Geometrical Modeling of a Microstructured Material

A Scaled Material Modeling

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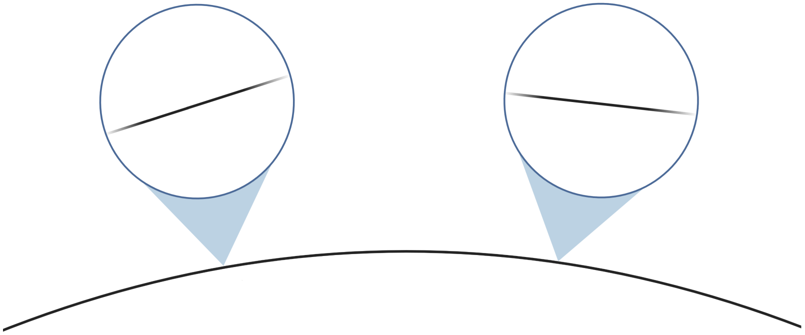
Introduction



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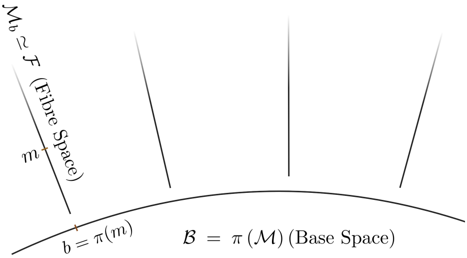


Figure: vertical representation

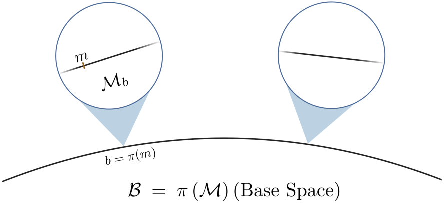


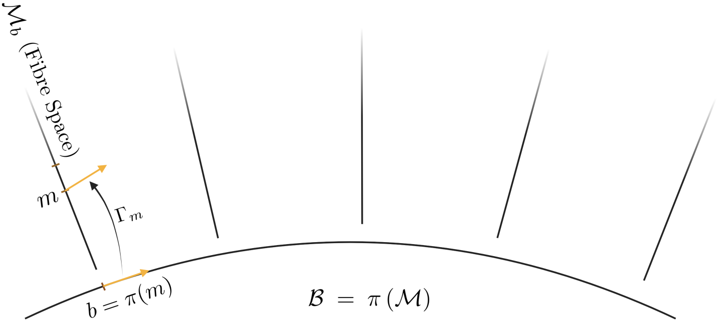
Figure: multi-scaled representation

Definition 1: Fibre bundle

Smooth manifolds $\mathcal{M}$ and $\mathcal{B}$ and a smooth projection $\pi : \mathcal{M} \to \mathcal{B}$ such that

$$\forall b \in \mathcal{B},\ \exists\, b \in \mathcal{U} \subset \mathcal{B}, \qquad \text{s.t.} \qquad \pi^{-1}(\mathcal{U}) \simeq \mathcal{U} \times \mathcal{F}$$

Ehresmann Connections



Definition 2: Connection – An horizontal/macroscopic lift

A smooth assignment at each $m \in \mathcal{M}$ of a smooth linear right inverse Γ_m of $\mathrm{d}\pi$:

$$\forall m \in \mathcal{M}, \qquad \mathrm{d}\pi \circ \Gamma_m = \mathrm{Id}_{\mathrm{T_b}\mathcal{B}}$$

Solder forms

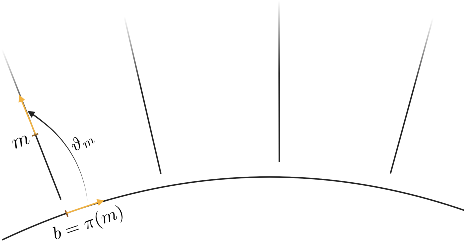


Figure: vertical representation

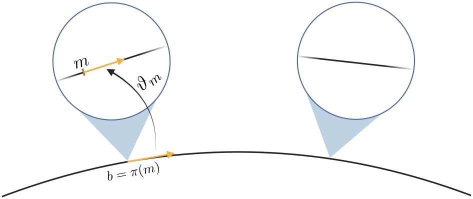


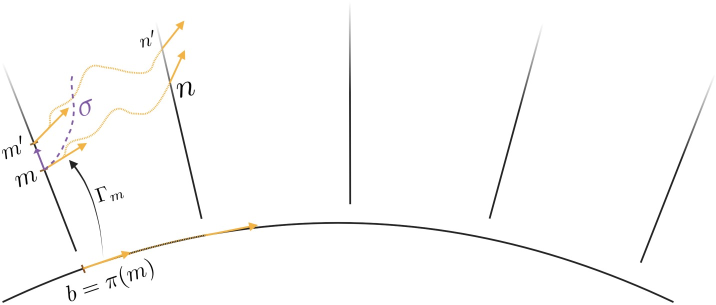
Figure: multi-scaled representation

Definition 3: Solder form – A vertical/microscopic lift

A smooth assignment at each $m \in \mathcal{M}$ of a smooth linear isomorphism:

$$\vartheta_m : \mathrm{T}_b\mathcal{B} \simeq \mathrm{V}_m(\mathcal{M}) \qquad \text{where } \mathrm{V}_m(\mathcal{M}) = \ker \mathrm{d}\pi_{|_{\mathrm{T}_m\mathcal{M}}}$$

Lifts and parallel transports



$$\mathbf{u} \in \mathrm{T}_b\mathcal{B}, \quad \sigma \in \mathrm{Sect}(\mathcal{M}), \quad \mathbf{A} : \mathcal{M} \to \mathcal{M}'$$

$$\nabla_{\mathbf{u}}\sigma \in \mathrm{V}_{\sigma(b)}(\mathcal{M}) = \mathcal{M}_b \qquad \nabla \cdot \sigma : \mathcal{B} \longrightarrow \mathbb{R}$$

$$\nabla_{\mathbf{u}}\mathbf{A} : \mathcal{M}_b \longrightarrow \mathcal{M}'_b \qquad \nabla \cdot \mathbf{A} \in \mathrm{Sect}(\mathcal{M}')$$

with:

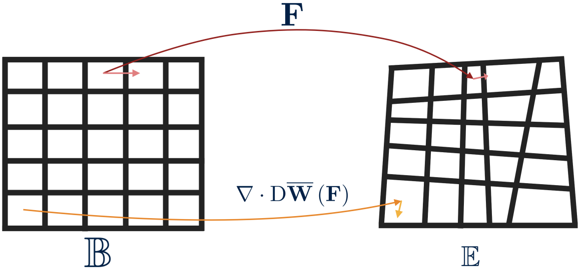
$$\nabla_{\mathbf{u}}A\sigma(b) = \nabla_{\mathbf{u}}(\mathbf{A}\sigma) - \mathbf{A}(\nabla_{\mathbf{u}}\sigma) \in \mathrm{V}_{\sigma(b)}(\mathcal{M}')$$

The Euclidean case

$$\varphi : \mathbb{B} \longrightarrow \mathbb{E} \qquad \mathbf{F} = \mathrm{d}\varphi : \mathrm{T}\mathbb{B} \longrightarrow \mathrm{T}\mathbb{E}$$

$$\overline{\mathbf{W}} : \mathcal{L}_\varphi\left(\mathrm{T}\mathbb{B}, \mathrm{T}\mathbb{E}\right) \ni \mathbf{F}_X \longmapsto \overline{\mathbf{W}}\left(\mathbf{F}_X\right) \in \mathbb{R}$$

$$\left[\mathrm{D}\overline{\mathbf{W}}\left(\mathbf{F}\right)\right]_X = \mathrm{d}\overline{\mathbf{W}}_X\left(\mathbf{F}_X\right) \in \mathcal{L}\left(\mathrm{T}^*_X\mathbb{B}, \mathrm{T}^*_x\mathbb{E}\right)$$



$X \in \mathbb{B}$ $\qquad$ $x = \varphi(X) \in \mathbb{E}$

A geometric formalism

Spatial metric:

$\mathbf{g} : \mathrm{T}\mathbb{E} \longrightarrow \mathrm{T}^*\mathbb{E}$ (linear, symmetrical and non-degenerate)

Material metric (right Cauchy-Green):

$\mathbf{C} = \mathbf{F}^T \mathbf{g} \mathbf{F} \in T_2^0(\mathbb{B})$ (linear, symmetrical and non-degenerate)

Isotropic behaviour:

$$\begin{aligned} \mathbf{W} :\ \mathcal{L}_{\mathrm{Id}}\left(\mathrm{T}\mathbb{B}, \mathrm{T}^*\mathbb{B}\right) &\longrightarrow \mathbb{R} \\ \mathbf{C}_X &\longmapsto \mathbf{W}\left(\mathbf{C}_X\right) \end{aligned} \qquad \begin{aligned} \mathrm{D}\mathbf{W} :\ T_2^0(\mathbb{B}) &\longrightarrow T_0^2(\mathbb{B}) \\ \mathbf{C} &\longmapsto \mathrm{D}\mathbf{W}\left(\mathbf{C}\right) \end{aligned}$$

where $\left[\mathrm{D}\mathbf{W}\left(\mathbf{C}\right)\right]_X = \mathrm{d}\mathbf{W}_X\left(\mathbf{C}_X\right) = \dfrac{1}{2}\mathbf{S}_X$

Summary

Geometry $(\mathbb{B}, \mathbf{G})$ and $(\mathbb{E}, \mathbf{g})$

Transformation: $\mathbf{F} : \mathrm{T}\mathbb{B} \rightarrow \mathrm{T}\mathbb{E}$

Behaviour: $\mathbf{W} : \mathcal{L}_{\mathrm{Id}}\left(\mathrm{T}\mathbb{B}, \mathrm{T}^{*}\mathbb{B}\right) \longrightarrow \mathbb{R}$

Equilibrium:
(without external forces)

$$
\begin{aligned}
\nabla \cdot \sigma &= \mathbf{0} \\
\nabla \cdot \left(\mathbf{F} \cdot \mathbf{S} \cdot \frac{\mathbf{F}^{-T}}{\det\left(\mathbf{F}\right)}\right) &= \\
\nabla \cdot \left(\mathbf{F} \cdot 2\mathrm{D}\mathbf{W}\left(\mathbf{C}\right) \cdot \frac{\mathbf{F}^{-T}}{\det\left(\mathbf{F}\right)}\right) &= \\
\nabla \cdot \left(\mathbf{F} \cdot 2\mathrm{D}\mathbf{W}\left(\mathbf{F}^{T}\mathbf{g}\mathbf{F}\right) \cdot \frac{\mathbf{F}^{-T}}{\det\left(\mathbf{F}\right)}\right) &=
\end{aligned}
$$

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Geometry $(\mathcal{M}, \mathbf{G})$ and $(\mathcal{E}, \mathbf{g})$ con. & sold. on $\mathrm{T}\mathcal{M}$ connexion on $\mathrm{T}\mathcal{E}$

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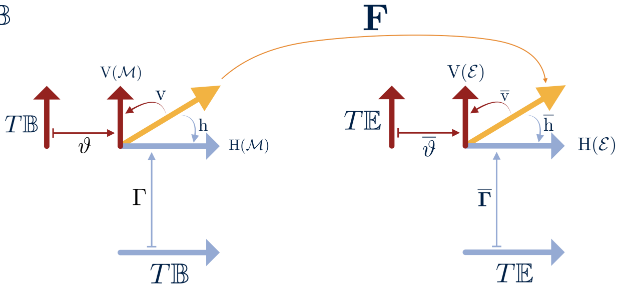
\end{aligned}$$

Scaled material modelling

$\pi : \mathcal{M} \longrightarrow \mathbb{B}$

$\overline{\pi} : \mathcal{E} \longrightarrow \mathbb{E}$



$$\mathbf{F}_{|m} = \begin{pmatrix} \overline{\boldsymbol{\Gamma}}_{\overline{m}} & \overline{\vartheta}_{\overline{m}} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}\varphi_{|m} & 0 \\ \mathbf{F}^{\mathrm{v}}_{\mathrm{h}}{}_{|m} & \mathbf{F}^{\mathrm{v}}_{\mathrm{v}}{}_{|m} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{\Gamma}^{-1}_{m}\mathrm{h} \\ \vartheta^{-1}_{m}\mathrm{v} \end{pmatrix} \qquad \varphi : \mathbb{B} \to \mathbb{E}$$

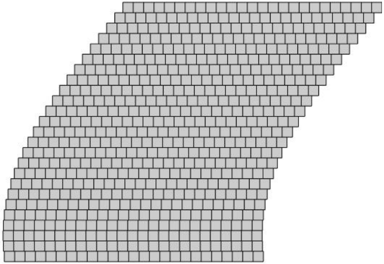
Scaling factor $\zeta \in [0,1]$: $\zeta = \frac{l}{L}$

Coupling term : $\mathbf{F}^{\mathrm{v}}_{\mathrm{h}} = \mathrm{d}\varphi - \mathbf{F}^{\mathrm{v}}_{\mathrm{v}}$

Example

$$\varphi : \begin{pmatrix} X \\ Y \end{pmatrix} \longmapsto \begin{pmatrix} X - \mathbf{u}(Y) \\ Y \end{pmatrix}$$

$$\mathbf{F}_{\mathrm{v}}^{\mathrm{v}} = \mathrm{Id} \qquad \|\mathbf{u}(Y)\| \ll 1$$



St. Vernant-Kirchhoff type of energy

$$\mathbf{W}(\mathbf{C}) = \frac{\lambda}{2}\mathrm{tr}\left[\frac{\mathbf{C}-\mathbf{G}}{2}\right]^2 + \mu \mathrm{tr}\left[\left(\frac{\mathbf{C}-\mathbf{G}}{2}\right)^2\right]$$

$$D\mathbf{W}(\mathbf{C})(\Delta\mathbf{C}) = \frac{\lambda}{2}\mathrm{tr}\left[\frac{\mathbf{C}-\mathbf{G}}{2}\right]\mathrm{tr}\left[\Delta\mathbf{C}\right] + \mu\mathrm{tr}\left[\frac{\mathbf{C}-\mathbf{G}}{2}\mathbf{G}^{-1}\Delta\mathbf{C}\right]$$

Example

$$[\nabla \cdot \sigma]_{(X,Y)} = [\nabla \cdot \sigma_{\text{elastic}}]_{(X,Y)} + \zeta \ddot{\mathbf{u}}(Y) \begin{pmatrix} \frac{3}{2}\zeta \dot{\mathbf{u}}^2(Y)(2\mu + \lambda) \\ \zeta \dot{\mathbf{u}}(Y)(2\mu + \lambda) \\ \mu + \frac{3}{2}\left(1 + \zeta^2\right)\dot{\mathbf{u}}^2(Y)(2\mu + \lambda) \\ 0 \end{pmatrix}$$

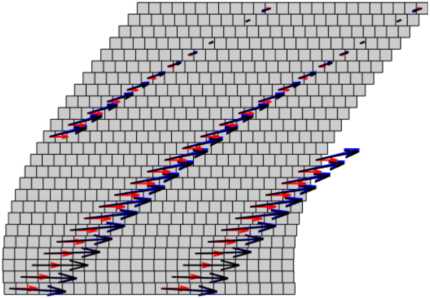


Figure: $\zeta = \frac{1}{2}$

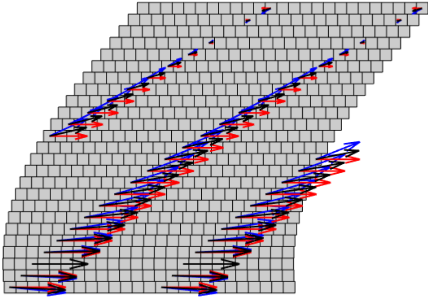


Figure: $\zeta = 1$

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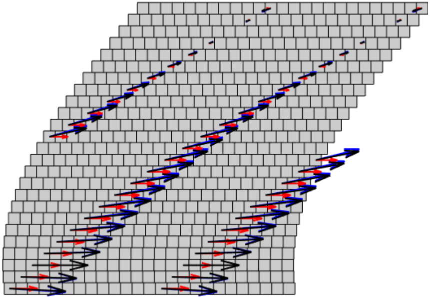
THANK YOU !



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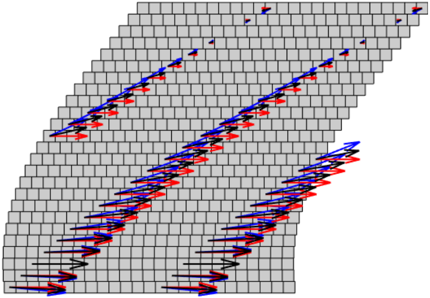

Figure: $\zeta = 1$