A stochastic Hamilton-Jacobi equation

Jacky Cresson

Nelson derivatives and stochastic derivatives

$$dX_t = v(t, X_t)dt + \sigma dW_t,$$

$$D_+[X_t] = \lim_{h \to 0^+} \mathbb{E}\left[\frac{X_{t+h} - X_t}{h} \mid \mathcal{P}_t\right], \quad D_-[X_t] = \lim_{h \to 0^+} \mathbb{E}\left[\frac{X_t - X_{t-h}}{h} \mid \mathcal{F}_t\right],$$

$$D_+[X_t] = v(t, X_t), \quad D_-[X_t] = v(t, X_t) - \sigma^2 \frac{\nabla p_t}{p_t}(X_t),$$

$$\mathscr{D}_\mu = \frac{D_+ + D_-}{2} + i\mu \frac{D_+ - D_-}{2}, \mu \in \{-1, 1\}.$$

E. Nelson (2001): Dynamical theory of Brownian motion. Princeton University Press. Second edition. Available online at `http://www.math.princeton.edu/~nelson/books/bmotion.pdf`.

Nelson E., Derivation of the Schrödinger equation from Newtonian mechanics, Physical Review, Vol. 150, No. 4, 1079-1084 (1966).

Algebraic properties

$$\mathscr{D}_\mu[g(X_t,t)] = \left(\partial_t + \mathscr{D}_\mu(X_t)\nabla + i\mu\frac{\sigma^2}{2}\Delta\right)[g](X_t,t),$$

$$\mathbb{E}\left[\mathscr{D}_\mu X \cdot Y + X \cdot \mathscr{D}_{-\mu} Y\right] = \frac{d}{dt}\mathbb{E}(X \cdot Y),$$

Lemma 1 (Composition) *For $\alpha = \pm 1$ and $\mu = \pm 1$, we have*

$$\begin{aligned} \mathscr{D}_{\alpha\mu} \circ \mathscr{D}_{\mu} &= \frac{1}{4}\left[(D_+^2 + D_-^2)(1-\alpha) + (D_+D_- + D_-D_+)(1+\alpha)\right] \\ &\quad + i\mu\frac{1}{4}\left[(D_+^2 - D_-^2)(1+\alpha) + (D_-D_+ - D_+D_-)(1-\alpha)\right]. \end{aligned}$$

Lemma 4 *Let X_t be a solution of the differential stochastic Newton equation. Then, we have $D_+^2 X = D_-^2 X$.*

Theorem 3 ([12],Theorem 5) *Let X of the form (24), verifying assumption (H), such that $b \in C^2(\mathbb{R}^d)$ with bounded derivatives and such that for all $t \in (0,T)$ the second order derivatives of $\nabla \ln p_t$ are bounded. Then we have the following equivalence :*

$$D_+^2 X = D_-^2 X \text{ for almost all t} \in (0,T) \iff b \text{ is a gradient.} \tag{65}$$

Lemma 5 (Action function) *Let X be a stochastic process solution of the differential stochastic Newton equation (28). Then, denoting $\mathscr{V} = \mathscr{D}X$ the complex speed of the process, there exists a function $\mathscr{A}(t, X)$ called the action function such that*

$$\mathscr{V} = \frac{\nabla \mathscr{A}(t, X_t)}{m}, \tag{68}$$

with

$$\mathscr{A}(t, X_t) = S(t, X_t) + i\mu R(t, X_t), \tag{69}$$

where

$$S(t, X) = mW(t, X) - \frac{1}{2} m\sigma^2 \ln(p_t), \quad R(t, X) = \frac{1}{2} m\sigma^2 \ln(p_t), \tag{70}$$

and the function W is given by Theorem 3.

Theorem 6 (A stochastic Hamilton-Jacobi equation) *The action function $\mathscr{A}(t, X)$ satisfies the nonlinear partial differential equation*

$$\partial_t \mathscr{A} + \frac{1}{2m} [\nabla \mathscr{A} \cdot \nabla \mathscr{A}] + i\mu \frac{\sigma^2}{2} \Delta \mathscr{A} = -U. \tag{71}$$

It must be noted that when the dynamics is not stochastic, meaning that $\sigma = 0$, then $\mathscr{A}$ is real and reduces to S as $R = 0$ in this case. As a consequence, the stochastic Hamilton-Jacobi equation is equivalent to

$$\partial_t S + \frac{1}{2m} [\nabla S \cdot \nabla S] = -U, \tag{72}$$

which corresponds to the classical Hamilton-Jacobi equation introducing the Hamiltonian function

$$H(p, x, t) = \frac{1}{2} p^2, \tag{73}$$

and rewriting equation (72) as

$$\partial_t S + H(\nabla S, x, t) = -U. \tag{74}$$

Lemma 6 (Modified Hamilton-Jacobi equation) *The first equation of system (81) can be rewritten as*

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 - m\frac{\sigma^4}{2}\frac{\Delta(\sqrt{p})}{\sqrt{p}} + U = 0 \tag{83}$$

called the modified Hamilton-Jacobi equation.

Lemma 7 (A continuity equation) *The density p_t of a stochastic process solution of the Newton stochastic differential equation satisfies the following continuity equation*

$$m\frac{\partial p}{\partial t} + \mathrm{div}(p\,\nabla S) = 0. \tag{88}$$

Definition 2 (Wave function) *Let $\sigma > 0$ and C be a non zero real constant. We call wave function and we denote by $\psi_{\sigma,\mu}$ the function defined by*

$$\psi_\mu(t,x) = e^{i\frac{\mathscr{A}_\mu(t,x)}{C}}. \tag{93}$$

Theorem 8 (Schrödinger formulation) *If X is a solution of the differential stochastic Newton equation then the wave function ψ satisfies the nonlinear partial differential equation*

$$iC\partial_t\psi - \mu\frac{\sigma^2 C}{2}\Delta\psi + \frac{C(\mu m\sigma^2 + C)}{2m}\frac{\nabla\psi\cdot\nabla\psi}{\psi} = \psi\cdot U. \tag{96}$$

Corollary 1 *Let X be a solution of the differential stochastic Newton equation, then taking the normalization constant*

$$C = -\mu m \sigma^2, \tag{106}$$

equation (96) reduces to the linear partial differential equation

$$-i\mu m\sigma^2 \partial_t \psi + m\frac{\sigma^4}{2}\Delta\psi = \psi \cdot U, \tag{107}$$

and

$$|\psi(t,x)|^2 = p_t(x). \tag{108}$$

If moreover, we consider the case $\mu = -1$ then equation (107) reduces to the classical linear Schrödinger equation

$$im\sigma^2 \partial_t \psi + m\frac{\sigma^4}{2}\Delta\psi = \psi \cdot U. \tag{109}$$

Let U be a given potential and let $\sigma > 0$. The main steps are the following:

- Write the Schrödinger equation (109) and compute the ground state solution ψ.
- Using formula (108), compute the density $p_t(x)$ of the stochastic process X.
- Compute the induced stochastic potential $U_{\sigma,induced}$ using formula (87).
- Identify the diffusion coefficient σ.

$$U(x,y,z) = \frac{-GMm}{r},$$

$$|\psi| = \frac{C}{m\sigma^4} e^{-2r/r_0}, \qquad \frac{1}{r_0} = \frac{GM}{2\sigma^4}.$$

$$p_t(x) = \frac{C^2}{m^2\sigma^8} e^{-4r/r_0}.$$

Lemma 9 (Induced potential-Kepler case) *Let $\mu = -1$ and U be given by the Kepler potential (114). Then the induced stochastic potential is given by*

$$U_{\sigma,induced} = -\frac{GMm}{r_0}\left(1 - \frac{r_0}{r}\right), \tag{118}$$

with r_0 given by (116).

Theorem 9 (Flat rotation curves) *At equilibrium the real part of the speed* $\mathscr{D}_\mu X$ *denoted by* $\mathbf{v}$ *has a constant norm equal to* v_0 *where* v_0 *is given by*

$$v_0^2 = \frac{2GM}{r_0}. \tag{123}$$

Lemma 10 (Diffusion coefficient) *The diffusion coefficient* σ *is given by*

$$\sigma^2 = \frac{GM}{v_0}. \tag{126}$$

- The central bulb which looks approximately like a sphere containing gazes and stars in a homogeneous way.
- The disk which is itself decomposed in two components: a fine disk which is dense and a rough one.
- The pair bulb-disk is contained in a halo of stars which is more or less a sphere but of low density.

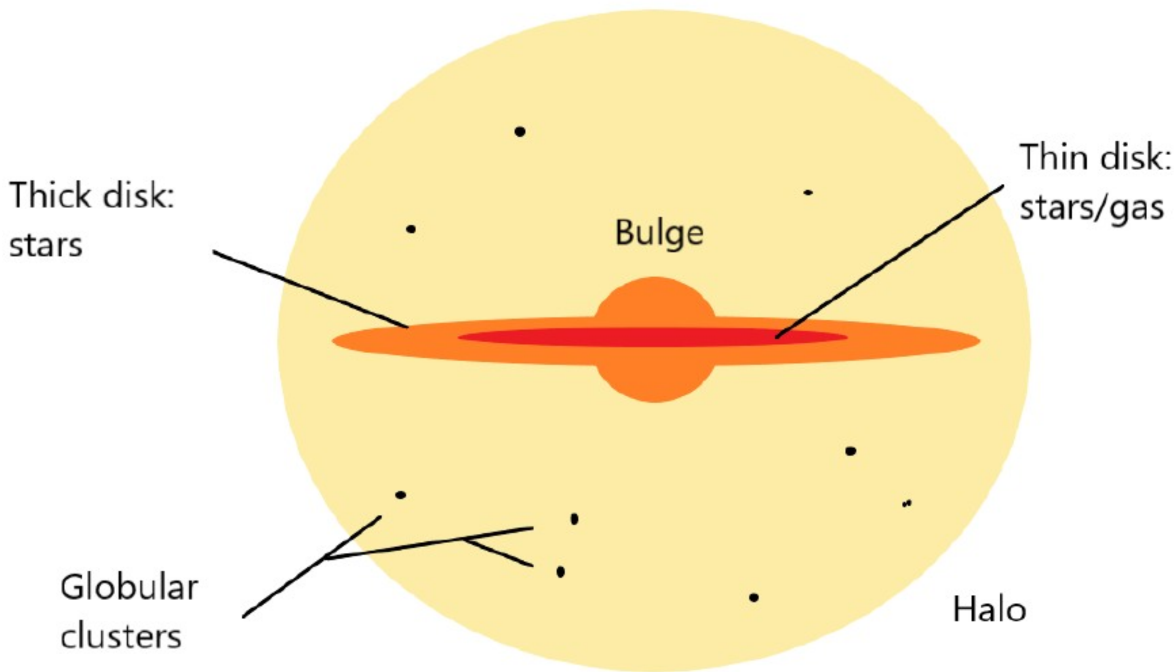


Figure 1: The Milky way.

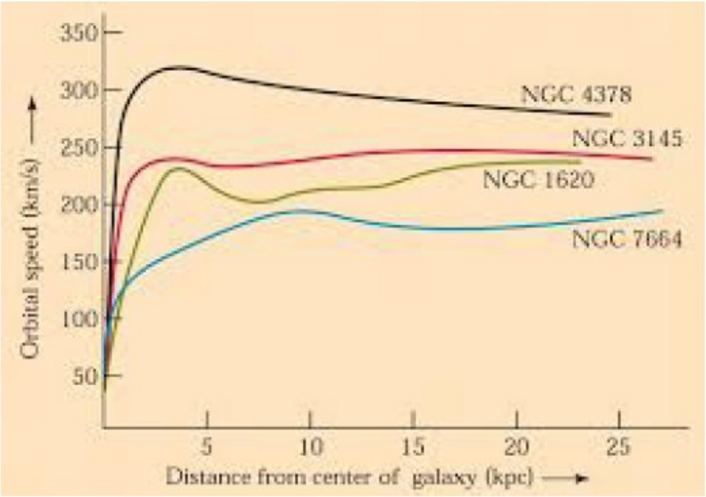
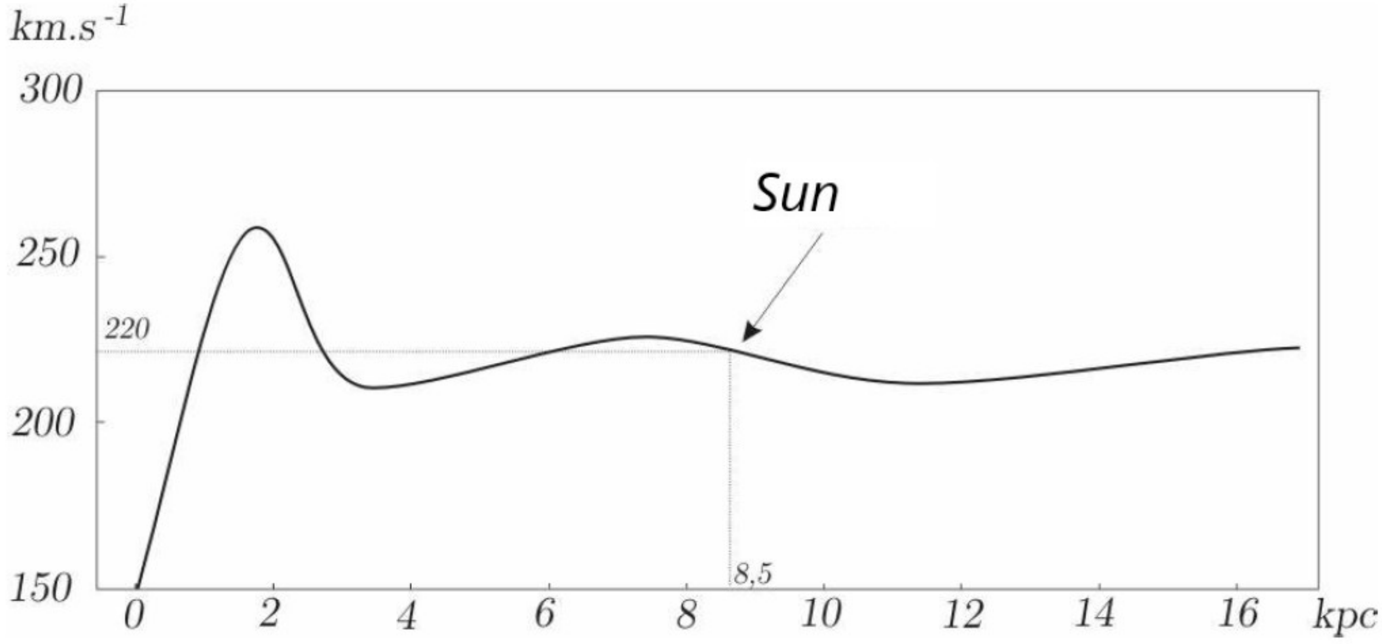


Figure 2: Some examples of rotation curves



Stochastic embedding

$$\frac{d^2 x_t}{dt} = -\nabla U(x_t),$$

$$\mathcal{L}(x_t) = \int_a^b \left(\frac{1}{2} \left(\frac{dx_s}{ds} \right)^2 - U(x_s) \right) ds,$$

$$\mathcal{D}_\mu^2 X_t = -\nabla U(X_t).$$

$$v = \frac{dx}{dt} \quad a = \frac{d^2 x}{dt^2}.$$

$$\mathcal{V} = \mathcal{D}X, \quad \mathcal{A} = \mathcal{D}^2 X.$$

J. Cresson and S. Darses, *Stochastic embedding of dynamical systems. J. Math. Phys.*, 48(7):072703, 54, 2007.

$$\mathcal{D}_i^2 = \frac{DD_* + D_*D}{2} + i\frac{D^2 - D_*^2}{2}.$$

Stochastic Lagrangian functionals

$$\mathcal{L}_{\mathrm{stoc}}(X_t) = \mathrm{E}\left[\int_a^b \left(\frac{1}{2}\mathcal{D}_\mu^2 X_s - U(X_s)\right)\, ds\right].$$

Critical points

$$\mathcal{D}_{-\mu}\mathcal{D}_\mu X_t = -\nabla U(X_t)$$

Changing the set of variations

$$\mathcal{N}^1 = \{X \in \Lambda^1, DX = D_* X\}.$$

$$\mathcal{D}_\mu^2 X_t = -\nabla U(X_t),$$

J. Cresson and S. Darses (2006), Plongement stochastique des systèmes lagrangiens, C.R. Acad. Sci. Paris Ser. I **342** (5), 333-336.

Invariance et théorème de Noether

$$L(\phi_s^{\text{stoc}}(X), \mathcal{D}\left(\phi_s^{\text{stoc}}(X)\right)) = L(X, \mathcal{D}).$$

Problem 1 **(Persistence of invariance).** — *Assume that a Lagrangian L is invariant under a group of symmetries $\{\phi_s\}_{s\in\mathbb{R}}$. Do we have the stochastic invariance of the Lagrangian L under the stochastic lifted one parameter group of diffeomorphisms $\{\phi_s^{\text{stoc}}\}_{s\in\mathbb{R}}$?*

Definition 7 **(Strong invariance).** — *Let $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$ be a one parameter group of diffeomorphisms. An admissible Lagrangian L is said to be strongly invariant under the action of Φ if*

$$L(t, x, v) = L(t, \phi_s(x), \phi_s(v)), \quad \forall s \in \mathbb{R},\ \forall t \in I,\ \forall x \in \mathbb{R}^d,\ \forall v \in \mathbb{R}^d.$$

Invariance et théorème de Noether

Definition 8 **($\mathcal{D}$-commutation).** — *Let $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$ be a one parameter group of diffeomorphisms. We say that the associated stochastic lifted group satisfies the $\mathcal{D}$-commutation property, if*

$$\mathcal{D}(\phi_s^{\text{stoc}}(X)_t) = \phi_s^{\text{stoc}}\Big(\mathcal{D}(X)_t\Big), \quad \forall s \in \mathbb{R}, \quad \forall t \in \mathbb{R}. \tag{25}$$

Theorem 4. — *Let L be an admissible Lagrangian system, strongly invariant under a one parameter group of diffeomorphisms $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$ satisfying the commutation property. Then, L is invariant under the stochastic lifted group of diffeomorphisms $\{\phi_s^{\text{stoc}}\}_{s\in\mathbb{R}}$.*

Théorème de Noether stochastique

Theorem 5. — *Let L be an admissible lagrangian with all second derivatives bounded, and invariant under a stochastic lifted one-parameter group of diffeomorphisms $\Phi_{\mathrm{stoc}} = \{\phi_s^{\mathrm{stoc}}\}_{s\in\mathbb{R}}$. Let $\mathcal{L}$ be the associated functional defined by (21) on $\Xi \cap \Lambda^1$. Let $X \in \Xi \cap \Lambda^1$ be a Λ^1-critical point of $\mathcal{L}$. Then, we have*

$$\frac{d}{dt} E\left[\partial_v L \cdot \left. \frac{\partial Y}{\partial s} \right|_{s=0} \right] = 0,$$

where

$$Y_s = \Phi_s(X). \tag{28}$$

J. Cresson, S. Darses, Théorème de Noether Stochastique, C.R. Acad. Sci. Paris, Ser. I 344 (2007) 259-264

L'hypothèse de stochastisation

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\text{Stochastisation}} & \mathcal{L}_{\text{stoc}} \\ \Big\downarrow{\scriptstyle \text{L.A.P.}} & & \Big\downarrow{\scriptstyle \text{S.L.A.P.}} \\ \frac{d^2x}{dt^2} = -\nabla U(x) & \xrightarrow{\text{Stochastisation}} & \mathcal{D}^2 X_t = -\nabla U(X_t), \end{array}$$

Stochastisation Hypothesis : *Let x_t be a physical process governed by a given ordinary differential equation. The stochastic extension of this equation is given by the differential stochastic embedding of the equation.*

Pourquoi les processus de diffusion ?

$$\lim_{t\downarrow 0} \frac{E^x[X_t - x]}{t} = b(x)$$
$$\lim_{t\downarrow 0} \frac{E^x[(X_t - x)^t (X_t - x)]}{t} = a(x).$$

Alors X_t satisfait une EDS au sens d'Itô

Plongement stochastique de l'équation de Newton

Let $\Psi(t,x)$ be the solution of the Schrödinger equation with initial condition Ψ_0:

$$(33) \qquad \left\{ \begin{array}{ccc} i\sigma^2\partial_t\Psi + \frac{\sigma^4}{2}\Delta\Psi & = & U\Psi \\ \Psi(0,\cdot) & = & \Psi_0. \end{array} \right.$$

Theorem 6. — *Set $\sigma \neq 0$. Let X^0 be a random variable with density $|\Psi_0|^2$. Then a solution $X \in \Lambda^2$ of the system*

$$(34) \qquad \left\{ \begin{array}{l} \mathcal{D}^2 X_t = -\nabla U(X_t) \\ X_0 = X^0 \end{array} \right.$$

is governed via $p_t(x) = |\Psi(t,x)|^2$ where Ψ is a solution of the linear Schrödinger equation (33). It is the Nelson diffusion of the type

$$(35) \qquad dX_t = \left(\Re\frac{\nabla\Psi}{\Psi} + \Im\frac{\nabla\Psi}{\Psi} \right)(t,X_t)dt + \sigma dW_t.$$

Theorem 7. — *Let $X \in \Sigma_\sigma$. We then have the following equivalence:*

$$D^2 X_t = D_*^2 X_t \ \textit{ for almost all } t \in (0,T) \quad \Longleftrightarrow \quad X \in \Sigma_\sigma^\nabla . \tag{36}$$

S. Darses and I. Nourdin, *Dynamical properties and characterization of gradient drift diffusions.*
Electron. Comm. Probab. 12 (2007) 390-400

Applications

Albeverio S., Blanchard Ph., Hoegh-Krohn R., A stochastic model for the orbits of planets and satellites: an interpretation of Titius-Bode law, Expositiones Mathematicae 4, 363-373 (1983).

Albeverio S., Blanchard Ph., Hoegh-Krohn R., Newtonian diffusions and planets, with a remark on non-standard Dirichlet forms and polymers,

Laskar J., Large-scale chaos in the solar system, Astron. Astrophys. 287, L9-L12 (1994).

Laskar J., On the spacing of planetary systems, Phys. Rev. Lett. 84? no. 15, 3240-3243, 2000.

Nottale L, New formulation of stochastic mechanics. Application to chaos, in "*Chaos and diffusion in Hamiltonian systems*", Proceedings of the fourth workshop in Astronomy and Astrophysics of Chamonix (France), 7-12 February 1994, Eds. D. Benest and C. Froeschlé (Editions Frontières), pp. 173-198 (1995).

Nottale L., The quantization of the solar system, Astron. Astrophys. 315, L9, 1996.

Nottale L., Scale relativity and quantization of the solar system, Astronomy and Astrophysics. 322, 1018-1025 (1997).

Nottale L., Scale-relativity, Fractal space-time and gravitational structures, in "Fractals and Beyond : complexity in the sciences", Ed. M.M. Novak, pp. 149-160, World Scientific, 1998.

Nebuleuse protoplanetaire

Structural assumptions : A protoplanetary nebula is made of :

- A central body of mass M acting by some potential U.
- A gas of some particles called grains around the central body.

Dynamical assumptions : The motion of a given grain is governed by :

- The potential U.
- Collisions between grains.

Assumptions on collisions : We assume that the collisions are :

- Randomly distributed
- Isotropic
- Homogeneous

Loi de répartition autour d'une étoile

$$\sqrt{a} = n\frac{2\sigma}{\sqrt{gM}},$$

$$\sigma = \frac{gM}{2\omega_0},$$

Estimation de la constant w_0 et statistiques

Efron B., Does an observed sequence of numbers follow a simple rule ? (another look at Bode's law), J. Am. Statist. Assoc. 66 (335), 552-559, 1971.

Good I.J., A subjective evaluation of Bode's law and an "objective" test for approximate numerical rationality, J. Am. Statis. Assoc. 64, 23-66, 1969.

W. Hayes, S. Tremaine, Fitting selected random planetary systems to Titius-Bode laws, Icarus 135, 549-557 (1998).

Prise en compte de la structure dynamique du système

$$\sigma_{\text{out}} = 5\sigma_{\text{in}}.$$

Sur la stabilité du système solaire ?

Newton's laws of motion "... predicted wonderfully planetary motion and, with perturbation, models the full set of planets for moderate period of time (e.g., maybe 10^8 years). But going out further (maybe to 10^9 years), the unmodeled effects begin to add up and the approximation is not useful. So where does this leave the mathematical study of the 3-body problem ? It makes the classical deterministic analysis of the 3-body gravitational equations about as relevant to the world as the continuum hypothesis! A major step in making the equation more relevant is to add a small stochastic term.".

David Mumford, 2001

Mumford D., The dawning of the age of stochasticity, in *Mathematics: Frontiers and perspectives*, V. Arnold, M. Atiyah, P. Lax, B. Mazur editors, AMS, 2000, 197-218.