \in A$, one defines:

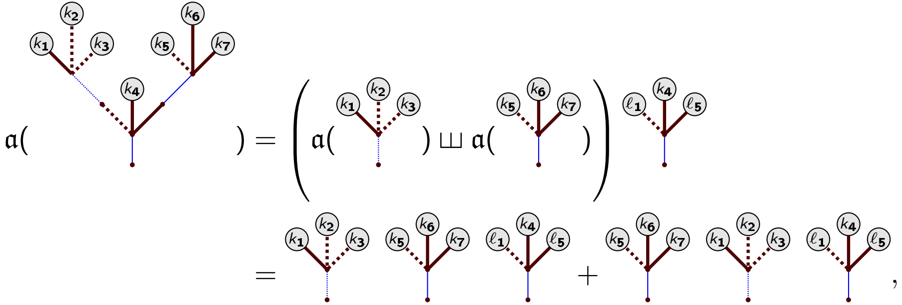
$$X_{st}(a_1 \cdots a_n) = \int_{s<t_1<\cdots<t_n<t} dX_{t_1}^{a_1} \cdots dX_{t_n}^{a_n}.$$

Proposition

One has for $u, v \in T(A)$:

$$X_{st}(u)X_{st}(v) = X_{st}(u \sqcup\!\sqcup v).$$

Arborification



where $\ell_1 = k_1 - k_2 - k_3$ and $\ell_5 = -k_5 + k_6 + k_7$.

Main results

Theorem (B. 24)

Main components of the normal form in Fourier space

$$\mathcal{N}_0^{(n)}(v)(k) = \sum_{T \in \hat{\mathcal{T}}_0^{n,k}} \frac{\Upsilon(T)(v)}{S(T)} \Psi(\mathfrak{a}(T)),$$

$$\mathcal{R}^{(n)}(v)(k) = \sum_{\hat{T} \in \hat{\mathcal{T}}_{res,0}^{n,k}} \frac{\Upsilon(\hat{T})(v)}{S(\hat{T})} \mathscr{F}(\hat{T}) \hat{\Psi}(\mathfrak{a}(\hat{T}))(t),$$

$$\mathcal{N}^{(n)}(v)(k) = \sum_{T \in \hat{\mathcal{T}}_0^{n,k}} \frac{\Upsilon(T)(v)}{S(T)} \mathscr{F}(T) \hat{\Psi}(\mathfrak{a}(T))(t).$$

Birkhoff Normal form

Consider the Hamiltonian for cubic NLS

$$\mathcal{H}_0 = H_0 + H_1 = \frac{i}{2} \sum_{k \in \mathbb{Z}^d} |k|^2 |u_k|^2 + \frac{i}{4} \sum_{k_1, \ldots, k_4 \in \mathbb{Z}^d} u_{k_1} \bar{u}_{k_2} u_{k_3} \bar{u}_{k_4} .$$

One has

$$H_1 = H_1^{\text{res}} + H_1^{\text{non-res}},$$

where

$$H_1^{\text{res}} = \frac{i}{4} \sum_{\substack{\mathrm{k} \in (\mathbb{Z}^d)^4 \\ \Phi_2(\mathrm{k}) \leq N}} u_{k_1} \bar{u}_{k_2} u_{k_3} \bar{u}_{k_4} , \quad H_1^{\text{non-res}} = \frac{i}{4} \sum_{\substack{\mathrm{k} \in (\mathbb{Z}^d)^4 \\ \Phi_2(\mathrm{k}) > N}} u_{k_1} \bar{u}_{k_2} u_{k_3} \bar{u}_{k_4} ,$$

and

$$\Phi_2(\mathrm{k}) = k_1^2 - k_2^2 + k_3^2 - k_4^2 .$$

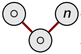
Decorated trees

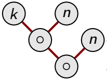
We introduce F_1 such that

$$\{H_0, F_1\} = -H_1^{\text{non-res}}, \qquad F_1 = H_{1,\Phi}^{\text{non-res}}.$$

Then

$$\mathcal{H}_1^3 = (H \circ F_1)^3 = H_0 + H_1^{\text{res}} + \{H_1, H_{1,\Phi}^{\text{non-res}}\} + \frac{1}{2}\{\{H_0, H_{1,\Phi}^{\text{non-res}}\}, H_{1,\Phi}^{\text{non-res}}\}$$

$$= (\Pi\, k) + (\Pi\, r) + (\Pi$$


$$) + \frac{1}{2}(\Pi$$


$$)$$

$$= (\Pi\, k) + \sum_{T \in \mathcal{T}_r^{<3}} \frac{(\Pi T)}{S(T)} + \sum_{T \in \mathcal{T}_\circ^3} \frac{(\Pi T)}{S(T)},$$

Planar binary trees for the Magnus expansion.

[Iserles-Norsett ; R. Soc. Lond. 99].

Main result

Theorem (Armstrong-Goodall-B. 25)

Let $m, \ell \in \mathbb{N}^*$ with $m < \ell$. One has

$$\mathcal{H}_m^\ell = (H \circ F_1 \circ \cdots \circ F_m)^\ell = H_0 + \sum_{T \in \mathcal{T}_r^{<m+2}} \frac{(\Pi T)}{S(T)} + \sum_{T \in \mathcal{T}_\circ^{m+2}} \frac{(\Pi T)}{S(T)} + \sum_{T \in \mathcal{T}_\circ^{m+2,\ell}} \frac{(\Pi T)}{S(T)}$$

Moreover, one has

$$F_i = \sum_{T \in \mathcal{T}_n^{i+1}} \frac{(\Pi T)}{S(T)}, \quad \{H_0, F_i\} = - \sum_{T \in \mathcal{T}_\circ^{i+1}} \frac{(\Pi T)^{\text{non-res}}}{S(T)}.$$

Perspectives

- Connections between three different fields:
  - Dispersive PDEs, [Guo-Kwon-Oh ; CMP 13] .
  - Dynamical Systems, [Ecalle-Valuet ; Ann. Fac. Sci. Toulouse 04] , [Fauvet-Menous ; Annales Sc. de l'Ecole Normale Sup. 17].
  - Numerical Analysis, [B.-Schratz ; Forum Pi 22].
- Link with Modified Energy, I-method and Birkhoff normal forms. .
- Computation of cancellations for dispersive PDEs with random initial data. [B.-Tolomeo ; 25].
- Combinatorics in Random Tensors, Wave Turbulence and stochastic dispersive equations.