Du bon usage des dimensions supplémentaires: de $3D$ à $4D$ et au-delà

Richard Kerner
LPTMC - Sorbonne Université, CNRS URA 7600, Paris, France

GDR Géométrie Différentielle et Mécaanique.
Atelier MMC et relativité, La Rochelle, le 6-7-2021

Introduction

- This talk is intended to illustrate a few applications of Differential Geometry to Physics.

Introduction

- This talk is intended to illustrate a few applications of Differential Geometry to Physics.
- It will be also shown how adding extra dimensions to the usual space-time formulation of the well-known theories may lead to interesting models and novel physical interpretations.

Introduction

- This talk is intended to illustrate a few applications of Differential Geometry to Physics.
- It will be also shown how adding extra dimensions to the usual space-time formulation of the well-known theories may lead to interesting models and novel physical interpretations.
- Two theories serve as examples, the classical Faraday-Maxwell electrodynamics and Einstein's General Relativity. Both rely crucially on geometry of the underlying space-time manifold, and are formulated in a geometric language.

Einstein's General Relativity in a nutshell:

The three realms: space, time and matter are intertwined in the equations paraphrasing Faraday-Maxwell's electromagnetism, resumed in Maxwell's equations

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0, \quad \partial_\mu F^{\mu\nu} = -j^\nu, \tag{1}$$

where $F_{\mu\nu}$ is the Faraday-Maxwell tensor uniting electric and magnetic fields in one geometrical entity, and $j^\mu = [c\rho, \mathbf{j}]$ is the 4-dimensional electric current density:

$$F_{0i} = E_i, \ \ F_{ij} = \epsilon_{ijk} B_k, \quad \partial_\mu j^\mu = 0 \tag{2}$$

The first set of homogeneous equations (1) becomes an identity if one introduces the four-potential $A_\mu(x^\lambda)$ such that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -F_{\nu\mu}$$

In Einstein's equations the gravitational forces are induced by the Riemann tensor $R_{\mu\nu\rho}{}^{\lambda}$ of the 4-dimensional space-time manifold, and the source is the energy-mementum tensor $T_{\mu\nu}$:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^3} \, T_{\mu\nu}, \quad \mu, \nu, ... = 0, 1, 2, 3, \tag{3}$$

where

$$R_{\mu\nu} = R_{\mu\lambda}{}^{\lambda}{}_{\nu}, \quad R = g^{\mu\nu} R_{\mu\nu},$$

and

$$R_{\mu\nu}{}^{\rho}{}_{\lambda} = \partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} - \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\lambda},$$

where $g_{\mu\nu}$ is the 4-dimensional metric tensor, and $\Gamma^\lambda_{\mu\nu}$ are the Christoffel connection coefficients defined by:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$$

Let $Y^\mu(x)$ be a 4-vector field on the 4-dimensional Riemannian manifold. Its partial derivatives do not transform as a tensor, but its covariant derivatives do:

$$\nabla_\mu Y^\lambda = \partial_\mu Y^\lambda + \Gamma^\lambda_{\mu\nu} Y^\nu. \tag{4}$$

Tbe Riemann tensor satisfies the *Bianchi identities* similar to the homogenerous Maxwell equations for the electromagnetic field $F_{\mu\nu}$:

$$\nabla_\rho R_{\mu\nu}{}^\lambda{}_\sigma + \nabla_\mu R_{\nu\rho}{}^\lambda{}_\sigma + \nabla_\nu R_{\rho\mu}{}^\lambda{}_\sigma = 0. \tag{5}$$

What we need to complete the theory is the equation of motion of a material point mass.

Let $x^\mu(s)$ be a one-parameter smooth curve in the 4-dimensional Riemannian manifold. It is called a *geodesic line* if it satisfies the following *geodesic equation*:

$$\frac{D^2 x^\lambda}{Ds^2} = \frac{d^2 x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0. \tag{6}$$

where the parameter s it the *proper length* of a four-dimensional curve:

$$ds^2 = g_{\mu\nu}\left(x^\lambda\right) dx^\mu dx^\nu, \tag{7}$$

Soon after Einstein's General Relativity was formulated in its final form (1915) David Hilbert has shown that Einstein's equations in vacuum can be derived from variational principle in 4 dimensions:

$$\delta \int \sqrt{\mid g \mid} R d^4 x = 0. \tag{8}$$

Looking for a time-independent global solution Einstein introduced a *cosmological constant* Λ in order to modify his equations in vacuo as follows:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \, R + \Lambda \, g_{\mu\nu} = 0, \tag{9}$$

which can be derived by adding the corresponding constant term to the Hilbert action integral:

$$\delta \int (\Lambda + R) \; \sqrt{\mid g \mid} \; d^4 x = 0. \tag{10}$$

- In 1921 Theodore Kaluza proposed an original mathematical scheme combining together Maxwell's and Einstein's theories.

- In 1921 Theodore Kaluza proposed an original mathematical scheme combining together Maxwell's and Einstein's theories.
- In 1926 Otto Klein improved it and discussed Kaluza's model from a quantum theoretical perspective, explaining the quantization of the electric charge. The result is known as the Kaluza-Klein theory, and is the first example of unification of several field theories.

- In 1921 Theodore Kaluza proposed an original mathematical scheme combining together Maxwell's and Einstein's theories.
- In 1926 Otto Klein improved it and discussed Kaluza's model from a quantum theoretical perspective, explaining the quantization of the electric charge. The result is known as the Kaluza-Klein theory, and is the first example of unification of several field theories.
- Further improvements and generalizations were brought by Yves Thiry in 1951 and Pascual Jordan in 1955.

- Kaluza's idea was to extend Einstein's theory to five dimensions; Klein added an assumption that the global topology should be not R^5 but rather $R^4 \times S^1$. In a 5-dimensional pseudo-Riemannian space-time a most general Kaluza-Klein metric is

$$g_{AB} = \begin{pmatrix} g_{\mu\nu} + g_{55} A_\mu A_\nu & g_{55} A_\mu \\ g_{55} A_\nu & g_{55} \end{pmatrix}, \qquad (11)$$

where $A, B = 0, 1, 2, 3, 5$ and $\mu, \nu = 0, 1, 2, 3$

- Kaluza's idea was to extend Einstein's theory to five dimensions; Klein added an assumption that the global topology should be not R^5 but rather $R^4 \times S^1$. In a 5-dimensional pseudo-Riemannian space-time a most general Kaluza-Klein metric is

$$g_{AB} = \begin{pmatrix} g_{\mu\nu} + g_{55} A_\mu A_\nu & g_{55} A_\mu \\ g_{55} A_\nu & g_{55} \end{pmatrix}, \tag{11}$$

where $A, B = 0, 1, 2, 3, 5$ and $\mu, \nu = 0, 1, 2, 3$

- The tensor $g_{\mu\nu}$ is the pseudo-Riemannian metric on the 4-dimensional submanifold, and g_{55} and A_μ are supposed to depend on coordinates, and behave as a scalar and a 4-vector fields. The generalized Kaluza-Klein theory with $g_{55}(x^\mu)$ leads to the *scalar-tensor* Brans-Dicke theory.

In its original version the model did not incorporate variable metric component $g_{55}(x^A)$, and the generalized 5-dimensional metric was reduced to:

$$g_{AB} = \begin{pmatrix} g_{\mu\nu} + A_\mu A_\nu & A_\mu \\ A_\nu & 1 \end{pmatrix}, \tag{12}$$

The variational principle was the same as for the 4-dimensional theory, with Riemann and Ricci tensors calculated in five dimensions from the 5-dimensional metric (11):

$$\delta \int \sqrt{\overset{(5)}{g}} \overset{(5)}{R} \, d^4x dx^5 = 0 \tag{13}$$

Integration along the fifth dimension yields a constant if we assume that the fifth dimension is a small circle. Then the variational principle reduces to a 4 dimensional one.

Three happy circumstances often referred to as "Kaluza-Klein miracles" deserve to be mentioned at this point:

1) $\det(\overset{(5)}{g}_{AB}) = \det(\overset{(4)}{g}_{\mu\nu})$;

2) The system of 15 equations resulting from the variational principle (13) for the 14 unknown functions $g_{\mu\nu} = g_{\nu\mu}$ and A_μ is not overdetermined, because the last equation

$$\overset{(5)}{R}_{55} - \frac{1}{2}\overset{(5)}{g}_{55}\overset{(5)}{R} = 0$$

reduces to trivial identity $0 = 0$;

3) The integrand in the five-dimensional variational principle is a sum of the four-dimensional Riemann scalar $\overset{(4)}{R}$ and the lagrangian of Maxwell's theory, $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$,

The metric (12) can be represented in a most concise and elegant manner as a symmetric 2-form spanned by the following basis of 1-forms in five dimensions, dual to the anholonomic vector basis:

$$e_A = (e_\mu, e_5) = (\partial_\mu - A_\mu \partial_5,\ \partial_5) \tag{14}$$

$$\theta^B = (\theta^\mu, \theta^5) = (dx^\mu,\ dx^5 + A_\mu dx^\mu). \tag{15}$$

One easily checks that $\theta^A(e_B) = \delta^A_B$, and that $\overset{(5)}{g}_{AB}\, \theta^A \otimes \theta^B$ gives indeed the components of (12):

$$\overset{(5)}{g}_{\mu\nu} = \overset{(4)}{g}_{\mu\nu} + A_\mu A_\nu, \quad \overset{(5)}{g}_{\mu 5} = \overset{(5)}{g}_{5\mu} = A_\mu, \quad \overset{(5)}{g}_{55} = 1.$$

The system of equations in 5 dimensions can be split in three parts:

$$\overset{(5)}{R}_{\mu\nu} - \frac{1}{2}\overset{(5)}{g}_{\mu\nu}\overset{(5)}{R} = 0 \rightarrow \overset{(4)}{R}_{\mu\nu} - \frac{1}{2}\overset{(4)}{g}_{\mu\nu}\overset{(4)}{R} = -T_{\mu\nu}, \tag{16}$$

$$\text{where} \quad T_{\mu\nu} = g^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho} - \frac{1}{4} g_{muv} \, F_{\lambda\rho} F^{\lambda\rho}$$

is the energy-momentum tensor of the electromagnetic field;

$$\overset{(5)}{R}_{\mu 5} - \frac{1}{2}\overset{(5)}{g}_{\mu 5}\overset{(5)}{R} = 0 \rightarrow \nabla_\mu F^{\mu\nu} = 0, \tag{17}$$

$$\overset{(5)}{R}_{55} - \frac{1}{2}\overset{(5)}{g}_{55}\overset{(5)}{R} = 0 \rightarrow 0 = 0. \tag{18}$$

- The Einstein-Hilbert lagrangian density (including the cosmological term) contains two invariants of the Riemann tensor, the 0-th order which is a constant, $I_0 = \text{Const.}$, and the 1-st order (linear) which is the Riemann curvature scalar $I_1 = R$.

- The Einstein-Hilbert lagrangian density (including the cosmological term) contains two invariants of the Riemann tensor, the 0-th order which is a constant, $I_0 = \text{Const.}$, and the 1-st order (linear) which is the Riemann curvature scalar $I_1 = R$.
- Although the Riemann tensor contains not only the first, but also the second partial derivatives of the metric, the variational principle leads to the second order differentials equations due to the Bianchi identities. Any other function of R, i.e. $f(R)$ replacing R as integrand in the variational principle will lead to third order differential equations, which is highly unrecommendable.

The invariants made of higher powers of the Riemann tensor san be easily found: for example, the second order contains the following combinations:

$$R_{ABCD}R^{ABCD}, \quad R_{AB}R^{AB} \quad \text{and} \quad R^2. \tag{19}$$

Each of these three quadratic expression produces third and fourth order derivative terms under variation, but there is one combination whose variation leads only to the second-order derivatives, although in non-linear combinations. It is known under the name of the *Gauss-Bonnet invariant* of the second order:

$$I_2 = R_{ABCD}R^{ABCD} - 4\, R_{AB}R^{AB} + R^2. \tag{20}$$

Other invariants, I_3, I_4, etc. can be produced, too, but we shall not use them here.

The way in which different invariants behave in a variation integral crucially depends on the dimension of the Riemannian manifold they are defined on. For example, the scalar curvature R reduces to a pure divergence in two dimensions, so that the integral is constant and the variation becomes identically null:

$$\int \sqrt{| g |} R d^2 x = 2\pi\chi = \mathrm{Constant}, \tag{21}$$

where χ is the Euler-Poincaré characteristic of the manifold over which the integral is taken. For a surface topologically equivalent with a 2-dimensional sphere $\chi = 2$ and the corresponding integral is equal to 4π, the total spherical angle.
It happens that the Gauss-Bonnet invariant I_2 reduces itself to pure divergence in 3 and 4 dimensions, so it is of no use in General Relativity. But not so in 5 dimensions, where it yields non-trivial equations under variation.

This fact was overlooked until late 80-ties. Let us see how the inclusion of the extra term proportional to I_2 alters the usual electrodynamic Lagrangian, thus modifying Maxwell's equation in vacuum. To this effect we shall consider just flat Minkowskian space with $g_{\mu\nu} = \text{diag}\,(+, - - -)$, so that the only field variables remaining in the definition of the 5-dimensional metric are the components of the 4-potential A_μ and their derivatives.
Our Lagrangian density in five dimensions will be

$$\mathcal{L} = (e^2 R + \epsilon \; I_2) \; \sqrt{\mid g \mid}. \tag{22}$$

where e^2 is a dimensional constant needed to make the whole sum homogeneous, and ϵ a small dimensionless parameter.

With Minkowskian metric $g_{\mu\nu}$ the four-dimensional Riemann tensor vanishes, and the only non-zero term in $\overset{(5)}{R}$ is the usual lagrangian of the electromagnetic field:

$$\overset{(5)}{R} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{23}$$

The second order Gauss-Bonnet invariant I_2 reduces itself to the following expression (without a part that is a total 4-divergence, and can be discarded in the variational principle):

The full Lagrangian density is now given by (we remind that now $\sqrt{\mid g \mid} = 1$):

$$\mathcal{L} = -\frac{1}{4}\, F_{\mu\nu} F^{\mu\nu} + \frac{3\epsilon}{16e^2}\, \left[(F_{\mu\nu} F^{\mu\nu})^2 - 2\, F_{\mu\lambda} F_{\nu\rho} F^{\mu\nu} F^{\lambda\rho} \right], \quad (24)$$

the resulting equations being

$$\partial_\lambda \left[F^{\lambda\rho} - \frac{3\epsilon}{2e^2}\, F_{\mu\nu} F^{\mu\nu} F^{\lambda\rho} + \frac{3\epsilon}{e^2}\, F_{\mu\nu} F^{\mu\lambda} F^{\nu\rho} \right] = 0, \quad (25)$$

and, of course, always valid

$$\partial_\mu F_{\lambda\rho} + \partial_\lambda F_{\rho\mu} + \partial_\rho F_{\mu\lambda} = 0. \quad (26)$$

Both the Lagrangian and the field equations can be written in terms of the elecrtric and magnetic fields, **E** and **B**, which enables one to have a better insight in their physical content.
Identifying the components of the Faraday-Maxwell tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as usual:

$$F_{0i} = E_i, \quad F_{ij} = \epsilon_{ijk} B_k,$$

we get:

$$\mathcal{L} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) + \frac{3\epsilon}{2e^2} (\mathbf{E} \cdot \mathbf{B})^2 \tag{27}$$

The field equation in terms of the fields **E** and **B** are:

$$\text{div } \mathbf{B} = 0, \quad \boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\text{div } \mathbf{E} = -\frac{3\epsilon}{e^2} \, \mathbf{B} \cdot [\boldsymbol{\nabla}(\mathbf{E} \cdot \mathbf{B})],$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \frac{3\epsilon}{e^2} \, \left[\mathbf{B} \cdot \frac{\partial (\mathbf{E} \cdot \mathbf{B})}{\partial t} - \mathbf{E} \times \boldsymbol{\nabla}(\mathbf{E} \cdot \mathbf{B}) \right]. \tag{28}$$

Non-linear fields act as sources of their own electric charge and current. Let us rewrite the non-homogeneous pair of Maxwell's equations:

$$\mathrm{div}\ \mathbf{E} = -\frac{3\epsilon}{e^2}\ \mathbf{B}\cdot[\boldsymbol{\nabla}(\mathbf{E}\cdot\mathbf{B})] = \rho_{ind},$$

$$\boldsymbol{\nabla}\times\mathbf{B} = \frac{\partial\mathbf{E}}{\partial t} + \frac{3\epsilon}{e^2}\ \left[\mathbf{B}\cdot\frac{\partial(\mathbf{E}\cdot\mathbf{B})}{\partial t} - \mathbf{E}\times\boldsymbol{\nabla}(\mathbf{E}\cdot\mathbf{B})\right] = \mathbf{j}_{ind}.$$

where we introduce the *induced electric charge density* ρ_{ind} and the *induced electric current* $\mathbf{j}_{ind}$ After some calculus and using the field equations one can prove that the conservation of the induced electric charge holds:

$$\frac{\partial\rho_{ind}}{\partial t} + \mathrm{div}\ \mathbf{j}_{ind} = 0. \qquad (29)$$

Let us form the following sum:

$$\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} \tag{30}$$

Replacing time derivatives of fields using the field equations, we arrive at the following conservation law:

$$\frac{\partial}{\partial t} \left[\frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2) + \frac{3\epsilon}{2e^2} (\mathbf{E} \cdot \mathbf{B})^2 \right] = \mathrm{div}(\mathbf{E} \times \mathbf{B}). \tag{31}$$

By the way, the expression for the energy density could be derived directly from the Lagrangian density: as we have

$$\mathcal{L} = \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) + \frac{3\epsilon}{2e^2}\left(\mathbf{E}\cdot\mathbf{B}\right)^2 \tag{32}$$

the Hamiltonian density $\mathcal{H}$ is defined as

$$\mathcal{H} = \mathbf{E}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{E}} - \mathcal{L}, \tag{33}$$

leading to the same expression

$$\mathcal{H} = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) + \frac{3\epsilon}{2e^2}\left(\mathbf{E}\cdot\mathbf{B}\right)^2$$

The Poynting vector conserves its form known from the Maxwellian theory, but the energy density is modified:

$$\mathbf{S} = \mathbf{E} \times \mathbf{B}, \qquad \mathcal{E} = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) + \frac{3\epsilon}{2e^2}\left(\mathbf{E}\cdot\mathbf{B}\right)^2, \tag{34}$$

with the continuity equation resuming the energy conservation satisfied by virtue of the equations of motion:

$$\frac{\partial \mathcal{E}}{\partial t} + \mathrm{div}\mathbf{S} = 0\,. \tag{35}$$

In the static case, when the fields do not depend explicitly on time t, the equations of non-linear Kaluza-Klein electrodynamics reduce to the following system:

$$\mathrm{div}\ \mathbf{B} = 0, \quad \boldsymbol{\nabla} \times \mathbf{E} = 0,$$

$$\mathrm{div}\ \mathbf{E} = -\frac{3\epsilon}{e^2}\ \mathbf{B} \cdot [\boldsymbol{\nabla}(\mathbf{E} \cdot \mathbf{B})],$$

$$\boldsymbol{\nabla} \times \mathbf{B} = -\frac{3\epsilon}{e^2}\ \mathbf{E} \times \boldsymbol{\nabla}(\mathbf{E} \cdot \mathbf{B}). \tag{36}$$

Let us try to find axially-symmetric configurations displaying both finite electric charge and finite magnetic moment; also, a kinetic momentum can be expected, parallel to the magnetic moment. In cylindric coordinates ρ, ϕ, z we expect the induced current density to be aligned on the $\mathbf{e}_\phi$-vector of the local frame, giving a current density circulating around the z-axis; the fields **E** and **B** should be contained in the $\rho - z$ planes orthogonal to $\mathbf{e}_\phi$.

Recalling the fact that the lines of strength of **B** must be closed, the best description of this configuration can be obtained using the toroidal curvilinear coordinates (μ, η, ϕ) defined as follows in terms of cylindric coordinates:

$$\rho = \frac{a \cosh \mu}{\cosh \mu - \cos \eta}, \;\; = \;\; z = \frac{a \sin \eta}{\cosh \mu - \cos \eta}, \quad \phi.$$

with $0 \leq \phi < 2\pi \, , \quad 0 \leq \eta < 2\pi \, , \quad 0 \leq \mu \leq \infty$. The coordinate lines of ϕ are concentric circles in the $(z = 0)$-plane, while the coordinate lines of the variable η are excentric tori concentrating around the circle $\rho = a$.

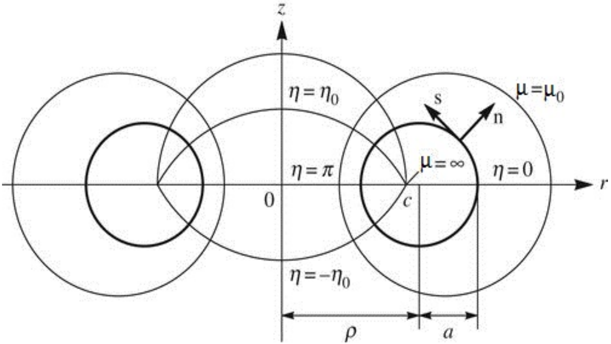

Figure: The toroidal coordinates

We shall suppose that the lines of force of the magnetic field coincide with the coordinate curves given by $\phi =$ Const. and $\mu =$ Const. The configuration we seek can be written as:

$$\mathbf{E} = E_\rho \, \mathbf{e}_\rho + E_z \, \mathbf{e}_z = E_\mu \mathbf{e}_\mu + E_\eta \, \mathbf{e}_\eta; \tag{37}$$

$$\mathbf{B} = B_\rho \, \mathbf{e}_\rho + B_z \, \mathbf{e}_z = B_\eta \, \mathbf{e}_\eta; \qquad \mathbf{j}_{ind} = j_{ind} \, \mathbf{e}_\phi. \tag{38}$$

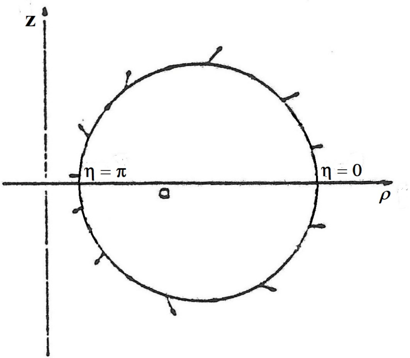

Figure: The electric field along a circle $\mu = \text{Const}$

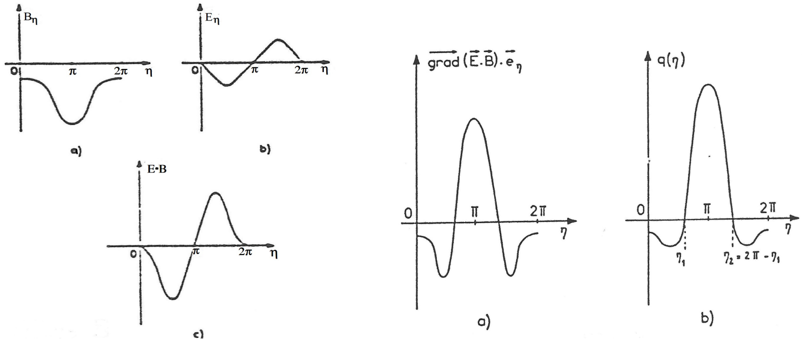

Figure: The field components E_ρ and B_η along a circle $\mu =$ Const.; the charge density along the same circle

It can be also shown that the whole problem can be reduced to determining just two unknown functions of the variables (μ, η), because $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ and $B_\mu = 0$, and because here $\mathbf{E} = -\boldsymbol{\nabla} V$, we have $\mathbf{A} = A(\mu, \eta)\, \mathbf{e}_\phi$, and $V = V(\mu, \eta)$.
Approximate solutions of this form have been found in 1987
At great distances, the fields **E** and **B** behave as if they were generated by a finite charge Q and a finite magnetic dipole **m** :

$$\mathbf{E}_\infty \simeq \frac{Q\, \mathbf{r}}{4\pi r^3}\,, \qquad \mathbf{B}_\infty \simeq \frac{\mathbf{m} \wedge \mathbf{r}}{4\pi r^3} \tag{39}$$

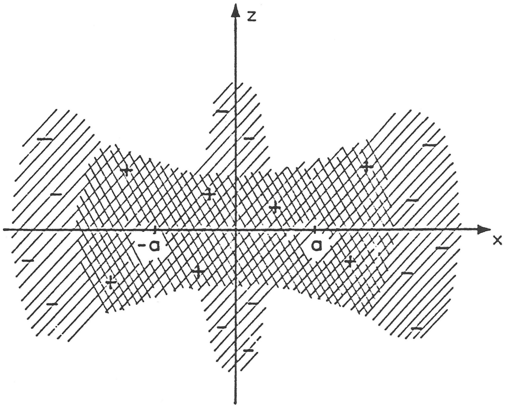

Figure: The induced charge density ρ_{ind} in the vicinity of the origin of coordinates

- The charge is concentrated around the circle $\rho = a$ and "smeared" in its vicinity; if it is chosen to be positive, there is a little "halo" of negative charge density farther away, imitating the vacuum polarization effect.

- The charge is concentrated around the circle $\rho = a$ and "smeared" in its vicinity; if it is chosen to be positive, there is a little "halo" of negative charge density farther away, imitating the vacuum polarization effect.
- The charge density's fall-off is vary rapid, behaving at short distances as r^{-9}, and then falling off exponentially; the same concerns the density of induced current $\mathbf{j}_{ind}$ which falls off as r^{-8}.

- The charge is concentrated around the circle $\rho = a$ and "smeared" in its vicinity; if it is chosen to be positive, there is a little "halo" of negative charge density farther away, imitating the vacuum polarization effect.
- The charge density's fall-off is vary rapid, behaving at short distances as r^{-9}, and then falling off exponentially; the same concerns the density of induced current $\mathbf{j}_{ind}$ which falls off as r^{-8}.
- The induced current behaves as if it was produced by the charge density rotating around the z-axis with the speed of light.

- An interesting feature of this solution is its $Z_2 \times Z_2$ symmetry. Any such solution displaying the total energy (mass) $\mathcal{E}$, the total charge Q, magnetic momentum $\mathbf{m}$ and the spin $\mathbf{s}$ is followed by three similar solutions with the same energy:

- An interesting feature of this solution is its $Z_2 \times Z_2$ symmetry. Any such solution displaying the total energy (mass) $\mathcal{E}$, the total charge Q, magnetic momentum **m** and the spin **s** is followed by three similar solutions with the same energy:
- either with the same charge, but with the spin and magnetic momentum in the opposite direction (both “down”), or a couple of solutions having the opposite charge, and spin and magnetic moment up or down, but always opposite to each other - just like with what we know about the electron and the positron.

The following table shows the properties of the four solutions:

Fields	*Energy*	*Charge*	**m**	*Spin*
E, **B**	$\mathcal{E}$	Q	**m**	**s**
E, $-$**B**	$\mathcal{E}$	Q	$-$**m**	$-$**s**
$-$**E**, **B**	$\mathcal{E}$	$-Q$	**m**	$-$**s**
$-$**E**, $-$**B**	$\mathcal{E}$	$-Q$	$-$**m**	**s**

Tab 1. The symmetry properties of four solutions.

Unfortunately, these solutions present a mild singularity on the circle $\rho = a$, which can not be avoided. Its presence can be proved by using Poincaré's lemma.

Deformations of embeddings - Introduction

- In order to take into account the perturbations of given solution of Einstein's equations we propose a novel approach using isometric embedding of an Einstein manifold into a flat (pseudo) Euclidean space.
- Instead of adding perturbations to the metric tensor

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \varepsilon \, h_{\mu\nu} + ...$$

and exploring the linearized equations satisfied by $h_{\mu\nu}$, we deform the embedding itself, which perturbs in turn the induced four-dimensional Riemannian metric induced on the embedded manifold.

Definitions

- The embeddings considered here are injective mappings of m-dimensional (pseudo)-Riemannian manifolds into N-dimensional (pseudo)-Euclidean spaces.

$$
\begin{aligned}
V_4 \quad \rightarrow \quad & E^N, x^\mu \rightarrow z^A(x^\mu), \\
& \mu, \nu, ... = 1, 2..., m., \quad N \leq m(m+1)/2.
\end{aligned}
$$

We require the smoothness and the differentiability (at least two times) of the embedding functions

The induced metric

The induced metric is easily obtained from the embedding functions $z^A(x^\lambda)$

- $$\begin{aligned} ds^2 &= \eta_{AB}\, dz^A dz^B \\ &= \eta_{AB} \frac{\partial z^A}{\partial x^\mu} \frac{\partial z^B}{\partial x^\nu} dx^\mu dx^\nu \equiv \overset{o}{g}_{\mu\nu}\, dx^\mu dx^\nu \end{aligned}$$

The induced metric

The induced metric is easily obtained from the embedding functions $z^A(x^\lambda)$

- $$ds^2 = \eta_{AB}\, dz^A dz^B = \eta_{AB} \frac{\partial z^A}{\partial x^\mu} \frac{\partial z^B}{\partial x^\nu} dx^\mu dx^\nu \equiv \overset{o}{g}_{\mu\nu}\, dx^\mu dx^\nu$$

- $$\overset{o}{g}_{\mu\nu} = \eta_{AB} \frac{\partial z^A}{\partial x^\mu} \frac{\partial z^B}{\partial x^\nu}$$

The induced metric

The induced metric is easily obtained from the embedding functions $z^A(x^\lambda)$

- $$\begin{aligned} ds^2 &= \eta_{AB}\, dz^A dz^B \\ &= \eta_{AB} \frac{\partial z^A}{\partial x^\mu} \frac{\partial z^B}{\partial x^\nu} dx^\mu dx^\nu \equiv \overset{o}{g}_{\mu\nu}\, dx^\mu dx^\nu \end{aligned}$$

- $$\overset{o}{g}_{\mu\nu} = \eta_{AB} \frac{\partial z^A}{\partial x^\mu} \frac{\partial z^B}{\partial x^\nu}$$

- $\overset{o}{g}{}^{\mu\nu}$ cannot be obtained directly from the embedding functions, but should be computed from the covariant metric as its inverse matrix.

The Christoffel symbols

The Christoffel symbols are easily expressed by means of first and second derivatives of the embedding functions $z^A(x^\mu)$

- $$\begin{aligned} \overset{o}{\Gamma}{}^{\lambda}_{\mu\nu} &\equiv \frac{1}{2}\, \overset{o}{g}{}^{\lambda\sigma}\,(x^\sigma)\left(\partial_\mu\, \overset{o}{g}_{\sigma\nu} + \partial_\nu\, \overset{o}{g}_{\mu\sigma} - \partial_\sigma\, \overset{o}{g}_{\mu\nu}\right) \\ &= \eta_{AB}\, \overset{o}{g}{}^{\lambda\sigma}\, \partial_\sigma z^A \partial^2_{\mu\nu} z^B \end{aligned}$$

The Christoffel symbols

The Christoffel symbols are easily expressed by means of first and second derivatives of the embedding functions $z^A(x^\mu)$

- $$\overset{o}{\Gamma}{}^{\lambda}_{\mu\nu} \equiv \frac{1}{2}\,\overset{o}{g}{}^{\lambda\sigma}(x^\sigma)\left(\partial_\mu\,\overset{o}{g}_{\sigma\nu} + \partial_\nu\,\overset{o}{g}_{\mu\sigma} - \partial_\sigma\,\overset{o}{g}_{\mu\nu}\right) = \eta_{AB}\,\overset{o}{g}{}^{\lambda\sigma}\,\partial_\sigma z^A \partial^2_{\mu\nu} z^B$$

- $$\partial^2_{\mu\nu} z^A = \nabla_\mu \nabla_\nu z^A + \overset{o}{\Gamma}{}^{\lambda}_{\mu\nu}\,\partial_\lambda z^B$$

The Christoffel symbols

The Christoffel symbols are easily expressed by means of first and second derivatives of the embedding functions $z^A(x^\mu)$

- $$\overset{o}{\Gamma}{}^\lambda_{\mu\nu} \equiv \frac{1}{2}\,\overset{o}{g}{}^{\lambda\sigma}(x^\sigma)\left(\partial_\mu\,\overset{o}{g}_{\sigma\nu} + \partial_\nu\,\overset{o}{g}_{\mu\sigma} - \partial_\sigma\,\overset{o}{g}_{\mu\nu}\right)$$
  $$= \eta_{AB}\,\overset{o}{g}{}^{\lambda\sigma}\,\partial_\sigma z^A \partial^2_{\mu\nu} z^B$$
- $$\partial^2_{\mu\nu} z^A = \nabla_\mu \nabla_\nu z^A + \overset{o}{\Gamma}{}^\lambda_{\mu\nu}\,\partial_\lambda z^B$$
- $$\eta_{AB}\,\overset{o}{g}{}^{\lambda\rho}\,\nabla_\rho z^A \nabla_\mu \nabla_\nu z^B = 0$$

The Riemann tensor

The Gauss-Codazzi equations express the Riemann tensor exclusively by means of *second derivatives* of the embedding functions:

- $$\eta_{AB}\left[\nabla_\mu\left(\nabla_\rho z^A \nabla_\nu \nabla_\sigma z^B\right) - \nabla_\nu\left(\nabla_\rho z^A \nabla_\mu \nabla_\sigma z^B\right)\right] = 0$$

The Riemann tensor

The Gauss-Codazzi equations express the Riemann tensor exclusively by means of *second derivatives* of the embedding functions:

- $$\eta_{AB}\left[\nabla_\mu\left(\nabla_\rho z^A \nabla_\nu \nabla_\sigma z^B\right) - \nabla_\nu\left(\nabla_\rho z^A \nabla_\mu \nabla_\sigma z^B\right)\right] = 0$$

- $$\left[\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right] \nabla_\rho z^A \equiv \overset{o}{R}_{\mu\nu}{}^{\lambda}{}_{\rho} \, \nabla_\lambda z^A$$

The Riemann tensor

The Gauss-Codazzi equations express the Riemann tensor exclusively by means of *second derivatives* of the embedding functions:

- $$\eta_{AB}\left[\nabla_\mu\left(\nabla_\rho z^A \nabla_\nu \nabla_\sigma z^B\right) - \nabla_\nu\left(\nabla_\rho z^A \nabla_\mu \nabla_\sigma z^B\right)\right] = 0$$

- $$\left[\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right] \nabla_\rho z^A \equiv \overset{o}{R}_{\mu\nu}{}^{\lambda}{}_{\rho}\, \nabla_\lambda z^A$$

- $$\overset{o}{R}_{\mu\nu\lambda\rho} = -\eta_{AB}\left[\nabla_\mu \nabla_\lambda z^A \nabla_\nu \nabla_\rho z^B - \nabla_\nu \nabla_\lambda z^A \nabla_\mu \nabla_\rho z^B\right]$$

Definition

Supposing that the deformation of geometry is provoked by small perturbations characterized by an infinitesimal parameter ε, we can expand the new embedding functions into a series of powers of ε:

$$z^A(x^\mu) \;\rightarrow\; \tilde{z}^A(x^\mu) = z^A(x^\mu) + \varepsilon\, v^{\,A}(x^\mu) + \varepsilon^2\, w^{\,A}(x^\mu) + \dots$$

$$\overset{o}{V}_4 \;\rightarrow\; \tilde{V}_4$$

$$\overset{\mathrm{o}}{g}_{\mu\nu} \;\rightarrow\; \tilde{g}_{\mu\nu}$$

$$\vdots$$

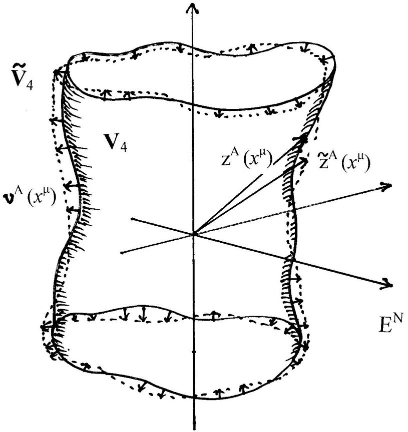

Figure: The initial embedding of a Riemannian manifold V_4 and an infinitesimal deformation producing the new embedding $\tilde{V}_4$

Deformation of the covariant metric

The (covariant) metric tensor of the deformed manifold can be expanded as series in powers of ε:

$$\begin{aligned}\tilde{g}_{\mu\nu} &= \eta_{AB}\frac{\partial \tilde{z}^A}{\partial x^\mu}\frac{\partial \tilde{z}^B}{\partial x^\nu} \\ &= \overset{\circ}{g}_{\mu\nu} + \varepsilon \overset{1}{g}_{\mu\nu} + \varepsilon^2 \overset{2}{g}_{\mu\nu} + \dots \\ &= \eta_{AB}\Big[\partial_\mu z^A \partial_\nu z^B + \varepsilon\Big(\partial_\mu z^A \partial_\nu v^B + \partial_\mu v^A \partial_\nu z^B\Big) \\ &+ \varepsilon^2\Big(\partial_\mu v^A \partial_\nu v^B + \partial_\mu z^A \partial_\nu w^B + \partial_\mu w^A \partial_\nu z^B\Big)\Big] + \dots\end{aligned}$$

Deformations which do not alter the geometry

A wide class of deformations of the embedding will not alter the induced metric:

- $$\tilde{z}^A = z^A + const.$$

Deformations which do not alter the geometry

A wide class of deformations of the embedding will not alter the induced metric:

- $$\tilde{z}^A = z^A + const.$$
- $$\tilde{z}^A = z^A + \varepsilon \Lambda^A_B z^B$$

Deformations which do not alter the geometry

A wide class of deformations of the embedding will not alter the induced metric:

- $$\tilde{z}^A = z^A + const.$$
- $$\tilde{z}^A = z^A + \varepsilon \Lambda^A_B z^B$$
- $$\eta_{AC} \Lambda^C_B + \eta_{BC} \Lambda^C_A = 0$$

The gauge degrees of freedom

The deformations of the embedding conserving the induced metric can be considered as "pure gauge" transformations, akin to the change of coordinates in the embedded V_4:

- $$g_{\mu\nu} = \overset{\circ}{g}_{\mu\nu} + \varepsilon\, h_{\mu\nu},$$

  $$\nabla_\mu\, h^{\mu\nu} = 0$$

The gauge degrees of freedom

The deformations of the embedding conserving the induced metric can be considered as "pure gauge" transformations, akin to the change of coordinates in the embedded V_4:

- $$g_{\mu\nu} = \overset{\circ}{g}_{\mu\nu} + \varepsilon \, h_{\mu\nu},$$
  $$\nabla_\mu \, h^{\mu\nu} = 0$$
- $$\eta_{AB} (\partial_\mu z^A) v^B = 0$$

Deformations which need a host space of higher dimension

Sometimes, extra conditions on the resulting deformed geometry can lead to an overdefined system which cannot be solved without extra degrees of freedom, i.e. introducing extra dimensions:

- $$E^{N+m} = E^N \oplus E^m$$

Deformations which need a host space of higher dimension

Sometimes, extra conditions on the resulting deformed geometry can lead to an overdefined system which cannot be solved without extra degrees of freedom, i.e. introducing extra dimensions:

- $$E^{N+m} = E^N \oplus E^m$$
- $$\eta_{\alpha\beta} = \begin{pmatrix} \eta_{AB} & 0 \\ 0 & \eta_{ij} \end{pmatrix},$$

Deformations which need a host space of higher dimension

Sometimes, extra conditions on the resulting deformed geometry can lead to an overdefined system which cannot be solved without extra degrees of freedom, i.e. introducing extra dimensions:

- $$E^{N+m} = E^N \oplus E^m$$
- $$\eta_{\alpha\beta} = \begin{pmatrix} \eta_{AB} & 0 \\ 0 & \eta_{ij} \end{pmatrix},$$
- $$\begin{aligned} v^\alpha &= [\, v^A,\, v^m \,] \\ z^\alpha &= [\, z^A,\, 0 \,] \end{aligned}$$

Deformations which need a host space of higher dimension

Sometimes, extra conditions on the resulting deformed geometry can lead to an overdefined system which cannot be solved without extra degrees of freedom, i.e. introducing extra dimensions:

- $$E^{N+m} = E^N \oplus E^m$$
- $$\eta_{\alpha\beta} = \begin{pmatrix} \eta_{AB} & 0 \\ 0 & \eta_{ij} \end{pmatrix},$$
- $$\begin{aligned} v^\alpha &= [\, v^A, \, v^m \,] \\ z^\alpha &= [\, z^A, \, 0 \,] \end{aligned}$$

so that the deformations towards other dimensions are visible only from the second order on

The contravariant metric

- $$\tilde{g}^{\mu\nu}\,\tilde{g}_{\nu\sigma} = \delta^{\mu}_{\sigma}$$

The contravariant metric

- $$\tilde{g}^{\mu\nu}\tilde{g}_{\nu\sigma} = \delta^{\mu}_{\sigma}$$

- $$\begin{aligned} \tilde{g}^{\mu\nu} &= \overset{o}{g}{}^{\mu\nu} + \varepsilon \overset{1}{g}{}^{\mu\nu} + \varepsilon^2 \overset{2}{g}{}^{\mu\nu} + \dots \\ &= \overset{o}{g}{}^{\mu\nu} - \varepsilon \overset{o}{g}{}^{\mu\rho} \overset{o}{g}{}^{\mu\sigma} \overset{1}{g}{}_{\rho\sigma} \\ &- \varepsilon^2 \left[\overset{o}{g}{}^{\mu\rho} \overset{o}{g}{}^{\mu\sigma} \overset{2}{g}{}_{\rho\sigma} - \overset{o}{g}{}^{\mu\rho} \overset{o}{g}{}^{\mu\sigma} \overset{o}{g}{}^{\lambda\kappa} \overset{1}{g}{}_{\rho\lambda} \overset{1}{g}{}_{\sigma\kappa} \right] + \dots \end{aligned}$$

The deformed Christoffel symbols

- $$\tilde{\Gamma}^{\mu}_{\nu\rho} = \frac{1}{2}\tilde{g}^{\mu\kappa}\left(\partial_{\nu}\tilde{g}_{\rho\kappa} + \partial_{\rho}\tilde{g}_{\nu\kappa} - \partial_{\kappa}\tilde{g}_{\rho\nu}\right)$$

The deformed Christoffel symbols

- $$\tilde{\Gamma}^{\mu}_{\nu\rho} = \frac{1}{2}\tilde{g}^{\mu\kappa}\left(\partial_{\nu}\tilde{g}_{\rho\kappa} + \partial_{\rho}\tilde{g}_{\nu\kappa} - \partial_{\kappa}\tilde{g}_{\rho\nu}\right)$$

- $$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \overset{\mathrm{o}}{\Gamma}{}^{\lambda}_{\mu\nu} + \varepsilon\,\overset{1}{\Gamma}{}^{\lambda}_{\mu\nu} + \varepsilon^2\overset{2}{\Gamma}{}^{\lambda}_{\mu\nu} + \dots\,,$$

The deformed Christoffel symbols

- $$\tilde{\Gamma}^{\mu}_{\nu\rho} = \frac{1}{2}\tilde{g}^{\mu\kappa}\left(\partial_\nu \tilde{g}_{\rho\kappa} + \partial_\rho \tilde{g}_{\nu\kappa} - \partial_\kappa \tilde{g}_{\rho\nu}\right)$$
- $$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\lambda}_{\mu\nu} + \varepsilon \overset{1}{\Gamma}{}^{\lambda}_{\mu\nu} + \varepsilon^2 \overset{2}{\Gamma}{}^{\lambda}_{\mu\nu} + \dots ,$$
- $$\overset{1}{\Gamma}{}^{\lambda}_{\mu\nu} = \eta_{AB}\, \overset{\circ}{g}{}^{\lambda\rho} \left[\nabla_\rho z^A \nabla_\mu \nabla_\nu \, v^B + \nabla_\rho v^A \nabla_\mu \nabla_\nu \, z^B \right].$$

The deformed Riemann tensor

- $$\tilde{R}_{\nu\mu\lambda\rho} = \eta_{AB} \left[\tilde{\nabla}_\mu \tilde{\nabla}_\lambda \tilde{z}^A \tilde{\nabla}_\nu \tilde{\nabla}_\rho \tilde{z}^B - \tilde{\nabla}_\nu \tilde{\nabla}_\lambda \tilde{z}^A \tilde{\nabla}_\mu \tilde{\nabla}_\rho \tilde{z}^B \right]$$

The deformed Riemann tensor

- $$\tilde{R}_{\nu\mu\lambda\rho} = \eta_{AB}\left[\tilde{\nabla}_\mu \tilde{\nabla}_\lambda \tilde{z}^A \tilde{\nabla}_\nu \tilde{\nabla}_\rho \tilde{z}^B - \tilde{\nabla}_\nu \tilde{\nabla}_\lambda \tilde{z}^A \tilde{\nabla}_\mu \tilde{\nabla}_\rho \tilde{z}^B\right]$$

- $$\tilde{\nabla}_\mu \tilde{\nabla}_\nu \tilde{z}^A = \tilde{\nabla}_\mu \tilde{\nabla}_\nu z^A + \varepsilon \tilde{\nabla}_\mu \tilde{\nabla}_\nu v^A + \dots$$
$$= \nabla_\mu \nabla_\nu z^A + \varepsilon\left[\nabla_\mu \nabla_\nu v^A - \overset{1}{\Gamma}{}^{\lambda}_{\mu\nu} \nabla_\lambda z^A\right] + \dots$$

The deformed Riemann tensor

- $$\tilde{R}_{\nu\mu\lambda\rho} = \eta_{AB} \left[\tilde{\nabla}_\mu \tilde{\nabla}_\lambda \tilde{z}^A \tilde{\nabla}_\nu \tilde{\nabla}_\rho \tilde{z}^B - \tilde{\nabla}_\nu \tilde{\nabla}_\lambda \tilde{z}^A \tilde{\nabla}_\mu \tilde{\nabla}_\rho \tilde{z}^B \right]$$

- $$\begin{aligned} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \tilde{z}^A &= \tilde{\nabla}_\mu \tilde{\nabla}_\nu z^A + \varepsilon \tilde{\nabla}_\mu \tilde{\nabla}_\nu v^A + \dots \\ = \nabla_\mu \nabla_\nu z^A &+ \varepsilon \left[\nabla_\mu \nabla_\nu v^A - \overset{1}{\Gamma}{}^{\lambda}_{\mu\nu} \nabla_\lambda z^A \right] + \dots \end{aligned}$$

- $$\overset{1}{R}_{\nu\mu\lambda\rho} = \Big[\nabla_\mu \nabla_\lambda z^A \nabla_\nu \nabla_\rho v^B + \nabla_\mu \nabla_\lambda v^A \nabla_\nu \nabla_\rho z^B - \nabla_\nu \nabla_\lambda z^A \nabla_\mu \nabla_\rho v^B - \nabla_\nu \nabla_\lambda v^A \nabla_\mu \nabla_\rho z^B \Big] \eta_{AB}$$

Deformation of the Ricci tensor and curvature scalar

- $$\tilde{R}_{\mu\nu} = \tilde{g}^{\kappa\sigma} \tilde{R}_{\kappa\mu\sigma\nu}$$
$$\tilde{R} = \tilde{g}^{\mu\nu} \tilde{R}_{\mu\nu}$$

Deformation of the Ricci tensor and curvature scalar

- $$\tilde{R}_{\mu\nu} = \tilde{g}^{\kappa\sigma} \tilde{R}_{\kappa\mu\sigma\nu}$$
  $$\tilde{R} = \tilde{g}^{\mu\nu} \tilde{R}_{\mu\nu}$$

- $$\overset{1}{R}_{\mu\nu} = \overset{o}{g}{}^{\kappa\sigma} \overset{1}{R}_{\kappa\mu\sigma\nu} + \overset{1}{g}{}^{\kappa\sigma} \overset{o}{R}_{\kappa\mu\sigma\nu}$$
  $$\overset{1}{R} = \overset{o}{g}{}^{\mu\nu} \overset{1}{R}_{\mu\nu} + \overset{1}{g}{}^{\mu\nu} \overset{o}{R}_{\mu\nu}$$

The deformed Einstein tensor

$$\tilde{G}_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R}$$

$$= \overset{\circ}{R}_{\mu\nu} - \frac{1}{2}\overset{\circ}{g}_{\mu\nu}\overset{\circ}{R}$$

$$+\varepsilon\left[\overset{1}{R}_{\mu\nu} - \frac{1}{2}\left(\overset{1}{g}_{\mu\nu}\overset{\circ}{R} + \overset{\circ}{g}_{\mu\nu}\overset{1}{R}\right)\right] + \ldots$$

First order equations in vacuo

- $$\overset{1}{G}_{\mu\nu} = \overset{1}{R}_{\mu\nu} - \frac{1}{2}\overset{\text{o}}{g}_{\mu\nu}\overset{1}{R} = \left(\delta^\rho_\mu\delta^\kappa_\nu - \frac{1}{2}\overset{\text{o}}{g}_{\mu\nu}\overset{\text{o}}{g}^{\rho\kappa}\right)\overset{1}{R}_{\rho\kappa}$$

First order equations in vacuo

- $$\overset{1}{G}_{\mu\nu} = \overset{1}{R}_{\mu\nu} - \frac{1}{2}\overset{o}{g}_{\mu\nu}\overset{1}{R} = \\ = \left(\delta^{\rho}_{\mu}\delta^{\kappa}_{\nu} - \frac{1}{2}\overset{o}{g}_{\mu\nu}\overset{o}{g}^{\rho\kappa}\right)\overset{1}{R}_{\rho\kappa}$$

- $$\overset{1}{R}_{\mu\nu} = 0$$

The first order corrections to Einstein's equations

$$\overset{1}{R}_{\nu\sigma} = U_{\nu\sigma}{}^{\mu\gamma}{}_A \nabla_\mu \nabla_\gamma v^A + V_{\nu\sigma}{}^{\mu}{}_A \nabla_\mu v^A = 0$$

$$\begin{aligned} U_{\nu\rho}{}^{\xi\gamma}{}_A \quad &\equiv \quad \eta_{AB} \overset{o}{g}{}^{\mu\kappa} \Big(\delta^\xi_\mu \delta^\gamma_\rho \nabla_\nu \nabla_\kappa z^B + \delta^\xi_\nu \delta^\gamma_\kappa \nabla_\mu \nabla_\rho z^B \\ &- \quad \delta^\xi_\nu \delta^\gamma_\rho \nabla_\mu \nabla_\kappa z^B - \delta^\xi_\mu \delta^\gamma_\kappa \nabla_\nu \nabla_\rho z^B \Big) \end{aligned}$$

$$V_{\nu\rho}{}^{\mu}{}_A \equiv \overset{o}{g}{}^{\mu\kappa} \overset{o}{g}{}^{\sigma\beta} \Big(\delta^\mu_\beta \nabla_\kappa z^B \nabla_\nu \nabla_\sigma z^C \nabla_\mu \nabla_\rho z^D + \delta^\xi_\kappa \nabla_\beta z^B \nabla_\nu \nabla_\sigma z^C \nabla_\mu \nabla_\rho z^D$$

$$- \delta^\xi_\beta \nabla_\kappa z^B \nabla_\mu \nabla_\sigma z^C \nabla_\nu \nabla_\rho z^D - \delta^\xi_\kappa \nabla_\beta z^B \nabla_\mu \nabla_\sigma z^C \nabla_\nu \nabla_\rho z^D \Big) \eta_{CD} \eta_{AB}$$

Einstein's equations with sources

- $$\tilde{G}_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

Einstein's equations with sources

- $$\tilde{G}_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

- $$\overset{1}{R}_{\mu\nu} - \frac{1}{2}\overset{\circ}{g}_{\mu\nu}\overset{1}{R} = -\frac{8\pi G}{c^4}\ \overset{1}{T}_{\mu\nu}$$

Einstein's equations with sources

- $$\tilde{G}_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$
- $$\overset{1}{R}_{\mu\nu} - \frac{1}{2}\overset{\circ}{g}_{\mu\nu}\overset{1}{R} = -\frac{8\pi G}{c^4}\ \overset{1}{T}_{\mu\nu}$$
- $$\overset{1}{R}_{\mu\nu} = -\frac{8\pi G}{c^4}\left(\delta^{\rho}_{\mu}\delta^{\kappa}_{\nu} - \frac{1}{2}\overset{\circ}{g}_{\mu\nu}\overset{\circ}{g}^{\rho\kappa}\right)\ \overset{1}{T}_{\rho\kappa}$$

Second order deformations: the starting point is the expansion

- $$\tilde{z}^A(x^\mu) = z^A(x^\mu) + \varepsilon\, v^A(x^\mu) + \varepsilon^2\, w^A(x^\mu) + ...,$$

Second order deformations: the starting point is the expansion

- $$\tilde{z}^A\,(x^\mu) = z^A\,(x^\mu) + \varepsilon\, v^A\,(x^\mu) + \varepsilon^2\, w^A\,(x^\mu) + ...,$$
- $$\varepsilon^2 \left(\partial_\mu v^A \partial_\nu v^B + \partial_\mu z^A \partial_\nu w^B + \partial_\mu w^A \partial_\nu z^B \right) + \ldots$$

Second order deformations: the starting point is the expansion

- $$\tilde{z}^A(x^\mu) = z^A(x^\mu) + \varepsilon\, v^A(x^\mu) + \varepsilon^2\, w^A(x^\mu) + ...,$$

- $$\varepsilon^2 \left(\partial_\mu v^A \partial_\nu v^B + \partial_\mu z^A \partial_\nu w^B + \partial_\mu w^A \partial_\nu z^B \right) + \dots$$

- $$\begin{aligned} \overset{2}{R}_{\mu\nu\sigma\rho} &= \eta_{AB} \Big[\nabla_\nu \nabla_\sigma z^A \nabla_\mu \nabla_\rho w^B + \nabla_\nu \nabla_\sigma w^A \nabla_\mu \nabla_\rho z^B - \\ &- \nabla_\mu \nabla_\sigma z^A \nabla_\nu \nabla_\rho w^B - \nabla_\mu \nabla_\sigma w^A \nabla_\nu \nabla_\rho z^B \Big] - \\ &- \overset{\mathrm{o}}{g}_{\kappa\lambda} \overset{1}{\Gamma}{}^{\kappa}_{\nu\sigma} \overset{1}{\Gamma}{}^{\lambda}_{\mu\rho} + \eta_{AB} \nabla_\nu \nabla_\sigma v^A \nabla_\mu \nabla_\rho v^B + \overset{\mathrm{o}}{g}_{\kappa\lambda} \overset{1}{\Gamma}{}^{\kappa}_{\mu\sigma} \overset{1}{\Gamma}{}^{\lambda}_{\nu\rho} - \\ &- \eta_{AB} \nabla_\mu \nabla_\sigma v^A \nabla_\nu \nabla_\rho v^B \end{aligned}$$

Second order deformation of the Ricci tensor

- $$\overset{2}{R}_{\nu\rho} = \overset{\mathrm{o}}{g}{}^{\mu\sigma} \overset{2}{R}_{\mu\nu\sigma\rho} + \overset{1}{g}{}^{\mu\sigma} \overset{1}{R}_{\mu\nu\sigma\rho} + \overset{2}{g}{}^{\mu\sigma} \overset{\mathrm{o}}{R}_{\mu\nu\sigma\rho}$$

Second order deformation of the Ricci tensor

- $$\overset{2}{R}_{\nu\rho} = \overset{o}{g}{}^{\mu\sigma} \overset{2}{R}_{\mu\nu\sigma\rho} + \overset{1}{g}{}^{\mu\sigma} \overset{1}{R}_{\mu\nu\sigma\rho} + \overset{2}{g}{}^{\mu\sigma} \overset{o}{R}_{\mu\nu\sigma\rho}$$

$$\overset{2}{R}_{\nu\rho} = 0 \Longrightarrow U_{\nu\sigma}{}^{\xi\gamma}{}_{A} \nabla_{\xi} \nabla_{\gamma} w^{A} + V_{\nu\sigma}{}^{\xi}{}_{A} \nabla_{\xi} w^{A} = B_{\nu\rho}(v^{A}, z^{A})$$

with $B_{\nu\rho}(v^A, z^A)$ combination of derivative of v^A and z^A functions.

-

Generalities

- Consider the Minkowskian space-time M_4 parametrized by cartesian coordinates $x^\mu = [ct, x, y, z]$

$$M_4 \rightarrow E^5_{1,4}$$

$$z^1 = ct,\ z^2 = x,\ z^3 = y,\ z^4 = z, \quad z^5 = 0,$$

Generalities

- Consider the Minkowskian space-time M_4 parametrized by cartesian coordinates $x^\mu = [ct, x, y, z]$

$$M_4 \to E^5_{1,4}$$

$$z^1 = ct,\; z^2 = x,\; z^3 = y,\; z^4 = z, \quad z^5 = 0,$$

- $$z^5 = \varepsilon\, v(x^\mu) + \varepsilon^2\, w(x^\mu) + \varepsilon^3\, h(x^\mu) + \ldots$$

The second and third order deformations

- $$\overset{2}{R}_{\mu\rho} = \overset{o}{g}{}^{\lambda\nu} \left[\nabla_\mu \nabla_\lambda v \, \nabla_\nu \nabla_\rho v - \nabla_\nu \nabla_\lambda v \, \nabla_\mu \nabla_\rho v \right] = 0$$

The second and third order deformations

- $$\overset{2}{R}_{\mu\rho} = \overset{\circ}{g}{}^{\lambda\nu} \left[\nabla_\mu \nabla_\lambda v \, \nabla_\nu \nabla_\rho v - \nabla_\nu \nabla_\lambda v \, \nabla_\mu \nabla_\rho v \right] = 0$$

- $$v(x^\mu) = f\,(k_\mu \, x^\mu)$$

The second and third order deformations

- $$\overset{2}{R}_{\mu\rho} = \overset{\circ}{g}{}^{\lambda\nu} \left[\nabla_\mu \nabla_\lambda v \, \nabla_\nu \nabla_\rho v - \nabla_\nu \nabla_\lambda v \, \nabla_\mu \nabla_\rho v \right] = 0$$

- $$v(x^\mu) = f\,(k_\mu \, x^\mu)$$

- $$\begin{aligned} \overset{3}{R}_{\nu\rho} &= \overset{\circ}{g}{}^{\mu\sigma} \, (\nabla_\mu \nabla_\rho v \nabla_\nu \nabla_\sigma w + \nabla_\mu \nabla_\rho w \nabla_\nu \nabla_\sigma v - \\ &- \nabla_\nu \nabla_\rho v \nabla_\mu \nabla_\sigma w - \nabla_\nu \nabla_\rho w \nabla_\mu \nabla_\sigma v) \end{aligned}$$

The plane wave solution

- ansatz: $z^5(x^\mu) = \varepsilon v(t,z) + \varepsilon^2 w(x,y)$

  with $v(t,z) = e^{i(\omega\, t - k\, z)}$

The plane wave solution

- ansatz: $z^5(x^\mu) = \varepsilon v(t,z) + \varepsilon^2 w(x,y)$

  with $v(t,z) = e^{i(\omega\, t - k\, z)}$

- From the components xx, xy, and yy of Einstein's equations *in vacuo*, $\overset{3}{R}_{\nu\rho} = 0$, we obtain:

$$\overset{\circ}{g}{}^{\mu\rho}\, \partial^2_{\mu\rho}\, v = 0$$

  which imposes the on-shell condition $\omega^2 = c^2 k^2$

The plane wave solution

- ansatz: $z^5(x^\mu) = \varepsilon v(t,z) + \varepsilon^2 w(x,y)$

  with $v(t,z) = e^{i(\omega\, t - k\, z)}$
- From the components xx, xy, and yy of Einstein's equations *in vacuo*, $\overset{3}{R}_{\nu\rho} = 0$, we obtain:

$$\overset{\circ}{g}^{\mu\rho}\, \partial^2_{\mu\rho}\, v = 0$$

  which imposes the on-shell condition $\omega^2 = c^2 k^2$
- From the remaining non-zero components we have:

$$\partial^2_{xx}\, w \; + \partial^2_{yy}\, w \; = \nabla^2 w(x,y) = 0$$

Obviously, the solution is given by a general second order polynomial in the variables (x, y), i.e.

$$w(x, y) = Ax^2 + Bxy + Cy^2 \tag{40}$$

Thde Laplace equation imposed on $w(x, y)$ leads to the coondition $A + C = 0$, thus leaving only two degrees of freedom for the transversal wave of the form

$$[\, A(x^2 - y^2) + Bxy \,] \cos(\omega t - kx) \tag{41}$$

which shows its quadrupolar character.
After rescaling, the second-order polynomial can be written in the complex form $x^2 + 2ixy - y^2 = (x + iy)^2$. After an arbitrary complex rotation $(x + iy) \to e^{i\phi}(x + iy)$, the expression $x^2 + 2ixy - y^2$ will acquire the phase $e^{2i\phi}$, typical for the fields of spin 2.

Global embeddings of (exterior) Schwarzschild solution are known since a long time, e.g. E. Kasner *Am. J. Math*, **43**, p.126, *ibid*, p. 130 (1921), Fronsdal *Physical Review*, **116** (3), p. 778 (1959), or Joe Rosen, *Rev. Mod. Phys.*, (1965).
All these embeddings use the six dimensional pseudo-Euclidean space:

$$V_4 \to E^6$$

with the flat metric: $\eta_{AB} = (+ - - - - -)$ or

$$\eta_{AB} = (+ + - - - -)$$

The embedding functions for the Schwarzschild solution

$$z^1 = MG \left(1 - \frac{2MG}{r}\right)^{\frac{1}{2}} \sinh\left(\frac{c\,t}{MG}\right),$$

$$z^2 = MG \left(1 - \frac{2MG}{r}\right)^{\frac{1}{2}} \cosh\left(\frac{c\,t}{MG}\right),$$

$$z^3 = \int \left[\frac{1 - \left(\frac{MG}{r}\right)^4}{1 - \frac{2MG}{r}} - 1\right]^{\frac{1}{2}} \mathrm{d}r,$$

$$z^4 = r \sin\theta \cos\phi,$$

$$z^5 = r \sin\theta \sin\phi,$$

$$z^6 = r \cos\theta$$

Induced covariant metric

- $$\overset{0}{g}_{\mu\nu} = \eta_{AB} \frac{\partial z^A}{\partial x^\mu} \frac{\partial z^B}{\partial x^\nu}$$

Induced covariant metric

- $$\overset{0}{g}_{\mu\nu} = \eta_{AB} \frac{\partial z^A}{\partial x^\mu} \frac{\partial z^B}{\partial x^\nu}$$

- $$ds^2 = \left(1 - \frac{2MG}{r}\right) \mathrm{d}t^2 - \left(1 - \frac{2MG}{r}\right)^{-1} \mathrm{d}r^2 - r^2 \mathrm{d}\theta^2 - r^2 \sin^2\theta \mathrm{d}\phi^2$$

More general embedding functions

$$
\begin{aligned}
z^1 &= \lambda\left(1-\frac{2MG}{r}\right)^{\frac{1}{2}} \sinh\left(\frac{c\,t}{\lambda}\right),\\
z^2 &= \lambda\left(1-\frac{2MG}{r}\right)^{\frac{1}{2}} \cosh\left(\frac{c\,t}{\lambda}\right),\\
z^3 &= \int\left[\frac{1-\frac{M^2G^2\lambda^2}{r^4}}{1-\frac{2MG}{r}}-1\right]^{\frac{1}{2}} \mathrm{d}r,\\
z^4 &= r\sin\,\theta\,\cos\,\phi,\\
z^5 &= r\sin\,\theta\,\sin\,\phi,\\
z^6 &= r\,\cos\,\theta
\end{aligned}
$$

The Minkowskian limit

$$\begin{array}{ccc} z(\lambda) & \xrightarrow{M \to 0} & z(M=0) \\ \downarrow & & \downarrow \\ g_{\mu\nu} & \xrightarrow{M \to 0} & \eta_{\mu\nu} \end{array}$$

Finding the orthogonal deformations

- $$\eta_{AB}\, v^A(x^\kappa)\partial_\mu z^B = 0$$

Finding the orthogonal deformations

- $$\eta_{AB}\, v^A(x^\kappa)\partial_\mu z^B = 0$$

- $$v^1(x^\mu)\partial_t z^1 - v^2(x^\mu)\partial_t z^2 =$$
$$\sqrt{1-\frac{2MG}{r}}\left(v^1\cosh\frac{ct}{MG} \quad - \quad v^2\sinh\frac{ct}{MG}\right) = 0$$

Finding the orthogonal deformations

- $$\eta_{AB}\, v^A(x^\kappa)\partial_\mu z^B = 0$$

- $$v^1(x^\mu)\partial_t z^1 - v^2(x^\mu)\partial_t z^2 =$$
$$\sqrt{1-\frac{2MG}{r}}\left(v^1 \cosh\frac{ct}{MG} \;-\; v^2 \sinh\frac{ct}{MG}\right) = 0$$

- $$\Rightarrow \begin{cases} v^1(x^\mu) = MG\, v(x^\mu)\, \sinh\frac{ct}{MG} \\ v^2(x^\mu) = MG\, v(x^\mu)\, \cosh\frac{ct}{MG} \end{cases}$$

First order deformation vector

$$
\begin{aligned}
v^1 &= MG\ v(x^\mu)\ \sinh\left(\frac{ct}{MG}\right), \\
v^2 &= MG\ v(x^\mu)\ \cosh\left(\frac{ct}{MG}\right) \\
v^3 &= -MG\frac{\left(M^2G^2v(x^\mu)+r^2\ w(x^\mu)\right)}{r^2\left[\frac{MG}{r^3}\left(\frac{2r^3-M^3G^3}{r-2MG}\right)\right]^{\frac{1}{2}}} \\
v^4 &= MG\ w(x^\mu)\ \sin\,\theta\ \cos\,\phi, \\
v^5 &= MG\ w(x^\mu)\ \sin\,\theta\ \sin\,\phi, \\
v^6 &= MG\ w(x^\mu)\ \cos\,\theta.
\end{aligned}
$$

Deformation corresponding to an infinitesima δM variation

$$z^A(x^\sigma, M+\delta M) = z^A(x^\sigma) + \frac{\delta M}{M} M \frac{\partial z^a(x^\sigma)}{\partial M} + \dots$$

$$\equiv z^A + \varepsilon v^A + \dots$$

$$\varepsilon \equiv \frac{\delta M}{M} \ll 1, \qquad v^A(x^\sigma) \equiv M \frac{\partial z^A}{\partial M}$$