Galerkin Variational Integrators

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Continuous case

$$\mathcal{L}(x) = \int_a^b L(x, \dot{x}) \; dt$$

Variational principle

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial x} = 0$$

Euler-Lagrange equation

Continuous case		**Discrete case**
$\mathcal{L}(x) = \int_a^b L(x, \dot{x})\ dt$	Discretization →	Discrete functional $\mathcal{L}_h$
Variational principle ↓		↓ Disc. var. principle
$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial x} = 0$	→ $\approx$	Numerical scheme
⋮		⋮
Euler-Lagrange equation		Discrete Euler-Lagrange equation

Variational Integrator

Continuous case | **Discrete case**

$$\mathcal{L}(x) = \int_a^b L(x, \dot{x})\ dt$$ —Discretization→ Discrete functional $\mathcal{L}_h$

Variational principle ↓ | ↓ Disc. var. principle

Variational integrator

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial x} = 0$$ —$\approx$→ Numerical scheme

Euler-Lagrange equation | Discrete Euler-Lagrange equation

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Variational principle

$$\downarrow$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial x} = 0$$

[1] Marsden & West. *Discrete mechanics and variational integrators*. Acta Numerica (2001)

Continuous case

Marsden-West approach [1]

$$\mathbb{L}(X_i, X_{i+1}) = L\left(X_i, \frac{X_{i+1} - X_i}{h}\right)$$

Action sum

$$\mathcal{L}(x) = \int_a^b L(x, \dot{x})\ dt$$

Discretization

$$\mathcal{L}_h(X) = \sum_{i=0}^{N-1} \mathbb{L}(X_i, X_{i+1})$$

Variational principle

Disc. var. principle

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial x} = 0$$

$\approx$

$$D_2\mathbb{L}(X_{i-1}, X_i) + D_1\mathbb{L}(X_i, X_{i+1}) = 0$$

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Discrete differential and integral calculus

Discrete derivatives and anti-derivatives: $\mathbb{T} = \{t_i\}_{i=0}^{i=N},\ t_{i+1} - t_i = h$

$$
\begin{array}{ccc}
\mathscr{P}_1([a,b],\mathbb{R}) & \xrightarrow{\frac{d^\pm}{dt}} & \mathscr{P}_0^\pm([a,b],\mathbb{R}) \\
\kappa_1(X) & & \frac{d^\pm}{dt}(\kappa_1(X)) \\
\uparrow \kappa_1 & & \downarrow \pi \\
\mathcal{F}(\mathbb{T},\mathbb{R}) & \xrightarrow{\Delta_\pm} & \mathcal{F}(\mathbb{T}^\pm,\mathbb{R}) \\
X & & \Delta_\pm X
\end{array}
$$

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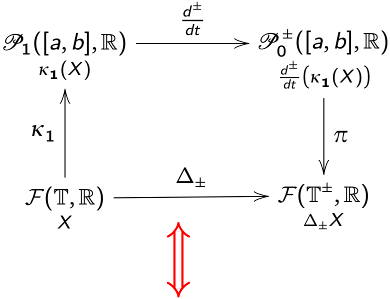

$$\Delta_\pm = \pi \circ \frac{d^\pm}{dt} \circ \kappa_1$$

$$\begin{cases} \Delta_+ X_i &= \dfrac{X_{i+1} - X_i}{h} \\ \Delta_- X_i &= \dfrac{X_i - X_{i-1}}{h} \end{cases}$$

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\mathcal{F}(\mathbb{T},\mathbb{R}) & \xrightarrow{\ \int_{\mathbb{T}}\Delta_\pm t\ } & \mathcal{F}(\mathbb{T},\mathbb{R}) \\
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$$\Updownarrow$$

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$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\Delta_\pm = \pi \circ \frac{d^\pm}{dt} \circ \kappa_1 \qquad\qquad \int_\mathbb{T} (\cdot)\Delta_\pm = \pi \circ \int_a^t \circ \ \kappa_0^\pm$$

$$\begin{cases} \Delta_+ X_i & = \dfrac{X_{i+1} - X_i}{h} \\ \Delta_- X_i & = \dfrac{X_i - X_{i-1}}{h} \end{cases} \qquad\qquad \begin{cases} \displaystyle\int_\mathbb{T} X \ \Delta_+ t & = \displaystyle\sum_{i=0}^{N-1} (t_{i+1} - t_i) X_i \\ \displaystyle\int_\mathbb{T} X \ \Delta_- t & = \displaystyle\sum_{i=1}^{N} (t_i - t_{i-1}) X_i \end{cases}$$

Continuous case	**Discrete case**

Fundamental theorem of the differential calculus

$$\int_a^t x'(s)\ ds = x(t) - x(a)$$

$$\frac{d}{dt}\int_a^t x(s)\ ds = x(t)$$

$$\int_{\mathbb{T}} (\Delta_+ X)\ \Delta_+ t = X - \mathbb{X}_0$$

$$\Delta_+ \circ \int_{\mathbb{T}} X\ \Delta_+ t = X$$

Integration by parts

$$\int_a^b x \cdot y'\ dt = -\int_a^b x' \cdot y\ dt + xy|_a^b$$

$$\int_{\mathbb{T}} (Y \cdot \Delta_+ X)\Delta_+ t = -\int_{\mathbb{T}} (\Delta_- Y \cdot X)\Delta_+ t + \mathbf{BC}$$

$$\int_{\mathbb{T}} (Y \cdot \Delta_- X)\Delta_- t = -\int_{\mathbb{T}} (\Delta_+ Y \cdot X)\Delta_- t + \mathbf{BC}$$

Continuous case

$$\mathcal{L}(x) = \int_a^b L\big(x(t), \dot{x}(t)\big)\, dt$$

Variational principle

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial x} = 0$$

Euler-Lagrange equation

Continuous case	**Discrete case**
$\mathcal{L}(x) = \int_a^b L\big(x(t), \dot{x}(t)\big)\ dt$	$\mathcal{L}_h(X) = \int_{\mathbb{T}} L(X, \Delta_+ X)\ \Delta_+ t$
↓ Variational principle	↓ Discrete var. principle
$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial x} = 0$	$\Delta_-\left(\frac{\partial L}{\partial v}(X, \Delta_+ X)\right) - \frac{\partial L}{\partial x}(X, \Delta_+ X) = 0$
⇣	⇣
Euler-Lagrange equation	Discrete E-L equation

Galerkin variational integrators [2]: over each subinterval $[t_i, t_{i+1}] \subset [a, b]$,

[2] J. Hall & M. Leok, Spectral variational integrators, Numer. Math. (2015)

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- Interpolate x using the Lagrange polynomials
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Euler-Lagrange equations: for $1 \le i \le N-1$

$$\sum_{a=1}^{A} hw_a \Big[\varphi_0(c_a) \frac{\partial L}{\partial x}(c_a h; \tilde{x}_i) + \frac{1}{h} \dot{\varphi}_0(c_a) \frac{\partial L}{\partial \dot{x}}(c_a h; \tilde{x}_i) \Big] + \sum_{a=1}^{A} hw_a \Big[\varphi_m(c_a) \frac{\partial L}{\partial x}(c_a h; \tilde{x}_{i-1}) + \frac{1}{h} \dot{\varphi}_m(c_a) \frac{\partial L}{\partial \dot{x}}(c_a h; \tilde{x}_{i-1}) \Big] = 0$$

$$\sum_{a=1}^{A} hw_a \Big[\varphi_j(c_a) \frac{\partial L}{\partial x}(c_a h; \tilde{x}_i) + \frac{1}{h} \dot{\varphi}_j(c_a) \frac{\partial L}{\partial \dot{x}}(c_a h; \tilde{x}_i) \Big] = 0, \quad 0 < j < m$$

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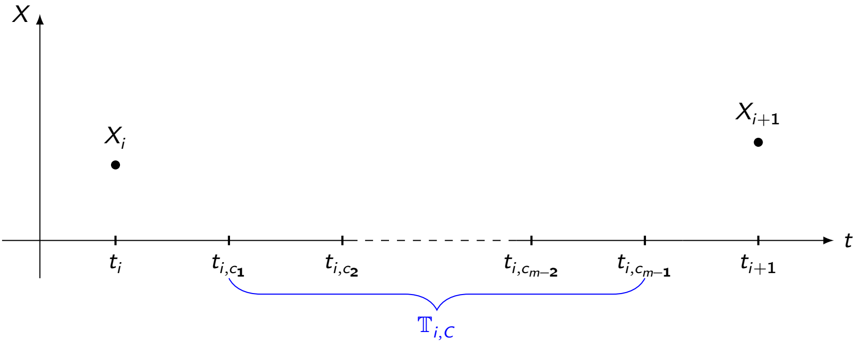
- $\mathbb{T} = \{t_i = a + i(b-a)/h,\ 0 \leq i \leq N\}$, $X \in \mathrm{C}(\mathbb{T}, \mathbb{R})$

Higher-order Variational Integrators

- $\mathbb{T} = \{t_i = a + i(b-a)/h,\ 0 \le i \le N\}$, $X \in \mathrm{C}(\mathbb{T}, \mathbb{R})$
- $\mathbb{T}_\eta = \{\eta_i,\ i = 0, \dots, m\}$: a time scale over $[0,1]$ with $\eta_0 = 0,\ \eta_m = 1.$
- $\mathbb{T}_{i,C}$: a control time scale over $]t_i, t_{i+1}[$ defined by

$$\mathbb{T}_{i,C} = \{t_{i,c_j} = t_i + \eta_j h,\ \eta_j \in \mathbb{T}_\eta^{\pm},\ j = 1, \dots, m-1\}.$$



- $\mathbb{T} = \{t_i = a + i(b-a)/h,\ 0 \leq i \leq N\}$, $X \in \mathrm{C}(\mathbb{T}, \mathbb{R})$
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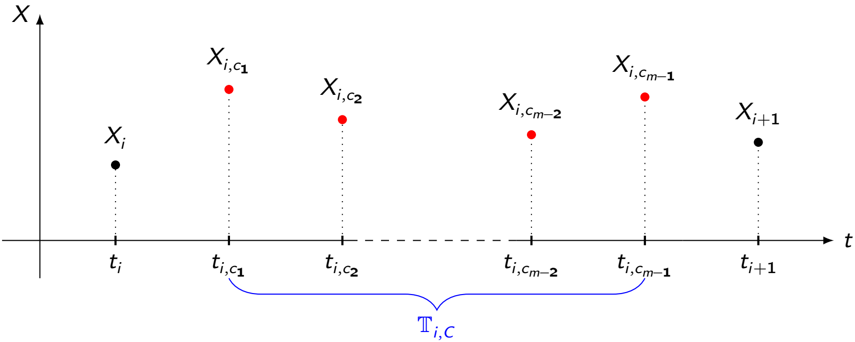
- $X_c \in \mathrm{C}(\mathbb{T}_C, \mathbb{R})$: a control function

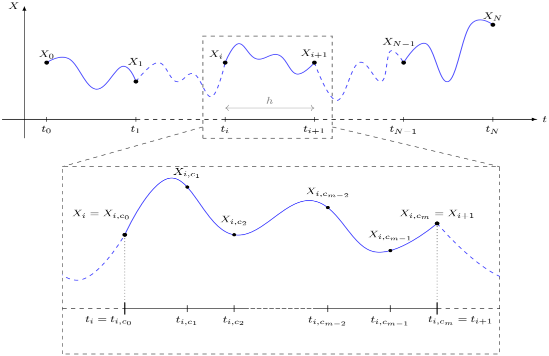


Interpolation map of degree m

- $\ell_j \in \mathscr{P}_m([0,1],\mathbb{R})$, the Lagrange polynomials
- $\kappa_{m,i} : C(\mathbb{T}\cup\mathbb{T}_C,\mathbb{R}) \longrightarrow \mathscr{P}_m([a,b],\mathbb{R})$ defined for all $t\in[t_i,t_{i+1}]$ by

$$\kappa_{m,i}(Z)(t) = \ell_0\left(\frac{t-t_i}{h}\right)X(t_i) + \ell_m\left(\frac{t-t_i}{h}\right)X(t_{i+1}) + \sum_{i=1}^{m-1}\ell_j\left(\frac{t-t_i}{h}\right)X_c(t_{i,c_j})$$

$Z \in C(\mathbb{T}\cup\mathbb{T}_C,\mathbb{R})$
$Z_{\mathbb{T}} = X,\ Z_{\mathbb{T}_C} = X_c$



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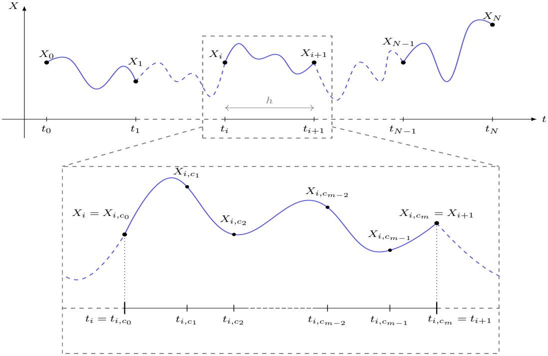
$Z \in C(\mathbb{T} \cup \mathbb{T}_C, \mathbb{R})$

$Z_{\mathbb{T}} = X, \; Z_{\mathbb{T}_C} = X_c$

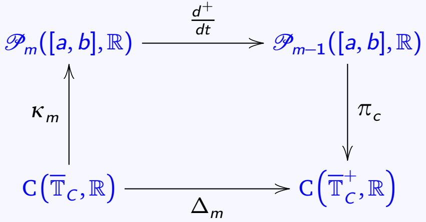


Discrete differential formula

Definition

Let $X \in \mathrm{C}(\mathbb{T},\mathbb{R})$ and Z the extension of X with respect to X_c. The derivative of Z is defined for all $t \in [t_i, t_{i+1}[$ by

$$\Delta_{m,i}\big(Z\big)(t) := \sum_{j=0}^{m} \ell'_{j,i}(t) Z(t_{i,c_j}).$$



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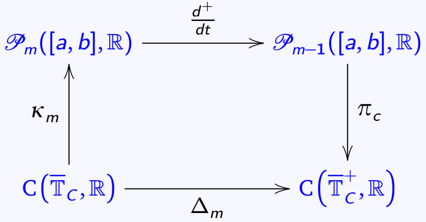
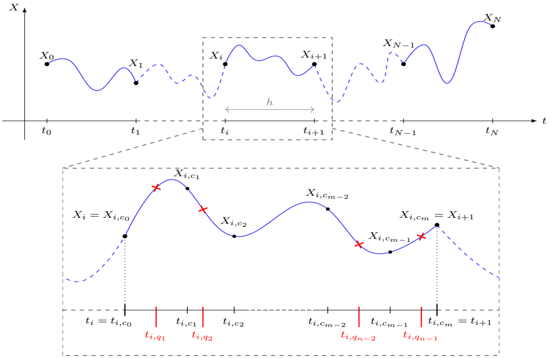


- For $m = 1 \implies \mathbb{T}_C = \emptyset$
- $\Delta_{1,i}(Z)(t) = (\Delta_+ X)_i = \dfrac{X_{i+1} - X_i}{h}$ for $i = 0, \ldots, N-1$

- $\mathbb{T}_q = \{q_i,\ i = 0,\ldots,n\}$, with $q_0 = 0,\ q_n = 1$: a quadrature
- $\mathbb{T}_{i,Q} = \{t_{i,q} = t_i + q_j h,\ q_j \in \mathbb{T}_q \setminus \{0,1\},\ j = 1,\ldots,n-1\},\ i = 0,\ldots,N-1$



Discrete integral formula

Definition (Quadrature formula)

Let $f \in \mathrm{C}(\mathbb{T} \cup \mathbb{T}_Q, \mathbb{R})$. The $(\mathbb{T}_q, w)$-integral of f over $\mathbb{T}$ denoted by $\int_a^t f(s)\ \Delta_{q,w} s$ is defined for all $t = t_k$ by

$$\begin{array}{ccc}
\mathscr{P}_m([a,b],\mathbb{R}) & \xrightarrow{\int_a^t} & \mathscr{P}_{m+1}([a,b],\mathbb{R}) \\
\uparrow \kappa_m & & \downarrow \pi_q \\
\mathrm{C}\left(\overline{\mathbb{T}}_C, \mathbb{R}\right) & \xrightarrow[(\mathbb{T}_q, w)\text{-integral}]{} & \mathrm{C}\left(\mathbb{T} \cup \mathbb{T}_Q, \mathbb{R}\right)
\end{array}$$

$w \in \mathrm{C}(\mathbb{T}_q, [0,1])$ is a weight function such that

$$\sum_{i=0}^{n} w(q_i) := \sum_{i=0}^{n} w_i = 1$$

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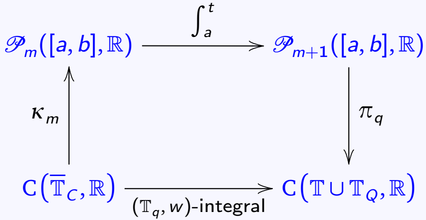
$$\int_a^{t_k} f(t)\ \Delta_{q,w} t = \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} f(t)\ \Delta_{q,w} t,$$

where

$$\int_{t_i}^{t_{i+1}} f(t)\ \Delta_{q,w} t = h \sum_{j=0}^{n} w_j f(t_i + q_j h).$$



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Theorem

Let $X \in \mathrm{C}(\overline{\mathbb{T}}_C, \mathbb{R})$ and $(\mathbb{T}_q, w)$ be a quadrature of order n over $[0,1]$. Then,

$$\int_0^1 \kappa_m \circ \Delta_m\big(X\big)(t)\ \Delta_{q,w} t = X(1) - X(0)$$

$$\Delta_m \circ \int_0^t \kappa_m\big(X\big)(s)\ \Delta_{q,w} s = X(t).$$

Theorem

Let $f \in \mathrm{C}(\mathbb{T} \cup \mathbb{T}_Q, \mathbb{R})$ and $X \in \mathrm{C}(\mathbb{T}, \mathbb{R})$ we have

$$\int_a^b f(t)\kappa_x\big(X\big)(t)\Delta_{q,w}t = \int_{\sigma(a)}^b X(t)\left[\int_t^{\sigma(t)} f^{\circ,-}(s)\Delta_{q,w}s\right]\Delta_+ t + \mathbb{F}_0, \quad (1)$$

$$\int_a^b f(t)\Delta_x X(t)\Delta_{q,w}t = \int_{\sigma(a)}^b X(t)\left[\int_t^{\sigma(t)} \Delta_q f^{\circ,-}(s)\Delta_{q,w}s\right]\Delta_+ t + \mathbb{G}_0, \quad (2)$$

Theorem

Let $f \in C(\mathbb{T} \cup \mathbb{T}_Q, \mathbb{R})$ and $X \in C(\mathbb{T}, \mathbb{R})$ we have

$$\int_a^b f(t)\kappa_x(X)(t)\Delta_{q,w}t = \int_{\sigma(a)}^b X(t)\left[\int_t^{\sigma(t)} f^{\circ,-}(s)\Delta_{q,w}s\right]\Delta_+ t + \mathbb{F}_0, \tag{1}$$

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where

$f^{\circ,-}(t) = \ell_0(t)f(t) + \ell_m(t)f^{\rho^n_{\mathbb{T}}}(t)$ then $\Delta_q f^{\circ,-}(t) = \ell_0'(t)f(t) + \ell_m'(t)f^{\rho^n_{\mathbb{T}}}(t)$.

Definition of discrete Lagrangian functional

Let $X \in C(\mathbb{T},\mathbb{R})$ and $X_c \in C(\mathbb{T}_C,\mathbb{R})$:

$$\mathcal{L}_{\eta,q,w}(X,X_c) = \int_a^b L\big(\kappa_m\big(Z\big)(t), \Delta_m\big(Z\big)(t)\big)\ \Delta_{q,w}t.$$

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$$\mathcal{L}_{\eta,q,w}(X, X_c) = \int_a^b L\big(\kappa_m(Z)(t), \Delta_m(Z)(t)\big)\ \Delta_{q,w} t.$$

Example: Let $\mathbb{T}_C = \emptyset$

- if $q = 0$, then $\mathcal{L}_{\eta,0,w}(X) = h \sum_{i=0}^{N-1} L\big(X(t_i), \Delta_+ X(t_i)\big)$
- if $q = \frac{1}{2}$, then $\mathcal{L}_{\eta,\frac{1}{2},w}(X) = h \sum_{i=0}^{N-1} L\left(\frac{X(t_i) + X(t_{i+1})}{2}, \Delta_+ X(t_i)\right)$

The Fréchet derivative of $\mathcal{L}_{\eta,q,w}$ at Z

$$D\mathcal{L}_{\eta,q,w}(Z)(H) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\big(\mathcal{L}_{\eta,q,w}(Z+\epsilon H) - \mathcal{L}_{\eta,q,w}(Z)\big)$$

The variation of $\mathcal{L}_{\eta,q,w}$ at Z along a direction $H = (V, V_c) \in \mathrm{C}_0(\mathbb{T}\cup\mathbb{T}_C, \mathbb{R})$ is

$$D\mathcal{L}_{\eta,q,w}(Z)(H) = \int_a^b \Bigg[\frac{\partial L}{\partial x}(\star(t))\,\kappa_x\big(V\big)(t) + \frac{\partial L}{\partial v}(\star(t))\,\Delta_x(V)(t) + \frac{\partial L}{\partial x}(\star(t))\,\kappa_c\big(V_c\big)(t) + \frac{\partial L}{\partial v}(\star(t))\,\Delta_c(V_c)(t)\Bigg]\Delta_{q,w}t.$$

where $\star(t) = \big(\kappa_m\big(Z\big)(t), \Delta_m\big(Z\big)(t)\big)$ and

$$\mathrm{C}_0\left(\mathbb{T}\cup\mathbb{T}_C, \mathbb{R}\right) = \left\{G \in \mathrm{C}\left(\mathbb{T}\cup\mathbb{T}_C, \mathbb{R}\right),\ G_{\mathbb{T}}(a) = G_{\mathbb{T}}(b) = 0\right\}.$$

Now, for $V=0$ or $V_c=0$, we get the following integral equations:

$$\int_a^b \left[\frac{\partial L}{\partial x}(\star(t))\,\kappa_x\big(V\big)(t)+\frac{\partial L}{\partial v}(\star(t))\,\Delta_x(V)(t)\right]\Delta_{q,w}t=0,\quad \forall V\in \mathrm{C}_0(\mathbb{T},\mathbb{R}), \tag{3}$$

$$\int_a^b \left[\frac{\partial L}{\partial x}(\star(t))\,\kappa_c\big(V_c\big)(t)+\frac{\partial L}{\partial v}(\star(t))\,\Delta_c(V_c)(t)\right]\Delta_{q,w}t=0,\quad \forall V_c\in \mathrm{C}(\mathbb{T}_C,\mathbb{R}). \tag{4}$$

Euler-Lagrange equations

The critical points of the functional $\mathcal{L}_{\eta,q,w}$ correspond to the solutions of the following Euler-Lagrange equations for all $t \in \mathbb{T}^{\pm}$

$$\int_t^{\sigma(t)} \left[\left(\frac{\partial L}{\partial x}(\star(s)) \right)^{\circ,-} + \Delta_q \left(\frac{\partial L}{\partial v}(\star(s)) \right)^{\circ,-} \right] \Delta_{q,w} s = 0$$

$$\int_t^{\sigma(t)} \ell_j(s) \frac{\partial L}{\partial x}(\star(s)) \, \Delta_{q,w} s = -\int_t^{\sigma(t)} \ell_j'(s) \frac{\partial L}{\partial v}(\star(s)) \, \Delta_{q,w} s, \quad 1 \leq j \leq m-1$$