Obstructions à l'integrabilité à la Ziglin et Morales-Ramis

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Integrability

Liouville–Arnold theorem (*complete integrability*):
Let n smooth functions F_i on a 2n-dimensional manifold M be in involution: $\{F_i, F_j\} = 0$. Consider the level set of functions F_i
$M_f = \{(\mathbf{q}, \mathbf{p}) : F_i(\mathbf{q}, \mathbf{p}) = f_i, i = 1, \ldots, n\}$.
Let the functions F_i be independent on M_f. Then:

- *M_f is invariant under the phase flow with the hamiltonian function $H = F_1(\mathbf{q}, \mathbf{p})$ ($\dot{p}_i = -\frac{\partial H}{\partial q_i}$, $\dot{q}_i = \frac{\partial H}{\partial p_i}$).*
- *If M_f is compact, then each connected component of it is diffeomorphic to an n-dimensional torus*
  $\mathbb{T}^n = \{(\varphi_1, \ldots, \varphi_n) \mod 2\pi\}$
- *The phase flow with the hamiltonian H defines on M_f a quasi-periodic motion: $\dot{\varphi} = \omega(\mathbf{f})$ (action-angle variables)*
- *Canonical hamiltonian equations can be integrated in quadratures.*

In this case the system is called *completely integrable* or *integrable in Liouville–Arnold sense*

Algebraic obstructions to integrability

- Polynomial first integrals:
  Yoshida H, Cel. Mech., 31, 363, 00/1983.
- Meromorphic first integrals – monodromy
  Ziglin S L, Fun. Anal. Appl, 16 and 17, 1982.
- Meromorphic first integrals – differential Galois
  Morales-Ruiz J J, Ramis J-P, Meth. Appl. Anal. 8(1), 33-95, 97-111, 2001.
- Galois / Malgrange groupoid,
  Casale G., Ann. Institut Fourier 56 n 3 (2006)

Variational equations

Consider in $\mathbb{C}^{2n}$ a system

$$\dot{x} = v(x) \tag{1}$$

with complex time. Fix some particular solution $x_0(t)$ of (1). Plug $x = x_0 + \xi$ to (1) and obtain

$$\dot{\xi} = A(t)\xi + \ldots, \qquad A = \frac{\partial v}{\partial x}(x_0(t))$$

Then the linearized system

$$\dot{\xi} = A(x_0(t))\xi$$

is called a system of *variational equations* along a particular solution $x_0(\cdot)$.

Theorem 1. (Ziglin's Lemma)

Let the monodromy group $\mathcal{M}$ of a curve x_0 contain a non-resonant transformation. Then the number of meromorphic first integrals of hamiltonian equations in the connected neighbourhood of x_0 functionally independent with H is less than or equal to the number of rational invariants of $\mathcal{M}$.

Theorem 2. (Ziglin)

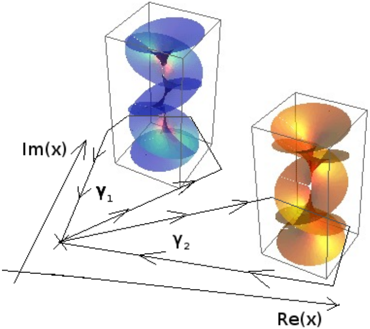
Let the monodromy group $\mathcal{M}$ of a curve x_0 contain a non-resonant transformation g. Then for the hamiltonian equations to have $(n-1)$ meromorphic first integrals in a connected neighbourhood of x_0, functionally independent with H, it is necessary that, any transformation $g' \in \mathcal{M}$ preserves the fixed point of g and maps all the eigendirections of g to eigendirections. If moreover no set of eigenvalues of g' forms a regular polygon centered in 0 on a complex plane, (with number of vertices ≥ 2), then g' preserves eigendirections of g (i.e. commutes with g).

Monodromy group (simple algebraic example)

Consider the equation

$$\xi^5 + \xi x + 1 = 0$$

The solution of $\xi = \xi(x)$ – 5-valued function.

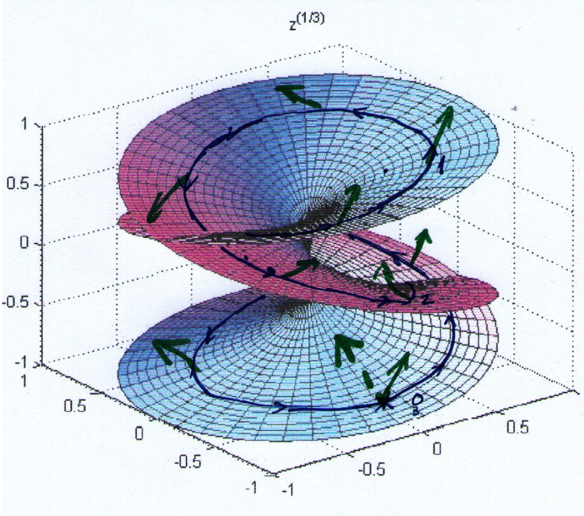


Change x along a closed loop (avoiding multiple roots)
$\Rightarrow$ possible permutation of roots.

Group structure: "product" $=$ concatenation of loops,
"inverse" $=$ following the loop in the opposite direction.
Monodromy group $\mathcal{M}$ $=$ all permutations of 5 elements.

Monodromy group

Initial system: $\dot{x} = v(x)$ with a particular solution $x_0(t)$ – Riemann surface. Variational equations along x_0: $\dot{\xi} = A(x_0(t))\xi$



Two mathematicians talking:

– I lost the key from the door, how do I get back home?..

– Is it a real door?

– Sure, why?

– Go to the complex space, go around it, then come back.

$$M_\gamma : (\xi_1, \ldots, \xi_n) \to (\tilde{\xi}_1, \ldots, \tilde{\xi}_n), \quad \mathcal{M} = \{M_\gamma, \forall \gamma \in \pi_1(x_0)\}$$

Properties

- Elements of the monodromy group are symplectic affine transformations $\Rightarrow$ eigenvalues go in couples: λ, λ^{-1}.

- Monodromy measures the difference of neighboring trajectories

- $\dot{\xi} = A(x_0(t))\xi$.
  Consider K = differential field of coefficients, L = extension of K by the solutions and all their derivatives.
  Fact: The Monodromy group is a subset (subgroup) of differential automorphisms of L acting trivially on K.

Ziglin's method

Standard application procedure:

1. For a given (initial) complexified system of differential equations construct an explicit particular solution.
2. Write down the system of variational equations. If possible reduce the order of this system to normal variational equations.
3. Find the singular points of the particular solution. (Not the branching points of the particular solution of time, but singularities on a Riemann surface).
4. Construct the monodromy matrices, corresponding to the loops around these singularities – they will generate the monodromy group.
5. Having computed the commutators of these matrices conclude on the possible integrability.

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Ways out

- "Richer" group (e.g. Morales–Ramis method via differential Galois theory = maximal group s.t. the fact above is true)

Morales–Ramis

Lemma. If the system of equation possesses a meromorphic first integral in the neighbourhood of a curve x_0 (particular solution), then the differential Galois group $\mathcal{G}$ of the system of normal variational equations possesses a rational first integral.

Theorem (Morales–Ramis) *Let the hamiltonian system be Liouville–Arnold integrable in the neighbourhood of its particular solution. Then the identity component of the differential Galois group of the corresponding system of normal variational equations is abelian.*

Remark 1. $\mathcal{M} \subset \mathcal{G}$, but construction of $\mathcal{G}$ is less explicit.

Remark 2. Kovacic algorithm for $\mathcal{G}$ (second order equation).

Ways out

G. Casale
J.-A. Weil, M. Barkatou, T. Cluzeau
. . .

- "Richer" group (e.g. Morales–Ramis method via differential Galois theory = maximal group s.t. the fact above is true)
- Higher variations (Ziglin–Morales–Ramis–Simo theory)
- Numerical methods:
  – Roekaerts D, Yoshida H, *J. Phys. A: Math. Gen.* **21**, 3547-3557, 1988.
  – Maciejewski A J, Godźiewski K, *Reports on Math. Phys.* **44**, 133-142, 1999.
  – C. Simó . Computer assisted studies in dynamical systems. In: Progress in nonlinear science. Vol. I:Mathematical problems of nonlinear dynamics (Eds.: L. M. Lerman, L. P. Shilnikov). Univ. Nizhny Novgorod Press, 2002, p. 152-165.

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Remark. Only the second difficulty...

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Effective algorithm of application of the Ziglin's method (1)

1) Write down an extended system of equations (not fixing any particular solution)

$$\begin{aligned} \dot{x} &= v(x) \\ \dot{\Xi} &= A(x)\Xi, \end{aligned} \tag{2}$$

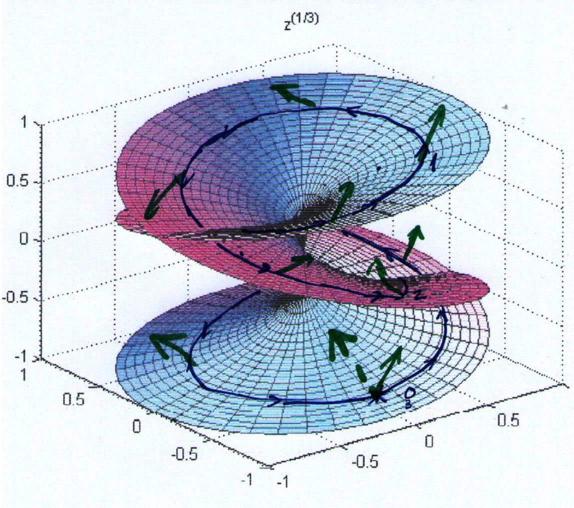
where the first line is the initial complexified system, second – matrix equation, where A is the matrix of the system of variational equations, depending explicitly on x, Ξ – unknown $2n \times 2n$ matrix (for $2n$ – dimension of the phase space of the initial system).

2) Choose a domain $\mathbb{C}$ and a grid Gr of points in it with some distinguished point t_0.

Monodromy group

Initial system: $\dot{x} = v(x)$

Variational equations: $\dot{\Xi} = A(x)\Xi$

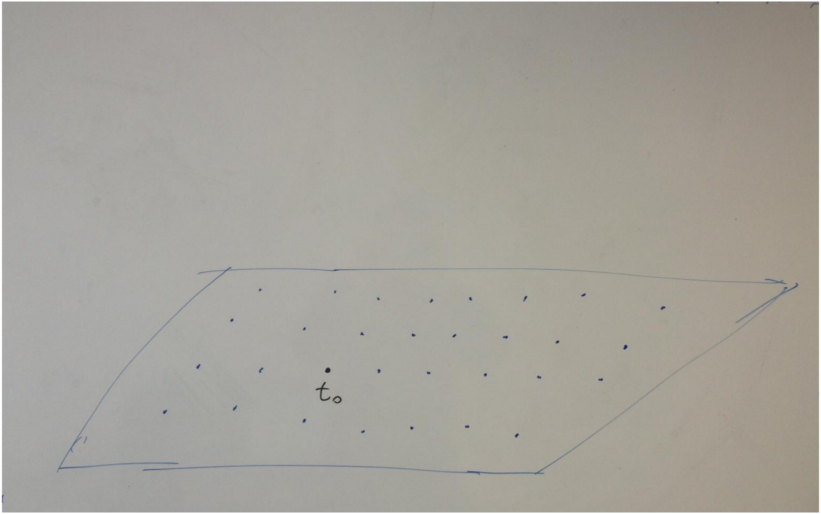


$M_\gamma\colon (\xi_1, \ldots, \xi_n) \to (\tilde{\xi}_1, \ldots, \tilde{\xi}_n), \quad \mathcal{M} = \{M_\gamma, \forall\gamma \in \pi_1(x_0)\}$

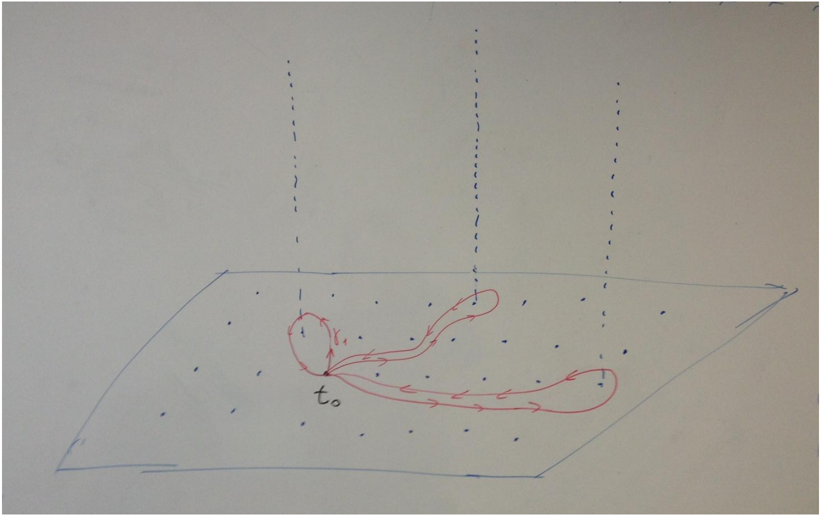
Effective algorithm of application of the Ziglin's method (2)

3) For each point from Gr choose a closed loop, going around it starting from t_0. Integrate the system (2) along this loop for the initial data $\Xi(t_0) = id$. Three options are possible:
   i. x and Ξ returned to initial values – this point gives trivial transformation from the monodromy group.
   ii. x didn't return to the initial value (within a given precision) – repeat integration along the same loop. If x didn't return to the initial value after several iterations, analize the density of this trajectory in the phase space.
   iii. Values of x returned to the initial data, but Ξ not – store the obtained matrix Ξ, it will be one of the generators of the monodromy group.
4) Compute pairwise commutators of the matrices from 3)iii. If there are non-zero commutators, conclude meromorphic non-integrability. If not, choose another point t_0 in 2) or initial value $x(t_0)$ in 3).

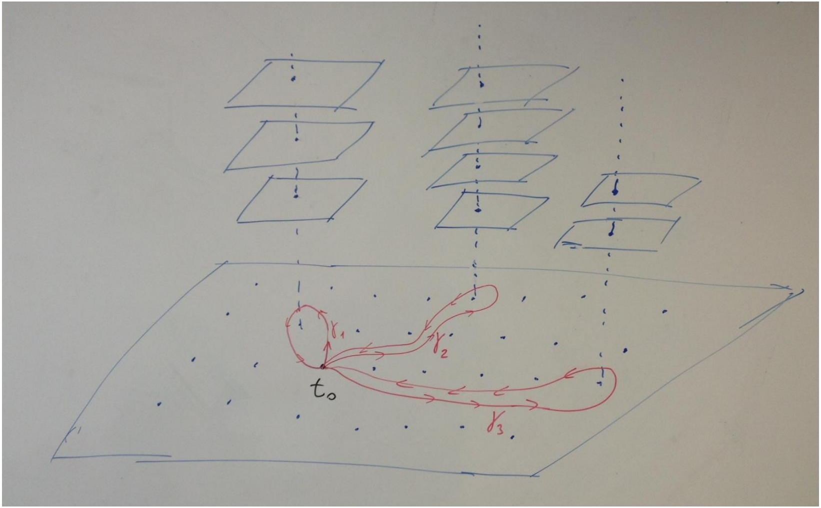
Effective algorithm steps 1-2



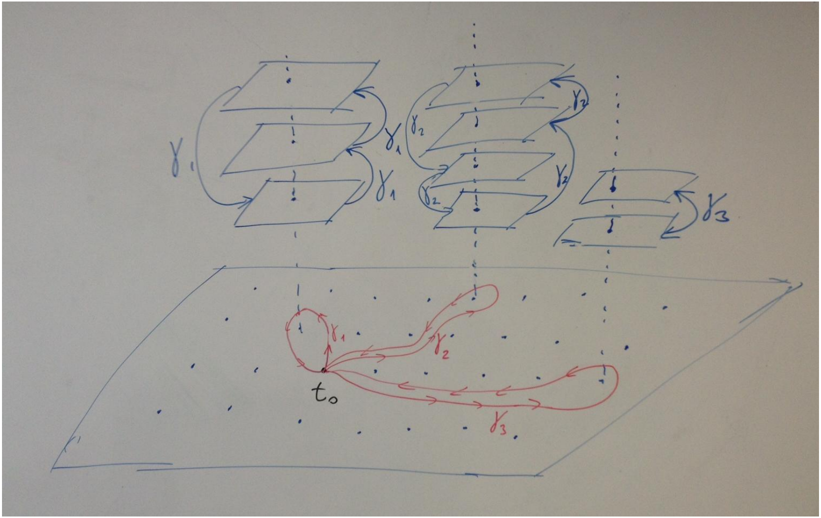
Effective algorithm step 3



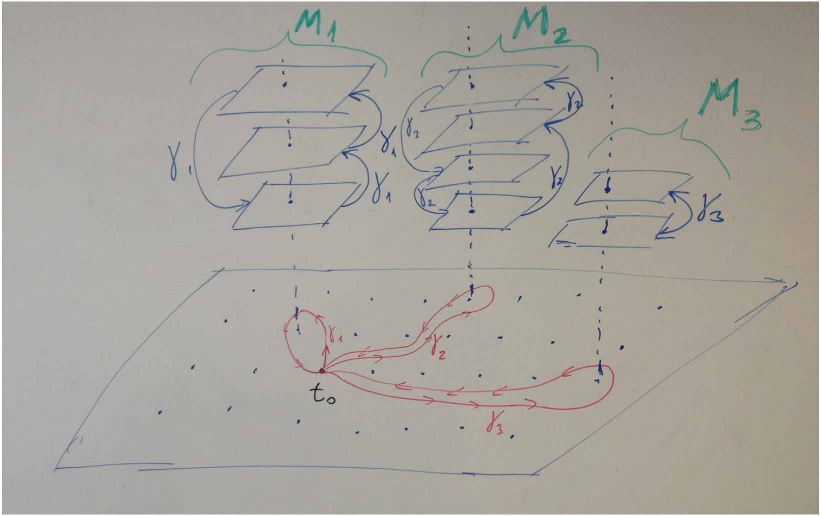
Effective algorithm step 3



Effective algorithm step 3 outcome 3.i or 3.iii



Effective algorithm step 4



Application: Henon–Heiles system

$$H = \frac{1}{2}(p_1^2 + p_2^2) - q_2^2(A + q_1) - \frac{\lambda}{3}q_1^3$$

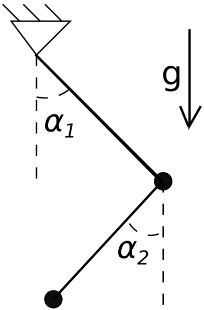
For $\lambda \neq 0$ not all the cases can be studied within the approach of Morales–Ramis – we apply the algorithm above.

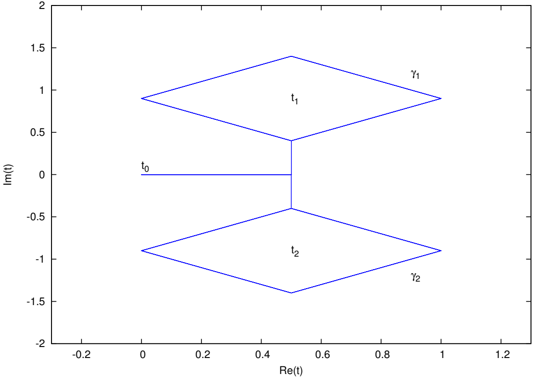
Example: for $\lambda = 1, \quad A = 0.25$ and initial data $(q_1, q_2, p_1, p_2) = (1, -0.4, -1.25, -0.3), t_0 = 1$. The monodromy matrices corresponding to the points $(0.2 + 2.5i)$ and $(0.2 - 2.5i)$ have the commutator

$$\begin{pmatrix} 0.390 - 0.913i & 0.658 - 3.463i & -1.099 + 2.566i & 0.655 - 2.223i \\ -0.937 + 2.301i & -0.637 + 2.599i & 1.311 - 1.866i & -1.720 + 7.996i \\ 0.315 - 0.760i & 0.274 - 1.212i & -0.520 + 0.878i & 0.576 - 2.514i \\ 0.464 - 1.079i & 0.797 - 4.199i & -1.335 + 3.133i & 0.765 - 2.564i \end{pmatrix}$$

that is meromorphic non-integrability.

Double pendulum – Ramis' question





$t_1 = 0.5 + 0.9i$
(3 times)

$$\begin{pmatrix} 20.72 - 17.12i & 15.79 - 1.34i & 4.94 + 29.46i & -14.55 + 10.90i \\ -17.67 + 12.78i & -12.24 - 0.06i & -1.93 - 24.87i & 12.91 - 7.99i \\ 12.28 + 6.91i & 3.55 + 7.78i & -13.07 + 7.86i & -8.18 - 5.42i \\ -11.84 - 12.56i & -1.31 - 10.39i & 19.32 - 4.06i & 8.59 + 9.31i \end{pmatrix}$$

$t_2 = 0.5 - 0.9i$
(3 times)

$$\begin{pmatrix} -18.72 - 17.12i & -15.79 - 1.34i & -4.94 + 29.46i & 14.55 + 10.90i \\ 17.67 + 12.78i & 14.24 - 0.06i & 1.93 - 24.87i & -12.91 - 7.99i \\ -12.28 + 6.91i & -3.55 + 7.78i & 15.07 + 7.86i & 8.18 - 5.42i \\ 11.84 - 12.56i & 1.31 - 10.39i & -19.32 - 4.06i & -6.59 + 9.31i \end{pmatrix}$$

Don't commute $\Rightarrow$ meromorphic non-integrability.

Application: Triple pendulum

(Reduced Routh system.)

Initial data $(\beta_1, \dot{\beta}_1, \beta_2, \dot{\beta}_2) = (0.3, -1, -0.15, 0.5)$, $\quad t_0 = 1$, with the value of cyclic integral $I_\beta = 1$,

Loops around $(0.2 + 0.5i)$ and $(0.2 - 0.5i)$ (6 times), give the commutator

$$\begin{pmatrix} 0.62 - 0.62i & 0.81 + 0.83i & -0.24 - 0.34i & -0.02 + 0.34i \\ -0.07 - 0.14i & 1.04 + 0.10i & -0.04 + 0.20i & -0.02 - 0.39i \\ 0.02 - 1.01i & 0.25 + 1.52i & 0.15 + 0.57i & 0.86 - 1.44i \\ -0.01 + 0.02i & -0.02 - 0.07i & -0.02 - 0.09i & 1.03 + 0.11i \end{pmatrix},$$

that is meromorphic non-integrability.

Remark. Balance between symbolic (Sage) and numerical computation.

Details

- Controllable precision?
- Inequation condition $\Rightarrow$ status of "computer assisted proof"
- Independent trajectories $\Rightarrow$ Parallelization

V.S.: "Effective algorithm of analysis of integrability via the Ziglin's method", Journal of Dynamical and Control Systems, Volume 20, Issue 4, pp 465-474, 2014;

V.S.: "Integrability of the double pendulum – the Ramis' question", arXiv:1303.4904 [math.DS];

Merci pour votre attention.

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