Homogeneous Second Order ODEs Under Fiber-Preserving Point Transformations: The Power Series Method

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Sommaire

Introduction

- We consider point equivalences of 2^{nd} order ODEs

$$y_{xx} = Q(x, y, y_x)$$

- under fiber-preserving transformations

$$\varphi : \begin{cases} \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \\ (x, y) \longmapsto (f(x), g(x, y)) =: (x', y'). \end{cases}$$

Introduction

Definition

Two 2^{nd} order ODEs $y_{xx} = Q(x, y, y_x)$ and $y'_{x'x'} = R(x', y', y'_{x'})$ are said to be *fiber-preserving-equivalent* if there is a diffeomorphism $\varphi : (x, y) \in \mathbb{C}^2 \longmapsto (f(x), g(x, y)) =: (x', y') \in \mathbb{C}^2$ such that

$$(\varphi^{(1)})^*(y'_{x'x'}) = y_{xx}. \tag{1}$$

or equivalently

$$\varphi^{(2)}\Big(\{(x, y, y_x, y_{xx}) \mid y_{xx} = Q(x, y, y_x)\}\Big) = \{(x', y', y'_{x'}, y'_{x'x'}) \mid y'_{x'x'} = R(x', y', y'_{x'})\}$$

Submanifold-Equation:

$$\Big\{(x, y, y_x, y_{xx}) \in \mathbb{C}^4 \mid y_{xx} = Q(x, y, y_x)\Big\} \subset J^2(\mathbb{C}^2_{x,y}) = \mathbb{C}^4_{x,y,p,y_{xx}}.$$

Introduction

What we want :

- Representatives of equivalence problem
- Symmetry groups associated to S:

$$\mathrm{LieSym}(S) := \left\{ \mathcal{L} = A(x)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y} \mid \mathcal{L}^{(2)}_{|S}. \text{ tangent to } S \right\}.$$

such that $\dim(\mathrm{LieSym}(S)) \geqslant \dim(S) = 3$.

Problem

Classify all homogeneous 2^{nd} order ODEs under fiber-preserving transformations, especially, find all (locally) homogeneous models.

Introduction

- In 1989, Hsu and Kamran solved this problem by the Élie Cartan's equivalence method = method in terms of one-forms
- One-forms generate too many calculations.
- In higher dimensions, equivalence problems will be too difficult to be solved.
- Now, see the foundations of the power series method, which lightens the computation by proceeding in terms of power series.

Introduction

- We work in local, so:

$$0 \in S.$$

-

$$T_0 S = \mathrm{Span}\{\mathcal{L}_{|0} \mid \mathcal{L} \in \mathrm{LieSym}(S)\}.$$

- Assume φ fixes $(0,0) \in \mathbb{C}^2$ and

$$Q(x,y,p) = Q_{0,0,0} + Q_{1,0,0}\, x^1 + Q_{0,1,0}\, y^1 + Q_{0,0,1}\, p^1 + \cdots$$

namely

$$Q(x,y,p) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} Q_{i,j,k}\, x^i\, y^j\, p^k \qquad (Q_{i,j,k} \in \mathbb{C}),$$

where $p := y_x$.

Introduction

Two notions are important:

- $$(\varphi^{(1)})^*(y'_{x'x'}) = y_{xx},$$

  equivalent to

  $$0 = f_x^3 R\left(f, g, \frac{g_x + g_y p}{f_x}\right) + g_x f_{xx} + g_y f_{xx} p - f_x g_{xx} - 2 f_x g_{xy} p - f_x g_{yy} p^2 - f_x g_y Q(x, y, p),$$

  the *fundamental equation* of this problem.

- $$\mathcal{L}^{(2)}_{|S}. \text{ tangent to } S,$$

  with

  $$\mathcal{L} = A(x)\frac{\partial}{\partial x} + B(x, y)\frac{\partial}{\partial y},$$

  where

  $$A(x) = A_0 + A_1 x + \cdots ,$$
  $$B(x, y) = B_{0,0} + B_{1,0} x + \cdots ,$$

Introduction

$$\mathcal{L}^{(2)}_{|S}.\ \text{tangent to } S,$$

is equivalent to

$$\mathcal{L}^{(2)}\Big(y_{xx} - Q(x,y,y_x)\Big)_{|y_{xx}=Q(x,y,y_x)} \equiv 0,$$

in $\mathbb{C}\{x,y,y_x\}$, and precisely :

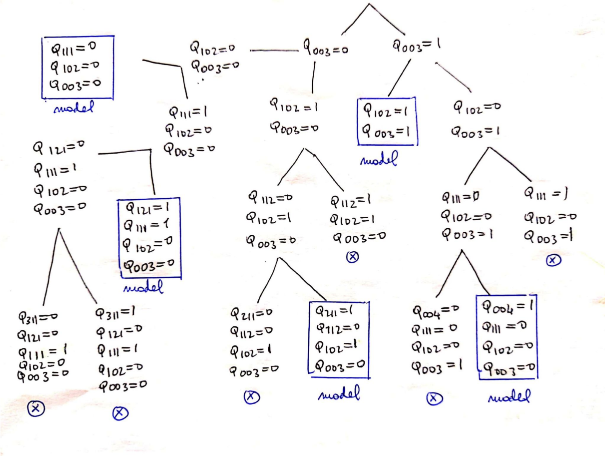
$$\begin{aligned} 0 = -B_{xx} + (A_{xx} - 2B_{xy})p + (2A_x - B_y)Q(x,y,p) - B_{yy}p^2 + \\ + AQ_x + BQ_y + \Big[B_y + (B_y - A_x)p\Big]Q_p, \end{aligned}$$

the *vectorial fundamental equation* of the problem.

Sommaire

Results

By these two equations, we can solve the problem :



Results

Theorem

In the branch $Q_{0,0,3} = Q_{1,0,2} = 1$, every homogeneous 2^{nd} order ODE is, in a unique way, fiber-preserving-equivalent to

$$\begin{aligned} Q = {} & xp^2 + p^3 + Q_{2,1,1}x^2yp + \\ & + \Big(1 + 4Q_{2,1,1}Q_{1,0,3} - \tfrac{8}{3}Q_{2,1,1}Q_{0,0,4} - \tfrac{1}{3}Q_{1,0,3}^3 + \tfrac{1}{3}Q_{2,0,2}Q_{1,0,3}^2 + \\ & - \tfrac{4}{9}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4} + 2Q_{2,0,2}^2Q_{1,0,3} - \tfrac{8}{9}Q_{2,0,2}^2Q_{0,0,4}\Big)xy^2p + Q_{2,0,2}x^2p^2 + \\ & + \Big(Q_{1,0,3}^2 - 3Q_{2,0,2}Q_{1,0,3} + \tfrac{4}{3}Q_{2,0,2}Q_{0,0,4}\Big)xyp^2 + Q_{1,0,3}xp^3 + \\ & + \Big(2Q_{1,0,3}^2 + \tfrac{8}{3}Q_{2,0,2}Q_{0,0,4} - 4Q_{1,0,3}Q_{0,0,4}\Big)yp^3 + Q_{0,0,4}p^4 + \\ & + \Big(-\tfrac{2}{9}Q_{2,1,1}Q_{1,0,3} + \tfrac{20}{9}Q_{2,1,1}Q_{2,0,2}\Big)x^3yp + \end{aligned}$$

Results

Theorem

$$+\Big(Q_{2,1,1}Q_{1,0,3}^2+5Q_{2,1,1}Q_{2,0,2}Q_{1,0,3}-4Q_{2,1,1}Q_{2,0,2}Q_{0,0,4}+2Q_{2,0,2}+$$
$$-\tfrac{2}{3}Q_{2,0,2}Q_{1,0,3}^3+\tfrac{2}{3}Q_{2,0,2}^2Q_{1,0,3}^2-\tfrac{8}{9}Q_{1,0,3}Q_{2,0,2}^2Q_{0,0,4}+4Q_{2,0,2}^3Q_{1,0,3}+$$
$$-\tfrac{16}{9}Q_{2,0,2}^3Q_{0,0,4}\Big)x^2y^2p+\Big(\tfrac{4}{3}Q_{1,0,3}^2-\frac{4}{9}Q_{1,0,3}^5+$$
$$+\tfrac{160}{9}Q_{2,0,2}Q_{1,0,3}Q_{2,1,1}Q_{0,0,4}-16Q_{2,0,2}Q_{1,0,3}^2Q_{2,1,1}+$$
$$-\tfrac{32}{9}Q_{1,0,3}^2Q_{2,1,1}Q_{0,0,4}-\tfrac{128}{27}Q_{2,0,2}Q_{0,0,4}^2Q_{2,1,1}-\tfrac{32}{27}Q_{1,0,3}^3Q_{2,0,2}Q_{0,0,4}+$$
$$+\tfrac{32}{27}Q_{1,0,3}^2Q_{2,0,2}^2Q_{0,0,4}-\tfrac{64}{81}Q_{2,0,2}^2Q_{0,0,4}^2Q_{1,0,3}+\tfrac{64}{9}Q_{2,0,2}^3Q_{1,0,3}Q_{0,0,4}+$$

Results

Theorem

$$-4Q_{2,0,2}Q_{1,0,3}+\tfrac{16}{9}Q_{2,0,2}Q_{0,0,4}+\tfrac{16}{3}Q_{1,0,3}^3Q_{2,1,1}+\tfrac{16}{9}Q_{1,0,3}^4Q_{2,0,2}+$$

$$+\tfrac{4}{3}Q_{1,0,3}^3Q_{2,0,2}^2-8Q_{2,0,2}^3Q_{1,0,3}^2-\tfrac{128}{81}Q_{2,0,2}^3Q_{0,0,4}^2\Big)xy^3p+$$

$$+\Big(\tfrac{2}{3}Q_{2,1,1}+\tfrac{10}{9}Q_{2,0,2}^2-\tfrac{1}{9}Q_{2,0,2}Q_{1,0,3}\Big)x^3p^2+\Big(\tfrac{7}{2}Q_{2,1,1}Q_{1,0,3}+$$

$$-\tfrac{4}{3}Q_{2,1,1}Q_{0,0,4}+Q_{2,0,2}Q_{1,0,3}^2-3Q_{2,0,2}^2Q_{1,0,3}+\tfrac{4}{3}Q_{2,0,2}^2Q_{0,0,4}\Big)x^2yp^2+$$

$$+\Big(4Q_{1,0,3}^2Q_{2,0,2}Q_{0,0,4}-8Q_{1,0,3}Q_{2,0,2}^2Q_{0,0,4}+\tfrac{16}{9}Q_{2,0,2}^2Q_{0,0,4}^2-\tfrac{2}{3}Q_{1,0,3}^3Q_{0,0,4}+$$

$$-\tfrac{8}{9}Q_{1,0,3}Q_{0,0,4}^2Q_{2,0,2}+\tfrac{1}{2}Q_{1,0,3}^4-\tfrac{9}{2}Q_{2,0,2}Q_{1,0,3}^3+9Q_{2,0,2}^2Q_{1,0,3}^2+\tfrac{3}{2}Q_{1,0,3}+$$

$$+6Q_{2,1,1}Q_{1,0,3}^2-4Q_{0,0,4}Q_{2,1,1}Q_{1,0,3}\Big)xy^2p^2+\Big(\tfrac{1}{3}Q_{1,0,3}^2+\tfrac{2}{3}Q_{2,0,2}Q_{1,0,3}\Big)x^2p^3+$$

$$+\Big(-2+\tfrac{8}{3}Q_{1,0,3}^3+\tfrac{8}{9}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}-\tfrac{16}{3}Q_{1,0,3}^2Q_{0,0,4}+\tfrac{4}{3}Q_{2,0,2}Q_{1,0,3}^2+$$

$$+\tfrac{16}{9}Q_{2,0,2}^2Q_{0,0,4}+4Q_{2,1,1}Q_{0,0,4}\Big)xyp^3+\Big(3Q_{1,0,3}^4+\tfrac{32}{3}Q_{1,0,3}^2Q_{2,0,2}Q_{0,0,4}+$$

Results

Theorem

$$-12Q_{1,0,3}^3Q_{0,0,4}+\tfrac{32}{9}Q_{2,0,2}^2Q_{0,0,4}^2-16Q_{1,0,3}Q_{0,0,4}^2Q_{2,0,2}+$$

$$+\tfrac{32}{3}Q_{1,0,3}^2Q_{0,0,4}^2-Q_{2,0,2}Q_{1,0,3}^3+\frac{8}{3}Q_{1,0,3}Q_{2,0,2}^2Q_{0,0,4}+2Q_{0,0,4}+$$

$$+8Q_{0,0,4}Q_{2,1,1}Q_{1,0,3}-\tfrac{16}{3}Q_{2,1,1}Q_{0,0,4}^2\Big)y^2p^3+$$

$$+\left(\tfrac{5}{3}Q_{1,0,3}Q_{0,0,4}-\tfrac{2}{3}Q_{2,0,2}Q_{0,0,4}\right)xp^4+\Big(3Q_{1,0,3}^2Q_{0,0,4}+4Q_{2,0,2}Q_{0,0,4}^2+$$

$$-\tfrac{20}{3}Q_{1,0,3}Q_{0,0,4}^2+Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}\Big)yp^4+\left(\tfrac{4}{3}Q_{0,0,4}^2-\tfrac{1}{5}Q_{1,0,3}Q_{0,0,4}\right)p^5+$$

$$+\sum_{i+j+k\geq 6}Q_{i,j,k}x^iy^jp^k,$$

Results

Theorem

with associated 3-dimensional Lie algebra generated by

$$e_1 = -\Big(-1 - \tfrac{1}{3}xQ_{1,0,3} + \tfrac{4}{3}xQ_{2,0,2} + x^2 Q_{2,1,1} \Big) \frac{\partial}{\partial x} +$$

$$+\bigg(\Big(-\tfrac{1}{3} Q_{1,0,3} - \tfrac{2}{3} Q_{2,0,2} \Big) y - Q_{2,1,1} xy + \tfrac{1}{2} y^2 +$$

$$+\Big(-\tfrac{1}{2} Q_{2,1,1} + \tfrac{1}{6} Q_{1,0,3}^2 - \tfrac{1}{2} Q_{2,0,2} Q_{1,0,3} + \tfrac{2}{9} Q_{0,0,4} Q_{2,0,2} \Big) y^3 +$$

$$+\Big(1/6 - \tfrac{1}{18} Q_{1,0,3}^3 - \tfrac{2}{27} Q_{1,0,3} Q_{2,0,2} Q_{0,0,4} + \tfrac{2}{3} Q_{2,1,1} Q_{1,0,3} +$$

$$-\tfrac{4}{9} Q_{2,1,1} Q_{0,0,4} + \tfrac{1}{18} Q_{1,0,3}^2 Q_{2,0,2} + \tfrac{1}{3} Q_{2,0,2}^2 Q_{1,0,3} - \tfrac{4}{27} Q_{2,0,2}^2 Q_{0,0,4} \Big) xy^3 +$$

Results

Theorem

$$+\Big(\tfrac{1}{8}Q_{1,0,3}+\tfrac{1}{24}Q_{1,0,3}^4-\tfrac{2}{27}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}^2+\tfrac{2}{3}Q_{1,0,3}Q_{2,1,1}Q_{0,0,4}+\tfrac{1}{3}Q_{1,0,3}^2Q_{2,0,2}Q_{0,0,4}+$$
$$-\tfrac{2}{3}Q_{1,0,3}Q_{2,0,2}^2Q_{0,0,4}-\tfrac{2}{3}Q_{2,0,2}Q_{2,1,1}Q_{0,0,4}+\tfrac{4}{27}Q_{2,0,2}^2Q_{0,0,4}^2-\tfrac{1}{18}Q_{1,0,3}^3Q_{0,0,4}+$$
$$-\tfrac{3}{8}Q_{1,0,3}^3Q_{2,0,2}+\tfrac{3}{4}Q_{1,0,3}^2Q_{2,0,2}^2\Big)y^4+$$
$$+\Big(\tfrac{1}{6}Q_{1,0,3}^2-\tfrac{1}{18}Q_{1,0,3}^5+\tfrac{20}{9}Q_{2,1,1}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}-\tfrac{4}{9}Q_{1,0,3}^2Q_{0,0,4}Q_{2,1,1}+$$
$$-\tfrac{4}{27}Q_{1,0,3}^3Q_{2,0,2}Q_{0,0,4}+\tfrac{4}{27}Q_{1,0,3}^2Q_{2,0,2}^2Q_{0,0,4}-2Q_{2,1,1}Q_{2,0,2}Q_{1,0,3}^2+\tfrac{8}{9}Q_{1,0,3}Q_{2,0,2}^3Q_{0,0,4}+$$

Results

Theorem

$$-\frac{16}{27}Q_{2,0,2}Q_{0,0,4}^2Q_{2,1,1}-\frac{8}{81}Q_{1,0,3}Q_{2,0,2}^2Q_{0,0,4}^2-\frac{1}{2}Q_{2,0,2}Q_{1,0,3}+\frac{2}{9}Q_{0,0,4}Q_{2,0,2}-Q_{1,0,3}^2Q_{2,0,2}^3+$$

$$-\frac{16}{81}Q_{2,0,2}^3Q_{0,0,4}^2+\frac{2}{3}Q_{2,1,1}Q_{1,0,3}^3+\frac{2}{9}Q_{1,0,3}^4Q_{2,0,2}+\frac{1}{6}Q_{1,0,3}^3Q_{2,0,2}^2\Big)xy^4+$$

$$+\Big(\frac{16}{45}Q_{2,1,1}Q_{2,0,2}^2Q_{1,0,3}Q_{0,0,4}-\frac{4}{5}Q_{0,0,4}Q_{1,0,3}^2Q_{2,0,2}Q_{2,1,1}+\frac{52}{27}Q_{0,0,4}^2Q_{1,0,3}Q_{2,0,2}Q_{2,1,1}+$$

$$+\frac{9}{40}Q_{1,0,3}^3-\frac{7}{360}Q_{1,0,3}^6-\frac{1}{20}+\frac{1}{6}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}-\frac{1}{12}Q_{2,1,1}Q_{2,0,2}Q_{1,0,3}^3+$$

$$+\frac{16}{405}Q_{1,0,3}^2Q_{0,0,4}^3Q_{2,0,2}-\frac{56}{405}Q_{1,0,3}Q_{0,0,4}^3Q_{2,0,2}^2-\frac{4}{27}Q_{1,0,3}^3Q_{0,0,4}^2Q_{2,0,2}+$$

$$+\frac{406}{405}Q_{1,0,3}^2Q_{2,0,2}^2Q_{0,0,4}^2-\frac{2}{5}Q_{2,1,1}^2Q_{1,0,3}Q_{0,0,4}-\frac{308}{405}Q_{1,0,3}Q_{2,0,2}^3Q_{0,0,4}^2+$$

$$-\frac{115}{54}Q_{1,0,3}^3Q_{2,0,2}^2Q_{0,0,4}+\frac{179}{135}Q_{1,0,3}^2Q_{2,0,2}^3Q_{0,0,4}+\frac{8}{45}Q_{2,0,2}^4Q_{1,0,3}Q_{0,0,4}+$$

$$+\frac{44}{135}Q_{1,0,3}^4Q_{2,0,2}Q_{0,0,4}-\frac{7}{45}Q_{0,0,4}Q_{1,0,3}^3Q_{2,1,1}-\frac{128}{135}Q_{0,0,4}^2Q_{2,0,2}^2Q_{2,1,1}+$$

Results

Theorem

$$-\frac{32}{45}Q_{0,0,4}^2Q_{1,0,3}^2Q_{2,1,1}-\frac{1}{5}Q_{2,1,1}Q_{1,0,3}+\frac{1}{30}Q_{2,1,1}Q_{0,0,4}-\frac{3}{40}Q_{1,0,3}^2Q_{2,0,2}+$$
$$-\frac{1}{10}Q_{2,0,2}^2Q_{1,0,3}+\frac{2}{15}Q_{2,0,2}^2Q_{0,0,4}+\frac{16}{135}Q_{2,0,2}^3Q_{0,0,4}^3+\frac{4}{135}Q_{1,0,3}^4Q_{0,0,4}^2+$$
$$+\frac{1}{30}Q_{1,0,3}^5Q_{0,0,4}-\frac{1}{3}Q_{1,0,3}^2Q_{0,0,4}+\frac{4}{15}Q_{2,1,1}^2Q_{0,0,4}^2-\frac{11}{45}Q_{1,0,3}^5Q_{2,0,2}+\frac{401}{360}Q_{1,0,3}^4Q_{2,0,2}^2+$$
$$-\frac{37}{60}Q_{1,0,3}^3Q_{2,0,2}^3-\frac{32}{405}Q_{2,0,2}^4Q_{0,0,4}^2+\frac{23}{60}Q_{1,0,3}^4Q_{2,1,1}\Big)y^5\Bigg)\frac{\partial}{\partial y}+$$
$$+\Bigg(-Q_{2,1,1}y+\Big(-\frac{2}{3}Q_{1,0,3}+\frac{2}{3}Q_{2,0,2}\Big)p+Q_{2,1,1}xp+yp+\Big(\frac{1}{6}-\frac{1}{18}Q_{1,0,3}^3+$$

Results

Theorem

$$-\frac{2}{27}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}+\frac{2}{3}Q_{2,1,1}Q_{1,0,3}-\frac{4}{9}Q_{2,1,1}Q_{0,0,4}+\frac{1}{18}Q_{1,0,3}^2Q_{2,0,2}+\frac{1}{3}Q_{2,0,2}^2Q_{1,0,3}+$$
$$-\frac{4}{27}Q_{2,0,2}^2Q_{0,0,4}\Big)y^3+\Big(-\frac{3}{2}Q_{2,1,1}+\frac{1}{2}Q_{1,0,3}^2+\frac{2}{3}Q_{0,0,4}Q_{2,0,2}-\frac{3}{2}Q_{2,0,2}Q_{1,0,3}\Big)y^2p+$$
$$+\Big(\frac{20}{9}Q_{2,1,1}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}+\frac{1}{6}Q_{1,0,3}^2-\frac{1}{18}Q_{1,0,3}^5-2Q_{2,1,1}Q_{2,0,2}Q_{1,0,3}^2+$$
$$+\frac{8}{9}Q_{1,0,3}Q_{2,0,2}^3Q_{0,0,4}-\frac{16}{27}Q_{2,0,2}Q_{0,0,4}^2Q_{2,1,1}-\frac{8}{81}Q_{1,0,3}Q_{2,0,2}^2Q_{0,0,4}^2-\frac{4}{9}Q_{1,0,3}^2Q_{0,0,4}Q_{2,1,1}+$$
$$-\frac{4}{27}Q_{1,0,3}^3Q_{2,0,2}Q_{0,0,4}+\frac{4}{27}Q_{1,0,3}^2Q_{2,0,2}^2Q_{0,0,4}+\frac{2}{9}Q_{0,0,4}Q_{2,0,2}-\frac{1}{2}Q_{2,0,2}Q_{1,0,3}+$$
$$-Q_{1,0,3}^2Q_{2,0,2}^3+\frac{2}{9}Q_{1,0,3}^4Q_{2,0,2}+\frac{1}{6}Q_{1,0,3}^3Q_{2,0,2}^2-\frac{16}{81}Q_{2,0,2}^3Q_{0,0,4}^2+\frac{2}{3}Q_{2,1,1}Q_{1,0,3}^3\Big)y^4+$$

Results

Theorem

$$+\left(\tfrac{1}{2}-\tfrac{1}{6}Q_{1,0,3}^3-\tfrac{2}{9}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}+2Q_{2,1,1}Q_{1,0,3}-\tfrac{4}{3}Q_{2,1,1}Q_{0,0,4}+\tfrac{1}{6}Q_{1,0,3}^2Q_{2,0,2}+\right.$$
$$\left.+Q_{2,0,2}^2Q_{1,0,3}-\tfrac{4}{9}Q_{2,0,2}^2Q_{0,0,4}\right)xy^2p+\left(\tfrac{1}{2}Q_{1,0,3}+\tfrac{1}{6}Q_{1,0,3}^4-\tfrac{8}{27}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}^2+\right.$$
$$+\tfrac{8}{3}Q_{1,0,3}Q_{2,1,1}Q_{0,0,4}+\tfrac{4}{3}Q_{1,0,3}^2Q_{2,0,2}Q_{0,0,4}-\frac{8}{3}Q_{1,0,3}Q_{2,0,2}^2Q_{0,0,4}+$$
$$-\tfrac{8}{3}Q_{2,0,2}Q_{2,1,1}Q_{0,0,4}+\tfrac{16}{27}Q_{2,0,2}^2Q_{0,0,4}^2-\tfrac{2}{9}Q_{1,0,3}^3Q_{0,0,4}-\tfrac{3}{2}Q_{1,0,3}^3Q_{2,0,2}+$$
$$\left.+3Q_{1,0,3}^2Q_{2,0,2}^2\right)y^3p+\left(\tfrac{80}{9}Q_{2,1,1}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}+\tfrac{2}{3}Q_{1,0,3}^2-\tfrac{2}{9}Q_{1,0,3}^5+\right.$$
$$-8Q_{2,1,1}Q_{2,0,2}Q_{1,0,3}^2+\tfrac{32}{9}Q_{1,0,3}Q_{2,0,2}^3Q_{0,0,4}-\tfrac{64}{27}Q_{2,0,2}Q_{0,0,4}^2Q_{2,1,1}+$$

Results

Theorem

$$-\frac{32}{81}Q_{1,0,3}Q_{2,0,2}^2Q_{0,0,4}^2-\frac{16}{9}Q_{1,0,3}^2Q_{0,0,4}Q_{2,1,1}-\frac{16}{27}Q_{1,0,3}^3Q_{2,0,2}Q_{0,0,4}+$$
$$+\frac{16}{27}Q_{1,0,3}^2Q_{2,0,2}^2Q_{0,0,4}+\frac{8}{9}Q_{0,0,4}Q_{2,0,2}-2Q_{2,0,2}Q_{1,0,3}-4Q_{1,0,3}^2Q_{2,0,2}^3+$$
$$+\frac{8}{9}Q_{1,0,3}^4Q_{2,0,2}+\frac{2}{3}Q_{1,0,3}^3Q_{2,0,2}^2-\frac{64}{81}Q_{2,0,2}^3Q_{0,0,4}^2+\frac{8}{3}Q_{2,1,1}Q_{1,0,3}^3\Big)xy^3p+$$
$$+\Big(\frac{16}{9}Q_{2,1,1}Q_{2,0,2}^2Q_{1,0,3}Q_{0,0,4}-4Q_{0,0,4}Q_{1,0,3}^2Q_{2,0,2}Q_{2,1,1}+$$
$$+\frac{260}{27}Q_{0,0,4}^2Q_{1,0,3}Q_{2,0,2}Q_{2,1,1}+\frac{9}{8}Q_{1,0,3}^3-\frac{7}{72}Q_{1,0,3}^6-\frac{1}{4}+\frac{5}{6}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}+$$

Results

Theorem

$$-Q_{2,1,1}Q_{1,0,3} + \tfrac{1}{6}Q_{2,1,1}Q_{0,0,4} - \frac{3}{8}Q_{1,0,3}^2 Q_{2,0,2} - \tfrac{1}{2}Q_{2,0,2}^2 Q_{1,0,3} + \tfrac{2}{3}Q_{2,0,2}^2 Q_{0,0,4} +$$
$$+\tfrac{23}{12}Q_{1,0,3}^4 Q_{2,1,1} + \tfrac{16}{27}Q_{2,0,2}^3 Q_{0,0,4}^3 + \tfrac{4}{27}Q_{1,0,3}^4 Q_{0,0,4}^2 + \tfrac{1}{6}Q_{1,0,3}^5 Q_{0,0,4} - \tfrac{5}{3}Q_{1,0,3}^2 Q_{0,0,4}$$

Results

Theorem

$$+\tfrac{4}{3}Q_{2,1,1}^2Q_{0,0,4}^2-\tfrac{11}{9}Q_{1,0,3}^5Q_{2,0,2}+\tfrac{401}{72}Q_{1,0,3}^4Q_{2,0,2}^2-\tfrac{37}{12}Q_{1,0,3}^3Q_{2,0,2}^3-\tfrac{32}{81}Q_{2,0,2}^4Q_{0,0,4}^2+$$
$$+\tfrac{406}{81}Q_{1,0,3}^2Q_{2,0,2}^2Q_{0,0,4}^2-2Q_{2,1,1}^2Q_{1,0,3}Q_{0,0,4}-\tfrac{308}{81}Q_{1,0,3}Q_{2,0,2}^3Q_{0,0,4}^2-\tfrac{5}{12}Q_{2,1,1}Q_{2,0,2}Q_{1,0,3}^3+$$
$$-\tfrac{7}{9}Q_{0,0,4}Q_{1,0,3}^3Q_{2,1,1}-\tfrac{128}{27}Q_{0,0,4}^2Q_{2,0,2}^2Q_{2,1,1}-\tfrac{32}{9}Q_{0,0,4}^2Q_{1,0,3}^2Q_{2,1,1}-\tfrac{20}{27}Q_{1,0,3}^3Q_{0,0,4}^2Q_{2,0,2}+$$
$$+\tfrac{16}{81}Q_{1,0,3}^2Q_{0,0,4}^3Q_{2,0,2}-\tfrac{56}{81}Q_{1,0,3}Q_{0,0,4}^3Q_{2,0,2}^2-\tfrac{575}{54}Q_{1,0,3}^3Q_{2,0,2}^2Q_{0,0,4}+\tfrac{179}{27}Q_{1,0,3}^2Q_{2,0,2}^3Q_{0,0,4}+$$
$$+\tfrac{8}{9}Q_{2,0,2}^4Q_{1,0,3}Q_{0,0,4}+\tfrac{44}{27}Q_{1,0,3}^4Q_{2,0,2}Q_{0,0,4}\Big)y^4p\Big)\tfrac{\partial}{\partial p}+$$
$$+\sum_{i+j+k\geq 6}C_{i,j,k}^{1,x}x^iy^jp^k\tfrac{\partial}{\partial x}+\sum_{i+j+k\geq 6}C_{i,j,k}^{1,y}x^iy^jp^k\tfrac{\partial}{\partial y}+\sum_{i+j+k\geq 6}C_{i,j,k}^{1,p}x^iy^jp^k\tfrac{\partial}{\partial p},$$

Results

Theorem

$$e_2 = \Bigg(\Big(2Q_{2,0,2}Q_{1,0,3} - \tfrac{4}{3}Q_{1,0,3}Q_{0,0,4}\Big)x + \Big(-4Q_{2,1,1}Q_{1,0,3} + \tfrac{8}{3}Q_{2,1,1}Q_{0,0,4} - \tfrac{1}{3}Q_{2,0,2}Q_{1,0,3}^2 +$$

$$-2Q_{2,0,2}^2Q_{1,0,3} + \tfrac{8}{9}Q_{2,0,2}^2Q_{0,0,4} - 1 + \tfrac{1}{3}Q_{1,0,3}^3 + \tfrac{4}{9}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}\Big)x^2 +$$

$$+\Big(-\tfrac{1}{12}Q_{2,1,1}\Big)x^4 + \Big(\tfrac{1}{90}Q_{2,1,1}Q_{1,0,3} - \tfrac{1}{9}Q_{2,0,2}Q_{2,1,1}\Big)x^5\Bigg)\frac{\partial}{\partial x}$$

$$+\Bigg(1 + \Big(-Q_{1,0,3}^2 + Q_{2,0,2}Q_{1,0,3} - \tfrac{4}{3}Q_{2,0,2}Q_{0,0,4} + \tfrac{4}{3}Q_{1,0,3}Q_{0,0,4}\Big)y +$$

$$+\Big(-1 + \tfrac{4}{9}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4} - 4Q_{2,1,1}Q_{1,0,3} + \tfrac{8}{3}Q_{2,1,1}Q_{0,0,4} - \tfrac{1}{3}Q_{2,0,2}Q_{1,0,3}^2 +$$

Results

Theorem

$$-2Q_{2,0,2}^2Q_{1,0,3}+\frac{8}{9}Q_{2,0,2}^2Q_{0,0,4}+\frac{1}{3}Q_{1,0,3}^3\Big)xy+\Big(-\frac{1}{2}+\frac{1}{6}Q_{1,0,3}^3+\frac{2}{9}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}+$$
$$-2Q_{2,1,1}Q_{1,0,3}+\frac{4}{3}Q_{2,1,1}Q_{0,0,4}-\frac{1}{6}Q_{2,0,2}Q_{1,0,3}^2-Q_{2,0,2}^2Q_{1,0,3}+\frac{4}{9}Q_{2,0,2}^2Q_{0,0,4}\Big)y^3+$$
$$+\Big(-\frac{8}{3}Q_{2,1,1}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}-\frac{1}{2}Q_{1,0,3}^2+\frac{1}{6}Q_{1,0,3}^5-\frac{8}{3}Q_{1,0,3}Q_{0,0,4}^2Q_{2,1,1}+$$
$$-\frac{4}{9}Q_{1,0,3}^2Q_{0,0,4}^2Q_{2,0,2}-\frac{4}{3}Q_{1,0,3}Q_{2,0,2}^3Q_{0,0,4}+\frac{16}{9}Q_{2,0,2}Q_{0,0,4}^2Q_{2,1,1}-\frac{16}{27}Q_{1,0,3}Q_{2,0,2}^2Q_{0,0,4}^2+$$
$$+\frac{16}{3}Q_{1,0,3}^2Q_{0,0,4}Q_{2,1,1}+\frac{7}{9}Q_{1,0,3}^3Q_{2,0,2}Q_{0,0,4}+\frac{20}{9}Q_{1,0,3}^2Q_{2,0,2}^2Q_{0,0,4}-\frac{2}{3}Q_{2,0,2}Q_{0,0,4}+$$
$$+Q_{1,0,3}Q_{0,0,4}-Q_{1,0,3}^3Q_{2,0,2}^2-\frac{1}{6}Q_{1,0,3}^4Q_{2,0,2}+\frac{16}{27}Q_{2,0,2}^3Q_{0,0,4}^2-2Q_{2,1,1}Q_{1,0,3}^3+$$

Results

Theorem

$$-\frac{1}{3}Q_{1,0,3}^4 Q_{0,0,4}\Big)y^4 + \Big(-\frac{304}{45}Q_{1,0,3}Q_{0,0,4}^3 Q_{2,1,1}Q_{2,0,2} + \frac{8}{5}Q_{2,1,1}Q_{2,0,2}^2 Q_{0,0,4}^2 Q_{1,0,3} +$$
$$-\frac{36}{5}Q_{2,1,1}Q_{1,0,3}^3 Q_{2,0,2}Q_{0,0,4} - 4Q_{2,1,1}Q_{1,0,3}^2 Q_{2,0,2}^2 Q_{0,0,4} + \frac{44}{3}Q_{1,0,3}^2 Q_{2,1,1}Q_{0,0,4}^2 Q_{2,0,2} +$$
$$-\frac{3}{10}Q_{0,0,4} - \frac{9}{20}Q_{1,0,3}^4 + \frac{3}{20}Q_{1,0,3}^7 - Q_{1,0,3}Q_{2,0,2}^2 Q_{0,0,4} - \frac{17}{10}Q_{1,0,3}^2 Q_{2,0,2}Q_{0,0,4} +$$
$$+\frac{64}{15}Q_{2,1,1}Q_{0,0,4}^3 Q_{1,0,3}^2 + \frac{32}{45}Q_{1,0,3}^3 Q_{2,0,2}Q_{0,0,4}^3 - \frac{24}{5}Q_{2,1,1}^2 Q_{1,0,3}^2 Q_{0,0,4} + \frac{3}{5}Q_{2,1,1}Q_{1,0,3}^4 Q_{2,0,2} +$$
$$-\frac{12}{5}Q_{2,1,1}Q_{1,0,3}Q_{0,0,4} + \frac{32}{5}Q_{2,1,1}^2 Q_{0,0,4}^2 Q_{1,0,3} + \frac{16}{45}Q_{1,0,3}^2 Q_{0,0,4}^3 Q_{2,0,2}^2 + \frac{4}{3}Q_{1,0,3}^5 Q_{0,0,4}Q_{2,0,2} +$$
$$-\frac{32}{15}Q_{1,0,3}^4 Q_{0,0,4}^2 Q_{2,0,2} + \frac{44}{5}Q_{1,0,3}^4 Q_{2,1,1}Q_{0,0,4} - \frac{28}{9}Q_{1,0,3}^3 Q_{2,0,2}^2 Q_{0,0,4}^2 + \frac{32}{45}Q_{2,0,2}^2 Q_{0,0,4}^3 Q_{2,1,1} +$$

Results

Theorem

$$-\frac{256}{135}Q_{2,0,2}^3Q_{0,0,4}^3Q_{1,0,3}-\frac{172}{15}Q_{1,0,3}^3Q_{0,0,4}^2Q_{2,1,1}+\frac{53}{15}Q_{2,0,2}^2Q_{1,0,3}^4Q_{0,0,4}+$$
$$-\frac{52}{15}Q_{2,0,2}^3Q_{1,0,3}^3Q_{0,0,4}-\frac{4}{5}Q_{2,0,2}^4Q_{1,0,3}^2Q_{0,0,4}+\frac{56}{9}Q_{2,0,2}^3Q_{1,0,3}^2Q_{0,0,4}^2-\frac{32}{45}Q_{2,0,2}^4Q_{1,0,3}Q_{0,0,4}^2+$$
$$+\frac{38}{15}Q_{1,0,3}Q_{0,0,4}^2Q_{2,0,2}-\frac{3}{5}Q_{1,0,3}^6Q_{0,0,4}-\frac{17}{20}Q_{2,0,2}^2Q_{1,0,3}^5+\frac{3}{10}Q_{2,0,2}^3Q_{1,0,3}^4-\frac{1}{5}Q_{2,0,2}Q_{1,0,3}^6+$$
$$-\frac{4}{15}Q_{2,0,2}^2Q_{0,0,4}^2+\frac{19}{10}Q_{1,0,3}^3Q_{0,0,4}+\frac{3}{20}Q_{2,0,2}Q_{1,0,3}^3-\frac{32}{15}Q_{2,1,1}^2Q_{0,0,4}^3+\frac{8}{15}Q_{1,0,3}^5Q_{0,0,4}^2+$$
$$+\frac{8}{5}Q_{2,1,1}Q_{0,0,4}^2-\frac{8}{5}Q_{1,0,3}^2Q_{0,0,4}^2-\frac{9}{5}Q_{2,1,1}Q_{1,0,3}^5+\frac{64}{135}Q_{2,0,2}^4Q_{0,0,4}^3\Big)y^5\Bigg)\frac{\partial}{\partial y}+$$

Results

Theorem

$$
+\Bigg(\Big(-1+\tfrac{1}{3}Q_{1,0,3}^{3}+\tfrac{4}{9}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}-4Q_{2,1,1}Q_{1,0,3}+\tfrac{8}{3}Q_{2,1,1}Q_{0,0,4}-\tfrac{1}{3}Q_{2,0,2}Q_{1,0,3}^{2}+
$$

$$
-2Q_{2,0,2}^{2}Q_{1,0,3}+\tfrac{8}{9}Q_{2,0,2}^{2}Q_{0,0,4}\Big)y+\Big(-Q_{1,0,3}^{2}-Q_{2,0,2}Q_{1,0,3}+\tfrac{8}{3}Q_{1,0,3}Q_{0,0,4}+
$$

$$
-\tfrac{4}{3}Q_{2,0,2}Q_{0,0,4}\Big)p+\Big(1-\tfrac{4}{9}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}+4Q_{2,1,1}Q_{1,0,3}-\tfrac{8}{3}Q_{2,1,1}Q_{0,0,4}+
$$

$$
+\tfrac{1}{3}Q_{2,0,2}Q_{1,0,3}^{2}+2Q_{2,0,2}^{2}Q_{1,0,3}-\tfrac{8}{9}Q_{2,0,2}^{2}Q_{0,0,4}-\tfrac{1}{3}Q_{1,0,3}^{3}\Big)xp+\Big(-\tfrac{3}{2}+\tfrac{1}{2}Q_{1,0,3}^{3}+
$$

$$
+\tfrac{2}{3}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}-6Q_{2,1,1}Q_{1,0,3}+4Q_{2,1,1}Q_{0,0,4}-\tfrac{1}{2}Q_{2,0,2}Q_{1,0,3}^{2}-3Q_{2,0,2}^{2}Q_{1,0,3}+
$$

$$
+\tfrac{4}{3}Q_{2,0,2}^{2}Q_{0,0,4}\Big)y^{2}p+\tfrac{1}{3}Q_{2,1,1}x^{3}p+\Big(-\tfrac{32}{3}Q_{2,1,1}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}-2Q_{1,0,3}^{2}+\tfrac{2}{3}Q_{1,0,3}^{5}+
$$

Results

Theorem

$$+\frac{64}{3}Q_{1,0,3}^2Q_{2,1,1}Q_{0,0,4}+\frac{28}{9}Q_{1,0,3}^3Q_{2,0,2}Q_{0,0,4}+\frac{64}{9}Q_{2,0,2}Q_{0,0,4}^2Q_{2,1,1}-\frac{64}{27}Q_{2,0,2}^2Q_{0,0,4}^2Q_{1,0,3}+$$
$$-\frac{32}{3}Q_{1,0,3}Q_{0,0,4}^2Q_{2,1,1}-\frac{16}{9}Q_{1,0,3}^2Q_{0,0,4}^2Q_{2,0,2}+\frac{80}{9}Q_{1,0,3}^2Q_{2,0,2}^2Q_{0,0,4}-\frac{16}{3}Q_{2,0,2}^3Q_{1,0,3}Q_{0,0,4}+$$
$$+4Q_{1,0,3}Q_{0,0,4}-\frac{8}{3}Q_{2,0,2}Q_{0,0,4}-4Q_{1,0,3}^3Q_{2,0,2}^2+\frac{64}{27}Q_{2,0,2}^3Q_{0,0,4}^2-\frac{4}{3}Q_{1,0,3}^4Q_{0,0,4}-\frac{2}{3}Q_{1,0,3}^4Q_{2,0}$$
$$8Q_{2,1,1}Q_{1,0,3}^3\Big)y^3p+\Big(-\frac{1}{18}Q_{2,1,1}Q_{1,0,3}+\frac{5}{9}Q_{2,0,2}Q_{2,1,1}\Big)x^4p+\Big(-\frac{304}{9}Q_{1,0,3}Q_{0,0,4}^3Q_{2,1,1}Q_{2,0}$$
$$+8Q_{2,1,1}Q_{2,0,2}^2Q_{0,0,4}^2Q_{1,0,3}-36Q_{2,1,1}Q_{1,0,3}^3Q_{2,0,2}Q_{0,0,4}-20Q_{2,1,1}Q_{1,0,3}^2Q_{2,0,2}^2Q_{0,0,4}+$$
$$+\frac{220}{3}Q_{1,0,3}^2Q_{2,1,1}Q_{0,0,4}^2Q_{2,0,2}-\frac{3}{2}Q_{0,0,4}-\frac{9}{4}Q_{1,0,3}^4+\frac{3}{4}Q_{1,0,3}^7-\frac{52}{3}Q_{2,0,2}^3Q_{1,0,3}^3Q_{0,0,4}+$$
$$-4Q_{2,0,2}^4Q_{1,0,3}^2Q_{0,0,4}+\frac{280}{9}Q_{2,0,2}^3Q_{1,0,3}^2Q_{0,0,4}^2-\frac{32}{9}Q_{2,0,2}^4Q_{1,0,3}Q_{0,0,4}^2-\frac{17}{2}Q_{1,0,3}^2Q_{2,0,2}Q_{0,0,4}+$$
$$+\frac{38}{3}Q_{1,0,3}Q_{0,0,4}^2Q_{2,0,2}+\frac{64}{3}Q_{2,1,1}Q_{0,0,4}^3Q_{1,0,3}^2+\frac{32}{9}Q_{1,0,3}^3Q_{2,0,2}Q_{0,0,4}^3-24Q_{2,1,1}^2Q_{1,0,3}^2Q_{0,0,4}+$$
$$+3Q_{2,1,1}Q_{1,0,3}^4Q_{2,0,2}-12Q_{2,1,1}Q_{1,0,3}Q_{0,0,4}+32Q_{2,1,1}^2Q_{0,0,4}^2Q_{1,0,3}+\frac{16}{9}Q_{1,0,3}^2Q_{0,0,4}^3Q_{2,0,2}^2+$$
$$-5Q_{1,0,3}Q_{2,0,2}^2Q_{0,0,4}+\frac{20}{3}Q_{1,0,3}^5Q_{0,0,4}Q_{2,0,2}-\frac{32}{3}Q_{1,0,3}^4Q_{0,0,4}^2Q_{2,0,2}+44Q_{1,0,3}^4Q_{2,1,1}Q_{0,0,4}+$$

Results

Theorem

$$-\frac{140}{9}Q_{1,0,3}^3Q_{2,0,2}^2Q_{0,0,4}^2+\frac{32}{9}Q_{2,0,2}^2Q_{0,0,4}^3Q_{2,1,1}-\frac{256}{27}Q_{2,0,2}^3Q_{0,0,4}^3Q_{1,0,3}-\frac{172}{3}Q_{1,0,3}^3Q_{0,0,4}^2Q_{2,1,1}-$$
$$+\frac{53}{3}Q_{2,0,2}^2Q_{1,0,3}^4Q_{0,0,4}-Q_{2,0,2}Q_{1,0,3}^6+\frac{64}{27}Q_{2,0,2}^4Q_{0,0,4}^3-3Q_{1,0,3}^6Q_{0,0,4}-\frac{17}{4}Q_{2,0,2}^2Q_{1,0,3}^5+$$
$$+\frac{3}{2}Q_{2,0,2}^3Q_{1,0,3}^4-\frac{4}{3}Q_{2,0,2}^2Q_{0,0,4}^2+\frac{19}{2}Q_{1,0,3}^3Q_{0,0,4}+\frac{3}{4}Q_{2,0,2}Q_{1,0,3}^3-\frac{32}{3}Q_{2,1,1}^2Q_{0,0,4}^3+$$
$$+\frac{8}{3}Q_{1,0,3}^5Q_{0,0,4}^2+8Q_{2,1,1}Q_{0,0,4}^2-8Q_{1,0,3}^2Q_{0,0,4}^2-9Q_{2,1,1}Q_{1,0,3}^5\Big)y^4p\Big)\frac{\partial}{\partial p}+$$
$$+\sum_{i+j+k\geq 6}C_{i,j,k}^{2,x}x^iy^jp^k\frac{\partial}{\partial x}+\sum_{i+j+k\geq 6}C_{i,j,k}^{2,y}x^iy^jp^k\frac{\partial}{\partial y}+\sum_{i+j+k\geq 6}C_{i,j,k}^{2,p}x^iy^jp^k\frac{\partial}{\partial p},$$

Results

Theorem

$$e_3 = \Big(\Big(-2Q_{1,0,3} + \tfrac{4}{3}Q_{0,0,4}\Big)x + \Big(3Q_{2,0,2}Q_{1,0,3} - \tfrac{4}{3}Q_{2,0,2}Q_{0,0,4} - Q_{1,0,3}^2\Big)x^2 - \tfrac{1}{3}x^3 +$$

$$-\tfrac{1}{6}Q_{2,0,2}x^4 + \Big(\tfrac{1}{90}Q_{2,0,2}Q_{1,0,3} - \tfrac{1}{12}Q_{2,1,1} - \tfrac{1}{9}Q_{2,0,2}^2\Big)x^5\Big)\frac{\partial}{\partial x} +$$

$$+\Big(x + \Big(-Q_{1,0,3} - \tfrac{4}{3}Q_{0,0,4}\Big)y + \Big(-Q_{1,0,3}^2 + 3Q_{2,0,2}Q_{1,0,3} - \tfrac{4}{3}Q_{2,0,2}Q_{0,0,4}\Big)xy +$$

$$+\tfrac{3}{2}y^2 + \Big(-2Q_{1,0,3}Q_{0,0,4} + \tfrac{2}{3}Q_{2,0,2}Q_{0,0,4} + \tfrac{1}{2}Q_{1,0,3}^2 + \tfrac{3}{2}Q_{2,0,2}Q_{1,0,3}\Big)y^3 +$$

$$+\Big(\tfrac{8}{3}Q_{0,0,4}^2Q_{1,0,3}^2 - \tfrac{8}{3}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}^2 - 2Q_{1,0,3}^3Q_{0,0,4} - \tfrac{5}{3}Q_{1,0,3}^2Q_{2,0,2}Q_{0,0,4} +$$

Results

Theorem

$$+\tfrac{8}{3}Q_{1,0,3}Q_{2,0,2}^2Q_{0,0,4}+\tfrac{1}{4}Q_{1,0,3}^4+\tfrac{5}{4}Q_{1,0,3}^3Q_{2,0,2}-\tfrac{4}{3}Q_{0,0,4}^2Q_{2,1,1}+2Q_{1,0,3}Q_{2,1,1}Q_{0,0,4}+$$

$$+\tfrac{1}{2}Q_{0,0,4}\Big)y^4+\tfrac{1}{12}Q_{2,1,1}x^4y+\Big(\tfrac{1}{6}-\tfrac{2}{27}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}+\tfrac{2}{3}Q_{2,1,1}Q_{1,0,3}-\tfrac{4}{9}Q_{2,1,1}Q_{0,0,4}+$$

$$+\tfrac{1}{18}Q_{2,0,2}Q_{1,0,3}^2+\tfrac{1}{3}Q_{2,0,2}^2Q_{1,0,3}-\tfrac{4}{27}Q_{2,0,2}^2Q_{0,0,4}-\tfrac{1}{18}Q_{1,0,3}^3\Big)x^3y^2+$$

$$+\Big(\tfrac{18}{5}Q_{0,0,4}Q_{1,0,3}^2Q_{2,0,2}Q_{2,1,1}+\tfrac{4}{3}Q_{0,0,4}^2Q_{1,0,3}Q_{2,0,2}Q_{2,1,1}-\tfrac{1}{20}Q_{1,0,3}^3+\tfrac{3}{20}Q_{1,0,3}^6+$$

$$+\tfrac{3}{20}+\tfrac{5}{6}Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}+\tfrac{16}{3}Q_{0,0,4}^3Q_{1,0,3}Q_{2,1,1}-\tfrac{112}{45}Q_{0,0,4}^3Q_{2,1,1}Q_{2,0,2}+$$

Results

Theorem

$$+\frac{896}{135}Q_{1,0,3}^2Q_{0,0,4}^3Q_{2,0,2}-\frac{64}{45}Q_{1,0,3}Q_{0,0,4}^3Q_{2,0,2}^2-\frac{12}{5}Q_{1,0,3}^3Q_{0,0,4}^2Q_{2,0,2}-\frac{388}{45}Q_{1,0,3}^2Q_{2,0,2}^2Q_{0,0,4}^2+$$
$$+\frac{128}{45}Q_{1,0,3}Q_{2,0,2}^3Q_{0,0,4}^2+\frac{76}{15}Q_{1,0,3}^3Q_{2,0,2}^2Q_{0,0,4}+\frac{4}{3}Q_{1,0,3}^2Q_{2,0,2}^3Q_{0,0,4}-3Q_{1,0,3}^4Q_{2,0,2}Q_{0,0,4}+$$
$$+\frac{14}{5}Q_{0,0,4}Q_{1,0,3}^3Q_{2,1,1}-\frac{148}{15}Q_{0,0,4}^2Q_{1,0,3}^2Q_{2,1,1}+\frac{3}{5}Q_{2,1,1}Q_{1,0,3}-\frac{2}{5}Q_{2,1,1}Q_{0,0,4}+$$
$$+\frac{1}{20}Q_{2,0,2}Q_{1,0,3}^2+\frac{3}{10}Q_{2,0,2}^2Q_{1,0,3}-\frac{2}{15}Q_{2,0,2}^2Q_{0,0,4}+Q_{1,0,3}^5Q_{2,0,2}-\frac{32}{9}Q_{0,0,4}^3Q_{1,0,3}^3+$$
$$-2Q_{0,0,4}^2Q_{1,0,3}+\frac{14}{15}Q_{0,0,4}^2Q_{2,0,2}-\frac{64}{135}Q_{2,0,2}^3Q_{0,0,4}^3+\frac{224}{45}Q_{1,0,3}^4Q_{0,0,4}^2-\frac{9}{5}Q_{1,0,3}^5Q_{0,0,4}+$$
$$+\frac{7}{10}Q_{1,0,3}^2Q_{0,0,4}-\frac{7}{20}Q_{1,0,3}^4Q_{2,0,2}^2\Big)y^5\bigg)\frac{\partial}{\partial y}$$

Results

Theorem

$$\Big(1 + \big(- Q_{1,0,3}^2 + 3Q_{2,0,2}Q_{1,0,3} - \tfrac{4}{3}\,Q_{2,0,2}Q_{0,0,4}\big)y + \big(Q_{1,0,3} - \tfrac{8}{3}\,Q_{0,0,4}\big)p +$$
$$+\big(- 3Q_{2,0,2}Q_{1,0,3} + \tfrac{4}{3}\,Q_{2,0,2}Q_{0,0,4} + Q_{1,0,3}^2\big)xp + 3yp + x^2p + \big(- 6Q_{1,0,3}Q_{0,0,4}+$$
$$+2Q_{2,0,2}Q_{0,0,4} + \tfrac{3}{2}\,Q_{1,0,3}^2 + \tfrac{9}{2}\,Q_{2,0,2}Q_{1,0,3}\big)y^2p + \tfrac{1}{3}\,Q_{2,1,1}x^3y+$$
$$+\big(\tfrac{1}{2} - \tfrac{2}{9}\,Q_{1,0,3}Q_{2,0,2}Q_{0,0,4} + Q_{2,0,2}^2Q_{1,0,3} + 2Q_{2,1,1}Q_{1,0,3} - \tfrac{4}{3}\,Q_{2,1,1}Q_{0,0,4} + \tfrac{1}{6}\,Q_{2,0,2}Q_{1,0,3}^2+$$
$$-\tfrac{4}{9}\,Q_{2,0,2}^2Q_{0,0,4} - \tfrac{1}{6}\,Q_{1,0,3}^3\big)x^2y^2 + \tfrac{2}{3}\,Q_{2,0,2}x^3p + \big(\tfrac{32}{3}\,Q_{0,0,4}^2Q_{1,0,3}^2 - \tfrac{32}{3}\,Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}^2+$$
$$-8Q_{1,0,3}^3Q_{0,0,4} - \tfrac{20}{3}\,Q_{1,0,3}^2Q_{2,0,2}Q_{0,0,4} + \tfrac{32}{3}\,Q_{1,0,3}Q_{2,0,2}^2Q_{0,0,4} + Q_{1,0,3}^4 + 5Q_{1,0,3}^3Q_{2,0,2}+$$
$$-\tfrac{16}{3}\,Q_{0,0,4}^2Q_{2,1,1} + 8Q_{1,0,3}Q_{2,1,1}Q_{0,0,4} + 2Q_{0,0,4}\big)y^3p + \big(- \tfrac{1}{18}\,Q_{2,0,2} * Q_{1,0,3} + \tfrac{1}{2}\,Q_{2,1,1}+$$
$$+\tfrac{5}{9}\,Q_{2,0,2}^2\big)x^4p + \big(\tfrac{1}{3} - \tfrac{4}{27}\,Q_{1,0,3}Q_{2,0,2}Q_{0,0,4} + \tfrac{2}{3}\,Q_{2,0,2}^2Q_{1,0,3} + \tfrac{4}{3}\,Q_{2,1,1}Q_{1,0,3} - \tfrac{8}{9}\,Q_{2,1,1}Q_{0,0,4}+$$
$$+\tfrac{1}{9}\,Q_{2,0,2}Q_{1,0,3}^2 - \tfrac{8}{27}\,Q_{2,0,2}^2Q_{0,0,4} - \tfrac{1}{9}\,Q_{1,0,3}^3\big)x^3yp + \big(18Q_{0,0,4}Q_{1,0,3}^2Q_{2,0,2}Q_{2,1,1}+$$
$$+\tfrac{20}{3}\,Q_{0,0,4}^2Q_{1,0,3}Q_{2,0,2}Q_{2,1,1} - \tfrac{1}{4}\,Q_{1,0,3}^3 + \tfrac{3}{4}\,Q_{1,0,3}^6 + \tfrac{3}{4} + \tfrac{25}{6}\,Q_{1,0,3}Q_{2,0,2}Q_{0,0,4}+$$

Results

Theorem

$$+\tfrac{80}{3}\,Q_{0,0,4}^3\,Q_{1,0,3}\,Q_{2,1,1}-\tfrac{112}{9}\,Q_{0,0,4}^3\,Q_{2,1,1}\,Q_{2,0,2}-\tfrac{388}{9}\,Q_{1,0,3}^2\,Q_{2,0,2}^2\,Q_{0,0,4}^2+\tfrac{128}{9}\,Q_{1,0,3}\,Q_{2,0,2}^3\,Q_{0,0,4}^2+$$

$$+14\,Q_{0,0,4}\,Q_{1,0,3}^3\,Q_{2,1,1}-\tfrac{148}{3}\,Q_{0,0,4}^2\,Q_{1,0,3}^2\,Q_{2,1,1}-12\,Q_{1,0,3}^3\,Q_{0,0,4}^2\,Q_{2,0,2}+\tfrac{896}{27}\,Q_{1,0,3}^2\,Q_{0,0,4}^3\,Q_{2,0,2}+$$

$$-\tfrac{64}{9}\,Q_{1,0,3}\,Q_{0,0,4}^3\,Q_{2,0,2}^2+\tfrac{76}{3}\,Q_{1,0,3}^3\,Q_{2,0,2}^2\,Q_{0,0,4}+\tfrac{20}{3}\,Q_{1,0,3}^2\,Q_{2,0,2}^3\,Q_{0,0,4}-15\,Q_{1,0,3}^4\,Q_{2,0,2}\,Q_{0,0,4}\check{r}+$$

$$+\tfrac{3}{2}\,Q_{2,0,2}^2\,Q_{1,0,3}+3\,Q_{2,1,1}\,Q_{1,0,3}-2\,Q_{2,1,1}\,Q_{0,0,4}+\tfrac{1}{4}\,Q_{2,0,2}\,Q_{1,0,3}^2-\tfrac{2}{3}\,Q_{2,0,2}^2\,Q_{0,0,4}+$$

$$-\tfrac{160}{9}\,Q_{0,0,4}^3\,Q_{1,0,3}^3-10\,Q_{0,0,4}^2\,Q_{1,0,3}+\tfrac{14}{3}\,Q_{0,0,4}^2\,Q_{2,0,2}-\tfrac{64}{27}\,Q_{2,0,2}^3\,Q_{0,0,4}^3+\tfrac{224}{9}\,Q_{1,0,3}^4\,Q_{0,0,4}^2+$$

$$-9\,Q_{1,0,3}^5\,Q_{0,0,4}+\tfrac{7}{2}\,Q_{1,0,3}^2\,Q_{0,0,4}+5\,Q_{1,0,3}^5\,Q_{2,0,2}-\tfrac{7}{4}\,Q_{1,0,3}^4\,Q_{2,0,2}^2\Big)\,y^4p\Big)\,\frac{\partial}{\partial p}+$$

$$+\sum_{i+j+k\geq 6}C_{i,j,k}^{3,x}\,x^iy^jp^k\,\frac{\partial}{\partial x}+\sum_{i+j+k\geq 6}C_{i,j,k}^{3,y}\,x^iy^jp^k\,\frac{\partial}{\partial y}+\sum_{i+j+k\geq 6}C_{i,j,k}^{3,p}\,x^iy^jp^k\,\frac{\partial}{\partial p},$$

Results

Theorem

satisfying

$$[e_1, e_2] = (2Q_{2,0,2}Q_{1,0,3} - \tfrac{4}{3}Q_{1,0,3}Q_{0,0,4})e_1 + (\tfrac{1}{3}Q_{1,0,3} + \tfrac{2}{3}Q_{2,0,2})e_2 + Q_{2,1,1}e_3,$$

$$[e_2, e_3] = (-Q_{1,0,3} - \tfrac{4}{3}Q_{0,0,4})e_2 + (-\tfrac{8}{3}Q_{1,0,3}Q_{0,0,4} - 4Q_{2,0,2}Q_{1,0,3})e_3,$$

$$[e_1, e_3] = (-2Q_{1,0,3} + \tfrac{4}{3}Q_{0,0,4})e_1 + e_2 + (\tfrac{2}{3}Q_{1,0,3} - \tfrac{2}{3}Q_{2,0,2})e_3,$$

Results

Theorem

and the set $\{Q_{1,0,3}, Q_{0,0,4}, Q_{2,0,2}, Q_{2,1,1}\}$ *satisfying equations generating an ideal with the Gröbner basis:*

$$\begin{aligned}
\mathbb{B}_1 &:= -Q_{1,0,3}(2Q_{0,0,4} - 3Q_{2,0,2}),\\
\mathbb{B}_2 &:= Q_{2,1,1}Q_{1,0,3},\\
\mathbb{B}_3 &:= -8Q_{0,0,4}Q_{2,0,2} + 12Q_{2,0,2}^2 + 27Q_{2,1,1},\\
\mathbb{B}_4 &= 4Q_{0,0,4}^2Q_{2,1,1} - 2Q_{0,0,4} + 3Q_{2,0,2},\\
\mathbb{B}_5 &:= -16Q_{0,0,4}^2Q_{1,0,3} + 16Q_{0,0,4}^2Q_{2,0,2} - 6Q_{0,0,4}Q_{1,0,3}^2 + 9Q_{1,0,3}^3 - 27.
\end{aligned}$$

Results

Theorem

Precisely, we obtain a homogeneous model if, and only if,

$$\mathbb{B}_i = 0. \qquad (1 \leqslant i \leqslant 5)$$

Sommaire

The power series method

Assume

$$\{y_{xx} = Q(x, y, y_x)\} \xrightarrow{\varphi^{(2)}} \{y'_{x'x'} = R(x', y', y'_{x'})\},$$

with

$$\varphi : (x, y) \in \mathbb{C}^2 \longmapsto \big(f(x), g(x, y)\big) =: (x', y'),$$

where

$$f(x) = \sum_{n=0}^{\infty} f_n x^n = \sum_{i=0}^{n+1} f_i x^i + \mathrm{O}_x(n+2),$$

$$g(x, y) = \sum_{n=0}^{\infty} \sum_{i+j=n} g_{i,j} x^i y^j = \sum_{i+j=0}^{n+2} g_{i,j} x^i y^j + \mathrm{O}_{x,y}(n+3),$$

$n \in \mathbb{N}$.

The power series method

- From the fundamental and vectorial fundamental equations:

$$0 = f_x^3 R\left(f, g, \frac{g_x + g_y p}{f_x}\right) + g_x f_{xx} + g_y f_{xx} p - f_x g_{xx} - 2 f_x g_{xy} p - f_x g_{yy} p^2 - f_x g_y Q(x, y, p),$$

  and

$$0 = -B_{xx} + (A_{xx} - 2B_{xy})p + (2A_x - B_y)Q(x, y, p) - B_{yy} p^2 + \\ + AQ_x + BQ_y + \left[B_y + (B_y - A_x)p\right] Q_p,$$

  we extract information for normalizing the power series coefficients of Q and R order by order:

- At each order n, by looking at the fundamental equation and by making appropriate choices of coefficients $f_{n+1}, g_{i,j}$ with $i + j = n + 2$, we will normalize the power series coefficients of Q and R.
  At the end, such normalizations will end up with normal forms Q^{normal} of Q and with associated diffeomorphisms

$$\{y_{xx} = Q(x, y, y_x)\} \overset{\varphi_{\square}^{(2)} = \varphi_r^{(2)} \circ \cdots \circ \varphi_1^{(2)}}{\longrightarrow} \{y'_{x'x'} = Q^{\mathrm{normal}}(x', y', y'_{x'})\}.$$

The power series method

Example

For the branching $Q_{0,0,3} = Q_{1,0,2} = 1$,

$$Q^{\mathrm{normal}} = Q \text{ in the theorem.}$$

The power series method

- To find the associated Lie algebra structure, we look the vectorial fundamental equations with

$$\mathcal{L} = \left(\sum_{i=0}^{n+1} A_i x^i + \mathrm{O}_x(n+2) \right) \frac{\partial}{\partial x} + \left(\sum_{i+j=0}^{n+2} B_{i,j} x^i y^j + \mathrm{O}_{x,y}(n+3) \right) \frac{\partial}{\partial y},$$

tangent to

$$\left\{ y_{xx} = Q(x, y, y_x) \right\}.$$

- At each order n, by looking at the vectorial fundamental equation, we will normalize coefficients $A_{n+1}, B_{i,j}$ with $i + j = n + 2$, hence we will normalize the coefficients of $\mathcal{L}$.
At the end of the process, we will obtain vector fields of the form

$$\mathcal{L}^{\mathrm{normal}} = \alpha\Big(A_0, B_{0,0}, B_{1,0}, A_1, A_2, B_{0,1}\Big) \frac{\partial}{\partial x} + \beta\Big(A_0, B_{0,0}, B_{1,0}, A_1, A_2, B_{0,1}\Big) \frac{\partial}{\partial y},$$

with all other coefficients A_* and $B_{*,*}$ normalized, where α and β are linear in their 6 arguments.

The power series method

Next, to obtain the associated Lie algebras in all branches, we will prolong the concerned vector field to $J^1(\mathbb{C}^2_{x,y}) = \mathbb{C}^3_{x,y,p}$, getting

$$\mathcal{L}^{\mathrm{normal},1}.$$

Then putting one of the arguments to 1 and the others to 0, we get generators e_i with $1 \leqslant i \leqslant 3$ or $1 \leqslant i \leqslant 6$ of the associated Lie algebra given in the theorems.
For example :

$$e_1 := \mathcal{L}^{\mathrm{normal},1}(1, 0, ..., 0).$$

The power series method

- Remark : We cannot normalize the coefficients order by order at the infinity.
- $$\pi_{\leqslant n}\left(\sum_{i,j,k} Q_{i,j,k}x^i y^j p^k\right) := \sum_{i+j+k\leqslant n} Q_{i,j,k}x^i y^j p^k \qquad (n\in\mathbb{N}),$$
- In fact, normalization of the coefficients of φ will create a decreasing and stationary sequence of subsets of

$$G_{\mathrm{stab}} = \left\{\varphi \mid \{y_{xx} = Q(x,y,y_x)\} \overset{\varphi^{(2)}}{\longrightarrow} \{y'_{x'x'} = R(x',y',y'_{x'})\}\right\} :$$

$$G_{\mathrm{stab}} =: G^0_{\mathrm{stab}} \supset G^1_{\mathrm{stab}} \supset \dots$$

$$\dots \supset G^n_{\mathrm{stab}} =: \left\{\varphi \mid \{y_{xx} = \pi_{\leqslant n}(Q)(x,y,y_x)\} \overset{\varphi^{(2)}}{\longrightarrow} \{y'_{x'x'} = \pi_{\leqslant n}(R)(x',y',y'_{x'})\}\right\} \supset \dots$$

$$\dots \supset G^r_{\mathrm{stab}} = G^{r+1}_{\mathrm{stab}} = \dots,$$

- At the order r,the process of normalization stops.
- In geometric terms, we make G_{stab} act on $\{y_{xx} = Q(x,y,y_x)\}$.

The power series method

In the branch

$$Q_{0,0,3} = Q_{1,0,2} = 1.$$

- Assume $Q_{0,0,0} = R_{0,0,0} = 0$
- and φ fixes $(0,0,0)$ in J^1 :

$$f_0 = g_{0,0} = g_{1,0} = 0.$$

- φ is a local diffeo :

$$f_1 \neq 0, \; g_{0,1} \neq 0.$$

- Nouvelle projection :

$$\pi_n\left(\sum_{i,j,k} Q_{i,j,k} x^i y^j p^k\right) = \sum_{i+j+k=n} Q_{i,j,k} x^i y^j p^k \qquad (n\in\mathbb{N}),$$

The power series method

Order $n = 0$,

$$Q = \mathrm{O}_{x,y,p}(1) \text{ and } R = \mathrm{O}_{x',y',p'}(1),$$

$$f(x) = \sum_{i=0}^{0+1} f_i x^i + \mathrm{O}_x(2) \text{ and } g(x,y) = \sum_{i+j=0}^{2} g_{i,j} x^i y^j + \mathrm{O}_{x,y}(3).$$

We make G_{stab} act on $\{y_{xx} = Q(x, y, y_x)\}$ on the order 0 by applying π_0 to the fundamental equation :

$$0 = \pi_0\Big(f_x^3 R + g_x f_{xx} + g_y f_{xx} p - f_x g_{xx} - 2 f_x g_{xy} p - f_x g_{yy} p^2 - f_x g_y Q\Big).$$

Therefore

$$0 = -2 f_1 g_{2,0},$$

hence by $f_1 \neq 0$, we get

$$g_{2,0} = 0.$$

The power series method

Order $n = 1$,

$$Q = \sum_{i+j+k=1} Q_{i,j,k} x^i y^j p^k + \mathrm{O}_{x,y,p}(2) \text{ and } R = \sum_{i+j+k=1} R_{i,j,k} x'^i y'^j p'^k + \mathrm{O}_{x',y',p'}(2),$$

$$f(x) = \sum_{i=0}^{1+1} f_i x^i + \mathrm{O}_x(3) \text{ and } g(x,y) = \sum_{i+j=0}^{3} g_{i,j} x^i y^j + \mathrm{O}_{x,y}(4).$$

We look the action of G_{stab} on $\{y_{xx} = Q(x, y, y_x)\}$ at the order 1 by applying π_1 to the fundamental equation, we get

$$(E1)\ \ 0 = R_{1,0,0} f_1^4 - Q_{1,0,0} f_1 g_{0,1} - 6 f_1 g_{3,0},$$

$$(E2)\ \ 0 = R_{0,1,0} f_1^3 g_{0,1} + R_{0,0,1} f_1^2 g_{1,1} - Q_{0,1,0} f_1 g_{0,1} - 2 f_1 g_{2,1} + 2 f_2 g_{1,1},$$

$$(E3)\ \ 0 = R_{0,0,1} f_1^2 g_{0,1} - Q_{0,0,1} f_1 g_{0,1} - 2 f_1 g_{1,1} + 2 f_2 g_{0,1}.$$

The power series method

$$(E1)\ \ 0 = R_{1,0,0} f_1^4 - Q_{1,0,0} f_1 g_{0,1} - 6 f_1 g_{3,0},$$
$$(E2)\ \ 0 = R_{0,1,0} f_1^3 g_{0,1} + R_{0,0,1} f_1^2 g_{1,1} - Q_{0,1,0} f_1 g_{0,1} - 2 f_1 g_{2,1} + 2 f_2 g_{1,1},$$
$$(E3)\ \ 0 = R_{0,0,1} f_1^2 g_{0,1} - Q_{0,0,1} f_1 g_{0,1} - 2 f_1 g_{1,1} + 2 f_2 g_{0,1}.$$

- By the freeness of $g_{3,0}$ in $(E1)$ we can normalize $R_{1,0,0} = 0$ and by equivalence:
$$Q_{1,0,0} = R_{1,0,0} = 0.$$
So :
$$g_{3,0} = 0.$$
- By looking at $(E3)$, and by a choice of $g_{1,1}$, we can normalize $R_{0,0,1} = 0$ and by equivalence:
$$Q_{0,0,1} = R_{0,0,1} = 0.$$
Therefore $(E3)$ gives
$$g_{1,1} = g_{0,1} \frac{f_2}{f_1}.$$
- By the same idea, from $(E2)$ we get:
$$Q_{0,1,0} = R_{0,1,0} = 0,$$
$$g_{2,1} = g_{0,1} \left(\frac{f_2}{f_1} \right)^2.$$

The power series method

Conclusion :

Lemma

The subgroup $G^1_{\mathrm{stab}} \subset G^0_{\mathrm{stab}}$ *which sends* $Q = \mathrm{O}_{x,y,p}(2)$ *to* $R = \mathrm{O}_{x',y',p'}(2)$ *consists of fiber-preserving transformations such that*

$$f_0 = 0,\ f_1 \neq 0,\ g_{0,0} = g_{1,0} = g_{2,0} = g_{3,0} = 0,\ g_{0,1} \neq 0,$$

$$g_{i,1} = g_{0,1}\left(\frac{f_2}{f_1}\right)^i \qquad (1 \leqslant i \leqslant 2).$$

For order $n \geqslant 2$, it is the same process :
1) Normalization of the $Q_{i,j}$ and $R_{i,j}$.
2) Reducing of G_{stab}.

The power series method

About the vectorial fundamental equation, with

$$\mathcal{L} = \left(\sum_{i=0}^{1+1} A_i x^i + \mathrm{O}_x(n+2) \right) \frac{\partial}{\partial x} + \left(\sum_{i+j=0}^{1+2} B_{i,j} x^i y^j + \mathrm{O}_{x,y}(n+3) \right) \frac{\partial}{\partial y},$$

By applying π_0 to the vectorial fundamental equation, we get:

$$B_{2,0} = 0.$$

Then by π_1, we get

$$B_{3,0} = 0,$$
$$B_{2,1} = 0,$$
$$B_{1,1} = A_2.$$

Hence

$$\mathcal{L} = \left(A_0 + A_1 x + A_2 x^2 + \mathrm{O}_x(n+2) \right) \frac{\partial}{\partial x} + \left(B_{0,0} + B_{1,0} x + B_{0,1} y + \mathrm{O}_{x,y}(n+3) \right) \frac{\partial}{\partial x}$$

About the vectorial fundamental equation, we do the same: at each order n:
Normalization of A_{n+1} and $B_{i,j}$, with $i + j = n + 2$.

Sommaire

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