Analyse topologique des données en Mécanique

Dina Razafindralandy†, Francisco Chinesta‡

† LaSIE, La Rochelle Université – UMR CNRS 7356
‡ PIMM, ENSAM – UMR CNRS 8006

GDR Géométrie Différentielle et Mécanique
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Plan

1. Rappels sur l'analyse topologique des données (TDA)

   $\longrightarrow$ Persistance topologique

2. Quelques applications en mécanique

Approche 1 : Nuage de points $\longrightarrow$ Maillage

- Maillage $\quad\Longrightarrow\quad$ Calcul différentiel discret

  (interpolation, résolution équation, calcul extérieur, . . .)

- Connecter les points “proches”

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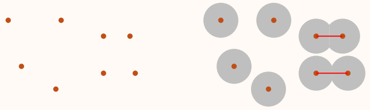
- Connecter les points "proches"
- Exemple: Proche $\iff$ distance $\leq 2r$

Alpha Complex



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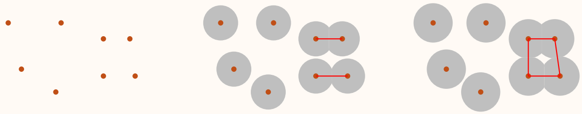

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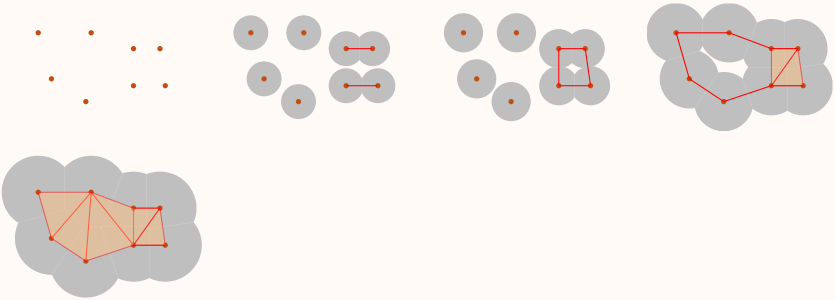

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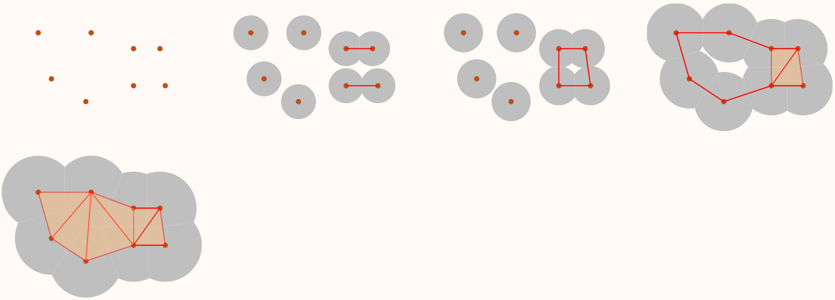
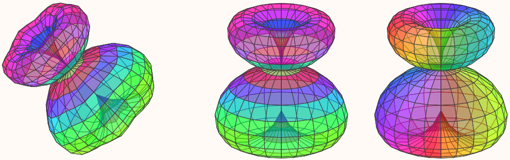


- Choix de r $\longrightarrow$ Persistance topologique

Approche 2 : Analyse de données sur un maillage avec une fonction de filtration

- Données sur un maillage
- Fonction de filtration : $f:\ \{\text{données}\} \longmapsto \mathbb{R}$



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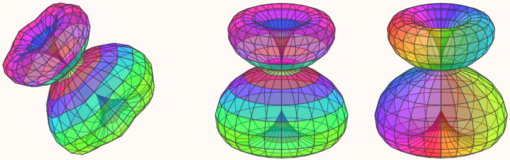


- Extraire des informations topologiques pertinentes

  Robuste aux bruits

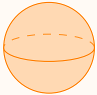
- Construire une empreinte à partir des invariants topologiques

  Réduction de dimension, Comparaison, Régression non-linéaire

$\longrightarrow$ Persistence topologique

Invariants topologiques ? Nombres de Betti $(\beta_k)_{k\in\mathbb{N}}$

$\beta_0 = 2,\ \beta_1 = 1,\ \beta_{k\geq 2} = 0$



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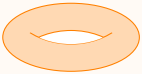

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- Espace projectif $\mathbb{P}^n(\mathbb{R})$: $\beta_0 = 1, \quad \beta_n = 1-(-1)^n, \quad \beta_k = 0$ sinon
- Bouteille de Klein: $(1, 2, 1)$
- $SO(3) : (1, 0, 1), \qquad SO(4) : (1, 0, 0, 2, 0, 0, 1), \qquad SO(5) : (1, 0, 0, 1, 0, 0, 0, 1)$

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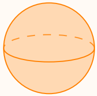

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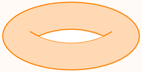

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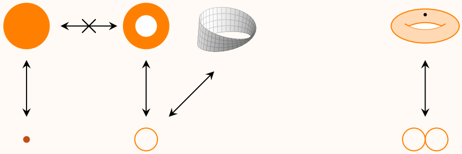
- Invariant par homomorphisme (ex: triangulation)
- Invariant par homotopie (déformation continue)



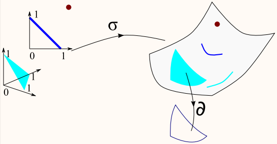
Plus sur les Nombres de Betti

k-simplex: $\sigma : \Delta_k \longrightarrow M$ continu

k-chaîne: $\sum_{k\text{-simplexe } \sigma} c_\sigma \sigma$

Bord: $\partial_k \sigma = \sum \sigma_{|face_{k-1}}$

somme des restrictions de σ aux $k-1$-faces de Δ_k



$\beta_k = \text{rank } H_k$ où $H_k = \frac{\ker \partial_k}{\operatorname{Im} \partial_{k+1}} = \frac{\{k\text{-cycle}\}}{\{k\text{-bord}\}}$ k-th homology group

Ex: 0-cycle = Σ points, 1-cycle = Σ lacets , 2-cycle = Σ surface sans bord

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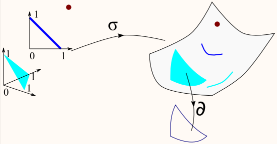

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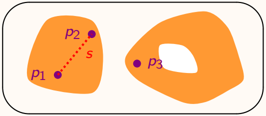

$p_1 \sim p_2$ car $p_1 + p_2 = \partial s$

$p_1 \not\sim p_3$

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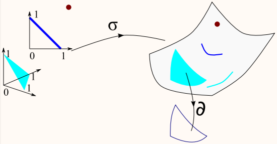

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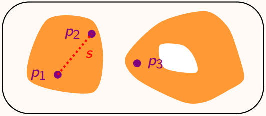

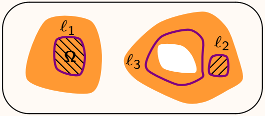

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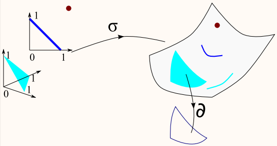

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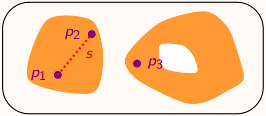

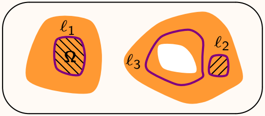

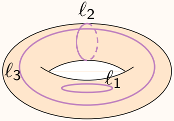

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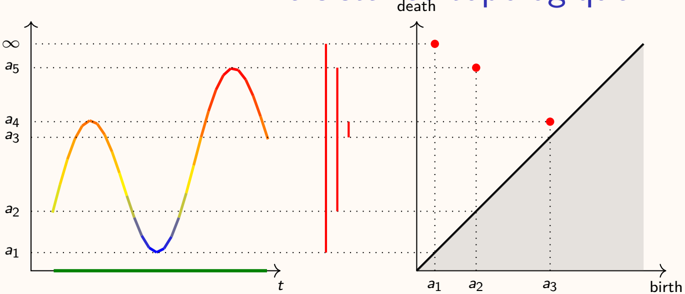
$\mathbb{T}^2$ est un 2-cycle mais pas un 2-bord

$H_2 = \text{span}\{\mathbb{T}^2\}, \quad \beta_2 = 1$

Persistance topologique



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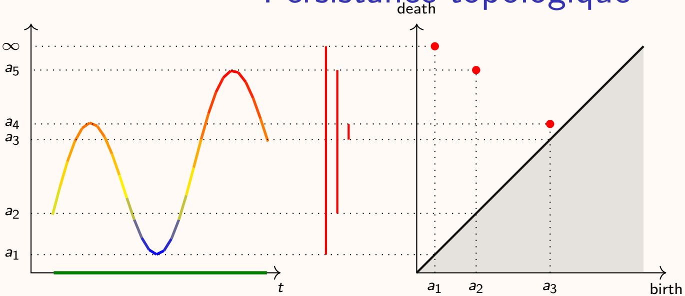

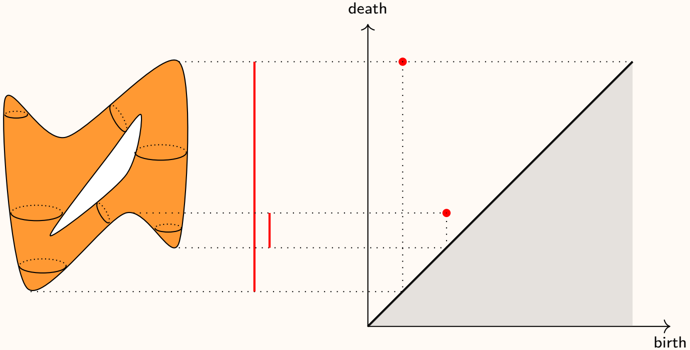

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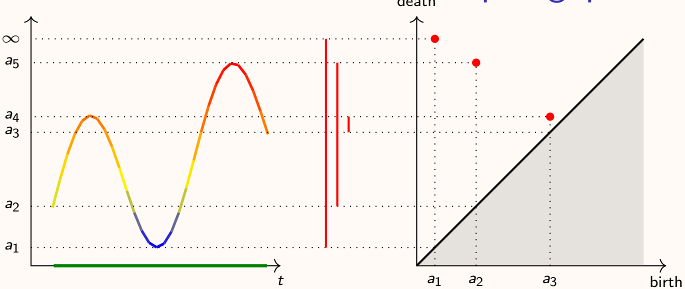
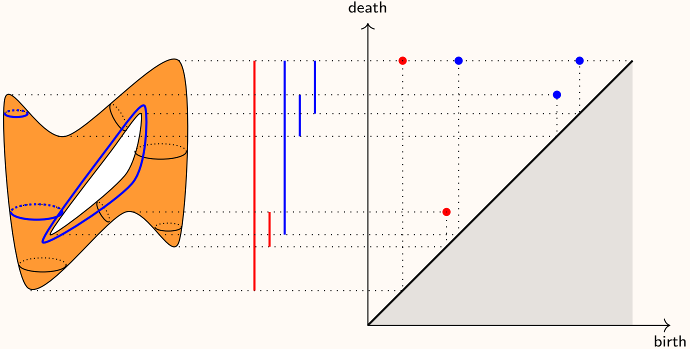
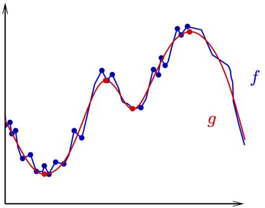
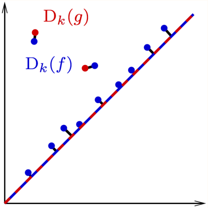




Métrique dans l'espace des diagrammes





Wasserstein distance

$$W_p(D, D') \; = \; \inf_{\beta \in B} \left(\sum_{x \in D} \|x - \beta(x)\|_p^p \right)^{1/p}$$

$B = \{ \textit{bijection } D_1 \cup \Delta \longrightarrow D_2 \cup \Delta \}$

Bottlneck distance: $W_\infty(D, D') \; = \; \inf_{\beta \in B} \sup_{x \in D} \|x - \beta(x)\|_\infty$

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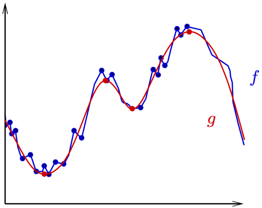

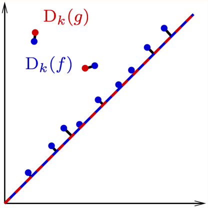

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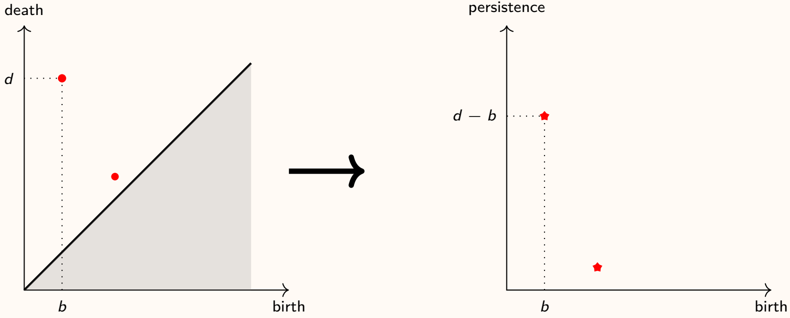
Stabilité. Tame functions $f, g : X \longrightarrow \mathbb{R}$

$$W_\infty(\ D(f)\ ,\ D(g)\ ) \leq \|f - g\|_\infty$$

Bottlneck distance [Cohen-Steiner, Edelsbrunner, Harer, 2007]. Wasterstein distance [Skraba, Turner, 2021]

Image de persistance



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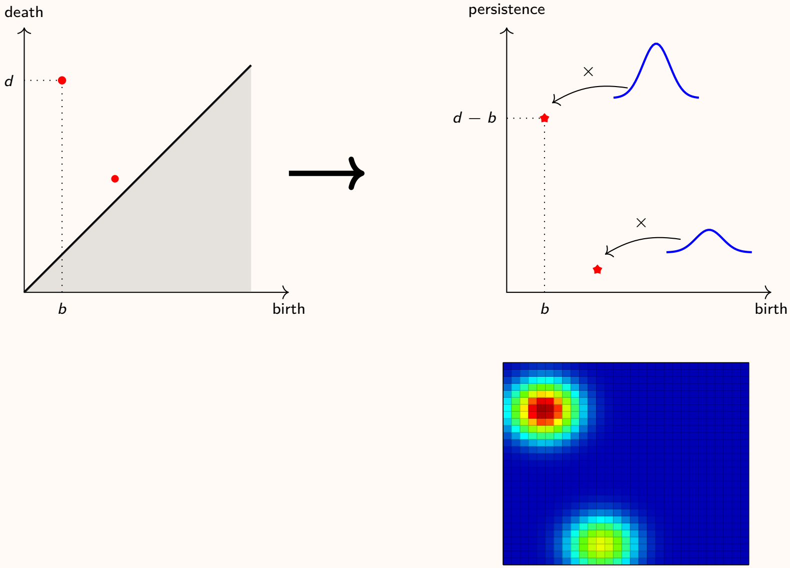

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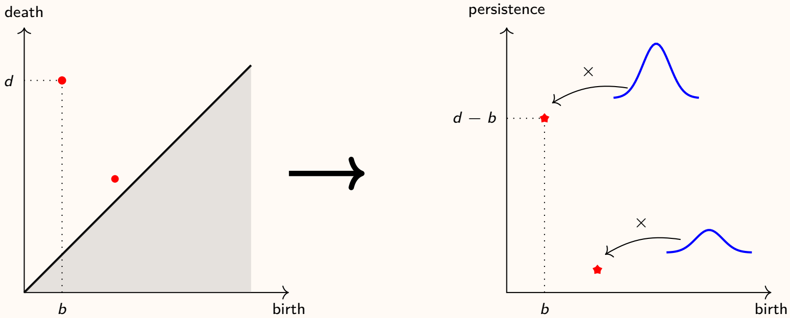
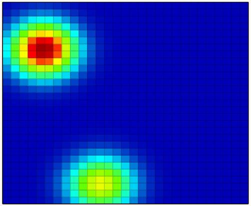


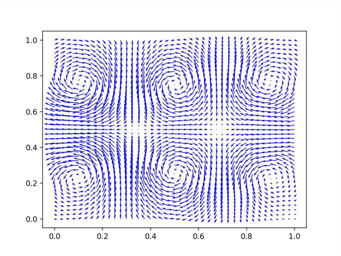


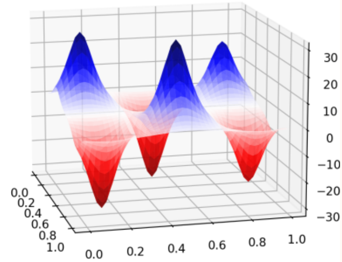
Stabilité

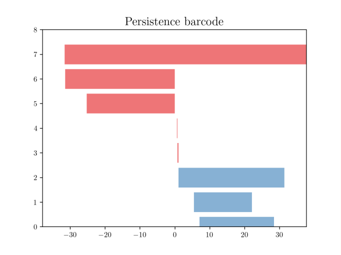
$$\|Image(Dia) - Image(Dia')\|_1$$
$$\leq C \cdot W_1(Dia, Dia')$$

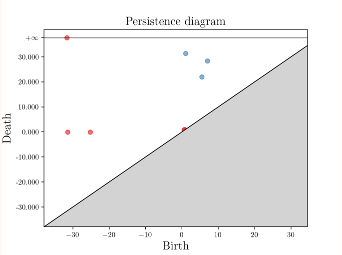
[Adams et al, 1997]

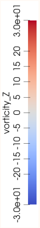
Écoulement tourbillonnaire. Filtration =vorticité



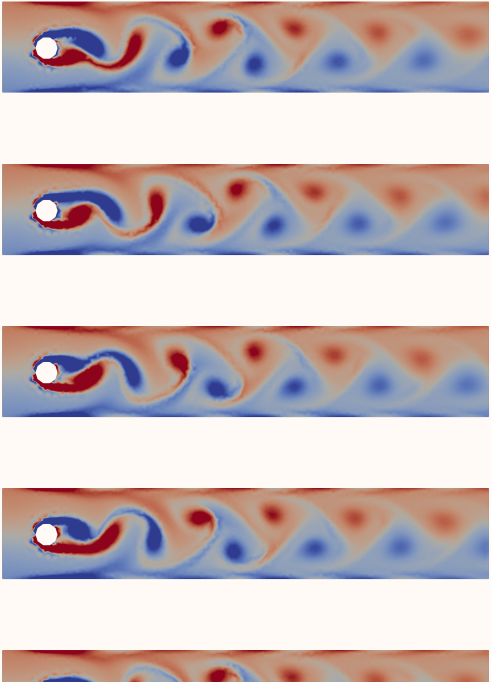


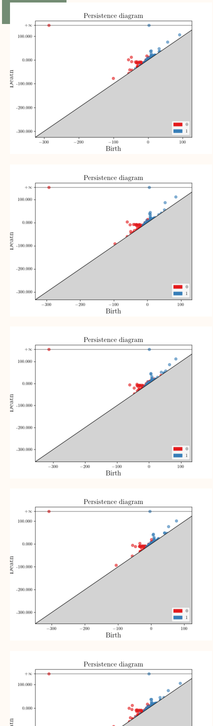




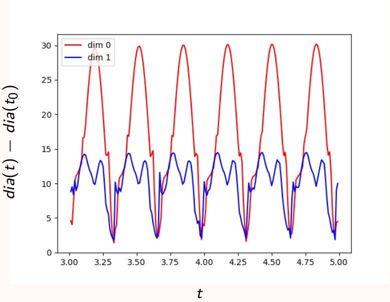


$\omega \in [-286, 300]$. Color limited to:



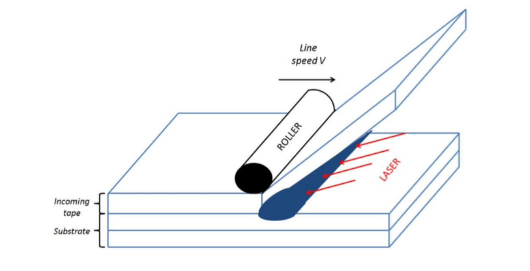


Evolution des diagrammes

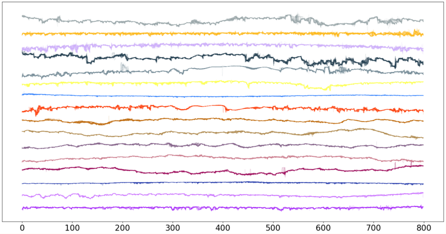


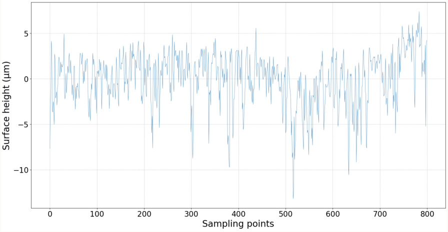
Manufacture de fuselage par ATP (Automated Tape Placement)

Consolidation thermo-plastique:



16 fournisseurs

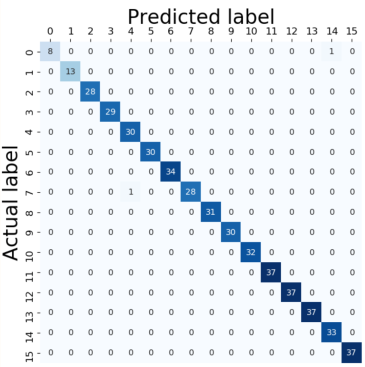


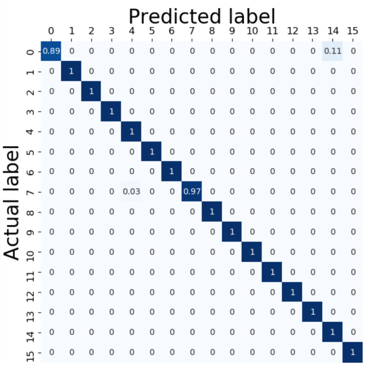


Distance Euclidienne directe ? Léger décallage en espace $\Longrightarrow$ Grande distance

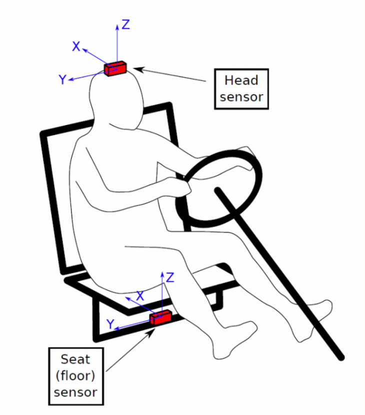
Prédictiion

- Base de données (images de persistance) construites avec 883 profils
- Prédiction pour 476 profils





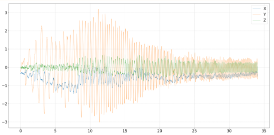
Conduite assistée

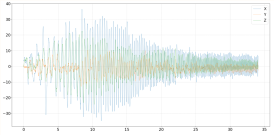


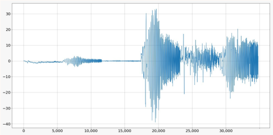
- 8 états
  1. Relaxed, Rigid seat, Driver
  2. Relaxed, Rigid seat, Passenger
  3. Relaxed, SAV seat, Driver
  4. Relaxed, SAV seat, Passenger
  5. Tense, Rigid seat, Driver
  6. Tense, Rigid seat, Passenger
  7. Tense, SAV seat, Passenger
  8. Tense, SAV seat, Driver
- 1 état $\longrightarrow$ 6 séries temporelles

  $(\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\theta}_x, \ddot{\theta}_y, \ddot{\theta}_z)$

  Concaténation en 1 seule série





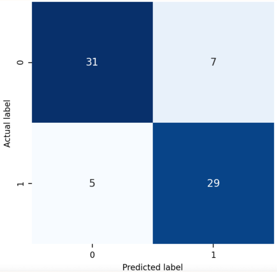


Prédiction

- Entraînement: 144 échantillons
- Prédiction: 72 échantillons

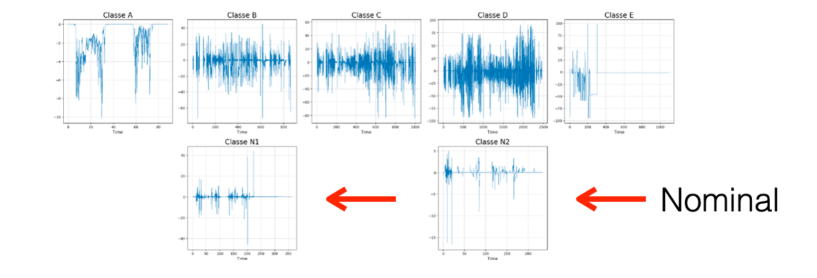
0 = Relaxed

1 = Tense

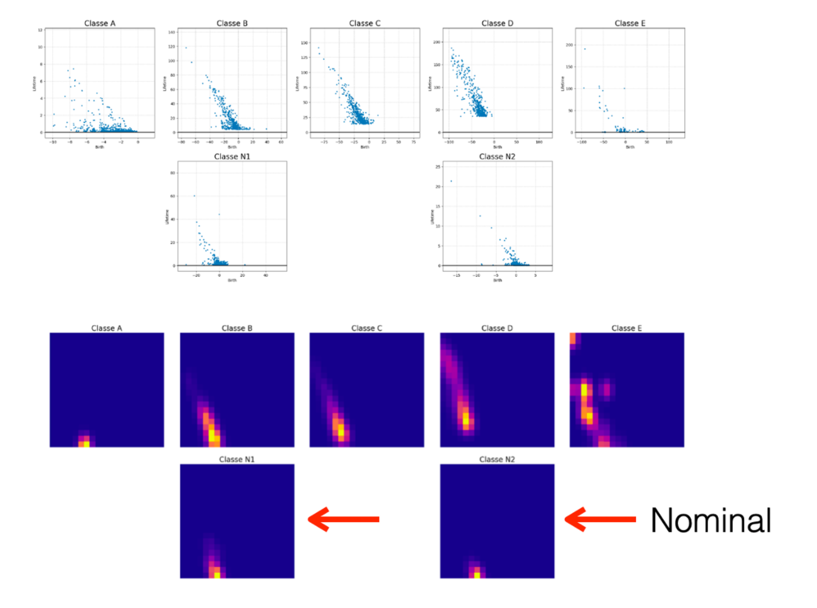


- 83% de bonnes réponses
- Enrichir le nombre d'échantillons (9 personnes)

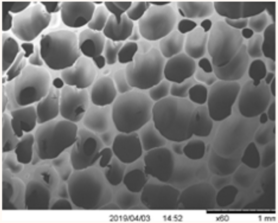
Défaut sur une voiture



Défaut sur une voiture



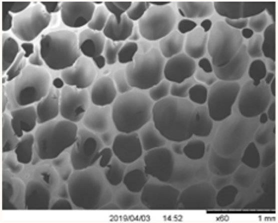
Microstructure

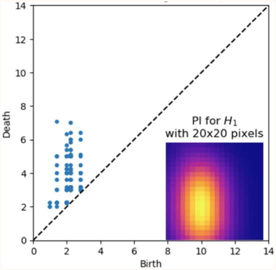


- Conductivité thermique de N échantillons (éléments finis ou expérience)

  Différentes distributions de pores

Microstructure



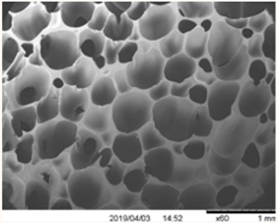


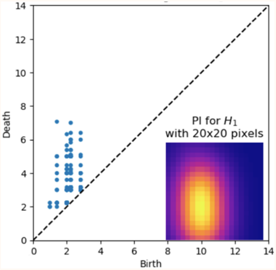
- Conductivité thermique de N échantillons (éléments finis ou expérience)

  Différentes distributions de pores

- Régression non linéaire
  - ★ Image de persistance: 20×20
  - ★ Analyse en composantes principales: $(\alpha_1, \alpha_2, \alpha_3)$

Microstructure





- Conductivité thermique de N échantillons (éléments finis ou expérience)

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- Prédiction pour un nouvel échantillon

Nb échantillons N	Erreur relative
13	0.076
16	0.056
19	0.046
35	0.037

Conclusion

- TDA: grande potentielle d'utilisation
- Empreinte compacte, sépare les bruits de l'information pertinente

  Comparaison robuste aux bruits
- Interpolation non-linéaire
- Détection/classification de tourbillons complexes
- Contrôle d'écoulements