Contraintes et déformations en hyperélasticité relativiste générale

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B. Kolev and R. Desmorat (2022).
Relativistic general covariant formulation of hyperelasticity after J.M. Souriau, hal-03861444.
https://hal.archives-ouvertes.fr/hal-03861444

INTRODUCTION

- The key notion is the conformation introduced by Souriau (1958, 1964), suitable for astrophysics, which does not need a spacetime.
- It has been rediscovered since (see Kijowski–Magli, 1992, 1997, Beig–Schmidt, 2003), but Souriau is not cited.
- We will show that if further assumptions are made (firstly such as a foliation of the World tube[1], secondly such as the consideration of a static spacetime), other strain measures can be defined.

[1] see for instance the (3+1)-formalism introduced for relativistic fluids in (Arnowitt et al., 1962, York, 1979, Gourgoulhon, 2012).

What is a differential form ?

A differential form ω of degree k on $\mathbb{R}^d$ (or more generally on a manifold) is a tensor field of order k which is alternate

$$\omega_{r_1 \cdots r_j \cdots r_i \cdots r_k} = -\omega_{r_1 \cdots r_i \cdots r_j \cdots r_k}.$$

- a 1-form on $\mathbb{R}^3$: $\alpha = (\alpha_i) = P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$,
- a 2-form on $\mathbb{R}^3$: $\omega = (\omega_{ij}) = \omega_{12}\,\mathrm{d}x \wedge \mathrm{d}y + \omega_{13}\,\mathrm{d}x \wedge \mathrm{d}z + \omega_{23}\,\mathrm{d}y \wedge \mathrm{d}z$,
- a 3-form on $\mathbb{R}^3$: $\omega = (\omega_{ijk}) = \omega_{123}\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.

Here $(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ is the dual basis of the canonical basis of $\mathbb{R}^3$ and

$$\mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}x \otimes \mathrm{d}y - \mathrm{d}y \otimes \mathrm{d}x,$$
$$\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = (\mathrm{d}x \otimes \mathrm{d}y \otimes \mathrm{d}z)^a,$$

is the alternate tensor product, called the wedge product.

The Riemannian volume form

A volume form on $\mathbb{R}^d$ (or more generally on a manifold of dimension d) is a d-form (maximal degree) which vanishes nowhere,

$$\mu = f\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^d, \quad \text{where } f(x,y,z) \neq 0.$$

On every (orientable) Riemannian manifold (M, g) there exists a unique volume form, noted vol_g which is characterized that its value is 1 when evaluated on every direct orthonormal basis.

- On $\mathbb{R}^3$, equipped with its natural Euclidean structure ($g = q$),

  $$\mathrm{vol}_q = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = (\mathrm{d}x \otimes \mathrm{d}y \otimes \mathrm{d}z)^a\,,$$

  in any system of (direct) orthogonal coordinates (x, y, z) (in which $q = (\delta_{ij})$)

OUTLINE

4D formalism for Continuum Mechanics

Souriau (1958–1964)

- The modeling of perfect matter adopted by Souriau is inspired by Gauge theory, where matter fields are described by sections of some vector bundle (here a trivial vector bundle).
- A perfect matter field is a smooth vector valued function

$$\Psi : \mathscr{M} \to V \simeq \mathbb{R}^3, \qquad m \mapsto \mathbf{X},$$

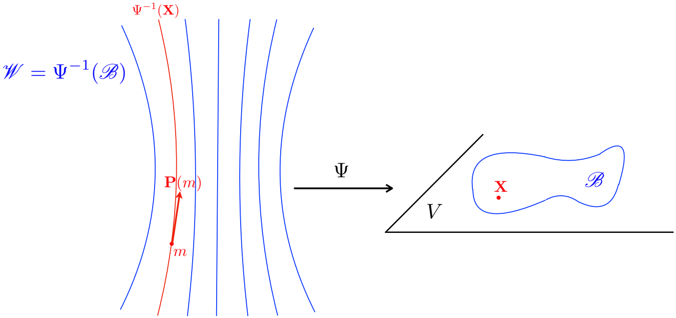
where $\mathscr{M}$ is the Universe, a four dimensional manifold, endowed with a Lorentzian metric g (of signature $(-,+,+,+)$).

$$\text{If } m = (x^\mu), \qquad X^I = \Psi^I(x^\mu), \qquad T\Psi = \left(\frac{\partial X^I}{\partial x^\mu}\right) \text{ rectangular.}$$

- The notation Ψ for the matter field is on purpose: Ψ is the wave function in Quantum Mechanics.

THE BODY, THE MASS MEASURE AND THE WORLD TUBE

- A continuous medium is then described by a 3D compact orientable manifold with boundary $\mathscr{B} \subset V \simeq \mathbb{R}^3$, the body, which labels the particles and is endowed with a volume form μ, the mass measure.
- It is further assumed that $T_m\Psi$ is of rank 3 at each point m of $\Psi^{-1}(\mathscr{B})$. Thus, $\mathscr{W} := \Psi^{-1}(\mathscr{B})$ is fibered by the particles World lines $\Psi^{-1}(\mathbf{X})$, $\mathbf{X} \in \mathscr{B}$, and is called for this reason the body's World tube.



A point of view reverse to 3D formalism

- In 3D formalism, a configuration is an embedding

$$p\colon \mathscr{B} \to \mathscr{E}$$

from the 3D body $\mathscr{B}$ to the 3D space $\mathscr{E}$.

- In the present 4D formalism, the main concept is a mapping

$$\Psi\colon \mathscr{M} \to \mathscr{B}$$

from the 4D Universe $\mathscr{M}$ to the 3D body $\mathscr{B}$.

A key difference is that, in Classical Continuum Mechanics, the embedding p and its tangent map

$$\mathbf{F} = Tp\colon T\mathscr{B} \to T\mathscr{E}$$

are invertible, whereas here, the matter field Ψ and its tangent map $T\Psi$ are not.

Current of matter – Rest mass density

- The pullback by Ψ of the mass measure μ on $\mathscr{B}$, $\Psi^*\mu$, is a 3-form. Since $T_m\Psi$ is assumed to be of rank 3 at each point of $\mathscr{W}$, there exists a nowhere vanishing (quadri-)vector field $\mathbf{P}$ on $\mathscr{W}$, such that

$$\Psi^*\mu = i_{\mathbf{P}}\mathrm{vol}_g,$$

  where $i_{\mathbf{P}}$ means the contraction of $\mathbf{P}$ with vol_g $\quad (i_{\mathbf{P}}\mathrm{vol}_g := \mathbf{P}\cdot\mathrm{vol}_g)$. This vector field $\mathbf{P}$ is the current of matter.
- To describe perfect matter, Souriau assumes furthermore that $\mathbf{P}$ is timelike,

$$\|\mathbf{P}\|_g^2 = g(\mathbf{P},\mathbf{P}) < 0 \quad \text{on the World tube } \mathscr{W}$$

- We can write

$$\mathbf{P} = \rho_r \mathbf{U}, \quad \text{with} \quad \|\mathbf{U}\|_g^2 = -1 \quad \text{and} \quad \rho_r := \sqrt{-\|\mathbf{P}\|_g^2}.$$

- This defines, on the World tube $\mathscr{W}$, the rest mass density $\rho_r > 0$.
- We have $T\Psi.\mathbf{P} = 0$ and $T\Psi.\mathbf{U} = 0$.

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Outline

CONFORMATION

The conformation $\mathbf{H}$ has been introduced by Souriau in 1958.

It is the cornerstone of the formulation of Relativistic Hyperelasticity at large scale, such as in the modeling of neutron stars (with a solid crust).

Definition

The conformation is the vector-valued function

$$\mathbf{H} : \mathscr{M} \to \mathbb{S}^2(V), \qquad m \mapsto \mathbf{H}(m) := (T_m\Psi)\, g_m^{-1} (T_m\Psi)^\star.$$

- For each $m \in \mathscr{W}$, $\mathbf{H}(m)$ is a positive definite quadratic form on $V^\star$ (consequence of $\mathbf{U}$ timelike).
- Since the mapping $\Psi : \mathscr{W} \to \mathscr{B}$ is not invertible, $\mathbf{H}$ is not the pushforward of g^{-1}.

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To be compared to the (3D) definition of the right Cauchy Green tensor

$$\mathbf{C} := \phi^{*} q = \mathbf{F}^{\star} q\, \mathbf{F} = (\mathbf{F}^{-1} q^{-1} \mathbf{F}^{-\star})^{-1}, \qquad \mathbf{F} := T\phi.$$

Variational Relativity

- Souriau has proposed a clear and detailed formulation of Hyperelasticity in General Relativity (part of his Variational Relativity).
- This formulation is inspired by Gauge Theory formulation of General Relativity (Palatini's Method).
- His approach consists in adding Lagrangians[2], each of them describing a physical phenomena, and looking for critical points of the total Lagrangian $\mathscr{L}$ (Principle of Least Action).

$$\mathscr{L}(g,\Psi) = \mathscr{H}(g) + \mathscr{L}^{\text{matter}}(g,\Psi),$$
$$\mathscr{L}^{\text{matter}}(g,\Psi) = \int L_0\left(g_{\mu\nu}, \Psi^I, \frac{\partial \Psi^I}{\partial x^\mu}\right) \mathrm{vol}_g .$$

[2] *i.e.*, functionals depending on tensorial fields (the metric g, gauge potentials $\mathbf{A}$, Γ,…, matter fields Ψ, …)

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$$\mathscr{L}(g,\Psi) = \mathscr{H}(g) + \mathscr{L}^{\text{matter}}(g,\Psi),$$
$$\mathscr{L}^{\text{matter}}(g,\Psi) = \int L_0(g_m, \Psi(m), T_m\Psi)\,\text{vol}_g.$$

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General Covariance

The main postulate of General Relativity is precisely that
Physical laws must be independent of the choice of coordinates.

This means that the Lagrangian $\mathscr{L}$ must be invariant by a (local) diffeomorphism φ, *i.e.*,

$$\mathscr{L}(\varphi^{*}g, \varphi^{*}\Psi) = \mathscr{L}(g, \Psi),$$

where, here,

$$\varphi^{*}g = (T\varphi)^{\star}\,(g \circ \varphi)(T\varphi), \quad \text{and} \quad \varphi^{*}\Psi = \Psi \circ \varphi.$$

General covariance of a perfect matter Lagrangian

Theorem (Souriau (1958))

Suppose that the Lagrangian

$$\mathscr{L}^{matter}(g, \Psi) = \int L_0(g_m, \Psi(m), T_m\Psi)\, \mathrm{vol}_g$$

is general covariant. Then, its Lagrangian density can be written as

$$L_0(g, \Psi, T\Psi) = L(\Psi, \mathbf{H}),$$

for some function L, where $\mathbf{H} = (T\Psi)\, g^{-1}(T\Psi)^{\star}$ *is the conformation.*

In Classical Continuum Mechanics, the energy density depends on the deformation ϕ only through the right Cauchy–Green tensor $\mathbf{C} = \phi^{*}q$. Here, $\mathbf{H}$ plays the role of $\mathbf{C}^{-1}$.

A STRESS-ENERGY TENSOR FOR PERFECT MATTER

The following splitting has been introduced by Souriau (1958, 1964) and De Witt (1962),

$$L(\Psi, \mathbf{H}) = \rho_r c^2 + E(\Psi, \mathbf{H}) = \rho_r c^2 + \rho_r e(\Psi, \mathbf{H}),$$

where ρ_r is the rest mass density, and E is the internal energy density. Then,

$$\mathbf{T} = -2\frac{\delta \mathscr{L}^{matter}}{\delta g} = \rho_r c^2 \mathbf{U} \otimes \mathbf{U} - \mathbf{S} = L\,\mathbf{U} \otimes \mathbf{U} - \mathbf{\Sigma},$$

such as $\operatorname{div}^g \mathbf{T} = 0$, where

$$\mathbf{S} := E\,g^{-1} - 2g^{-1}(T\Psi)^\star \frac{\partial E}{\partial \mathbf{H}}(T\Psi)g^{-1}, \qquad \mathbf{S}.\mathbf{U}^\flat = E\,\mathbf{U},$$

$$\mathbf{\Sigma} := -2\rho_r\,g^{-1}(T\Psi)^\star \frac{\partial e}{\partial \mathbf{H}}(T\Psi)g^{-1}, \qquad \mathbf{\Sigma}.\mathbf{U}^\flat = 0,$$

correspond to two choices of a (4D) relativistic stress tensor.

Outline

The existence of a unit timelike vector field $\mathbf{U} = \mathbf{P}/\rho_r$ allows to perform the related orthogonal decompositions of g and g^{-1},

$$g = h - \mathbf{U}^\flat \otimes \mathbf{U}^\flat, \qquad g^{-1} = h^\sharp - \mathbf{U} \otimes \mathbf{U}, \qquad \text{on } \mathscr{W},$$

where the tensor fields h and $h^\sharp = g^{-1} h g^{-1}$ are uniquely defined by

$$h\mathbf{U} = 0, \quad \text{and} \quad h = g \quad \text{on} \quad \mathbf{U}^\perp.$$

$\mathbf{U}^\perp$: three-dimensional (necessarily spacelike) orthogonal subbundle to $\mathbf{U}$.

Both h and $h^\sharp$ have signature $(0, +, +, +)$.

Remark

These orthogonal decompositions are highlighted at the beginning of most works on Relativistic Fluids or Solids (Eckart, 1940, Lichnerowitz, 1955, Carter–Quintana, 1972, Kijowski–Magli, 1997).

Link with the conformation

Souriau did not need h nor $h^\sharp$ to derive the general covariant formulation of Relativistic Hyperelasticity . There are two reasons for it.

1. the four-dimensional symmetric second-order tensors h and $h^\sharp$ are strongly related to the conformation $\mathbf{H} := (T\Psi)\, g^{-1}(T\Psi)^\star$.

Lemma

On the World tube $\mathscr{W}$, we have

$$\mathbf{H} = (T\Psi)\, h^\sharp (T\Psi)^\star, \quad \textit{and} \quad h = (T\Psi)^\star\, \mathbf{H}^{-1} T\Psi.$$

where $h = g + \mathbf{U}^\flat \otimes \mathbf{U}^\flat$ *and* $h^\sharp = g^{-1} h g^{-1}$.

2. h and $h^\sharp$ do not appear naturally in the derivation of a general covariant formulation of Relativistic Hyperelasticity, contrary to the conformation $\mathbf{H}$ (Souriau's 1958 theorem).

FIXED METRIC ON $\mathscr{B}$ – FROZEN "METRIC" ON $\mathscr{W}$

The definition of a strain in (hyper)elasticity is usually obtained by comparing two metrics.

- If the body $\mathscr{B}$ is endowed with a fixed Riemannian metric γ_0, the later can be used to define a strain tensor in Relativistic Hyperelasticity.
- Carter–Quintana (1972) and Maugin (1978) furthermore introduce the pullback by Ψ of γ_0, the so-called frozen metric given by

$$h_0 := \Psi^* \gamma_0 = (T\Psi)^\star (\gamma_0 \circ \Psi) T\Psi.$$

h_0 is defined on the World tube $\mathscr{W}$ and is of signature $(0, +, +, +)$.

Remark

Conversely, given a quadratic form h_0 on $\mathscr{W}$ with signature $(0, +, +, +)$, the question of when it can be realized as the pullback by Ψ of a fixed Riemannian metric γ_0 on the body, has been investigated by Kijowski and Magli.

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Generalized Euler-Almansi strain tensor

A generalization of the Euler-Almansi strain tensor (Carter–Quintana, 1972, Maugin, 1978) is the four-dimensional symmetric covariant tensor field,

$$\mathbf{e} := \frac{1}{2}(h - h_0).$$

Note that $\mathbf{e} = 0$ for $h = h_0$ and that $\mathbf{e}$ is degenerate since $\mathbf{eU} = 0$.

Remark due to Maugin

Since the linear tangent map $T\Psi$ plays a role similar to that of the inverse of $\mathbf{F} = Tp$ in Classical Continuum Mechanics, the frozen metric h_0 plays a role similar to that of the inverse, sometimes called the finger deformation tensor, of the left Cauchy–Green tensor $\mathbf{b} := \mathbf{F}\gamma_0^{-1}\mathbf{F}^\star$.

Strain tensors directly built from $\mathbf{H}$

Other choices for strain tensors similar to the ones of Classical Continuum Mechanics can be made, for instance the more relevant ones

$$\mathfrak{E} := \frac{1}{2}\left(\mathbf{H}^{-1} - \mathbf{H}_0^{-1}\right) \quad \text{or} \quad \widehat{\mathfrak{E}} := -\frac{1}{2}\log\left(\mathbf{H}\,\mathbf{H}_0^{-1}\right),$$

where

$$\mathbf{H}_0 := \gamma_0^{-1} \circ \Psi.$$

- The first one generalizes the Green–Lagrange strain, whereas the second one generalizes the logarithmic strain.
- They both vanish when $\mathbf{H}^{-1} = \mathbf{H}_0^{-1} = \gamma_0 \circ \Psi$.
- These strain tensors are three-dimensional second-order tensors.
- Like the conformation, they are not tensor fields on $\mathscr{B}$ but vector valued functions defined on the World tube $\mathscr{W}$ with values in $\mathbb{S}^2 V$.

Note that $\mathbf{H}_0$ is related to h_0 by

$$h_0 = (T\Psi)^\star \, \mathbf{H}_0 \, (T\Psi)$$

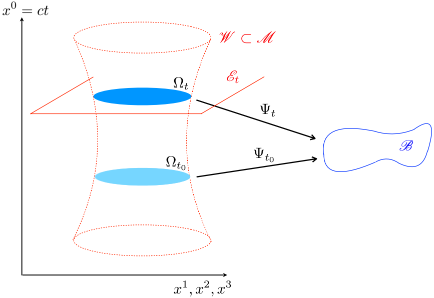
and that $\mathfrak{E}$ is connected to $\mathbf{e}$ by

$$\mathbf{e} = (T\Psi)^\star \, \mathfrak{E} \, (T\Psi) \quad \text{on } \mathscr{W}.$$

OUTLINE

Foliation of $\mathscr{W}$ by spacelike hypersurfaces Ω_t

Introduction of an observer (Arnowitt et al, 1962, York, 1979, Gourgoulhon, 2012)



The 3D submanifolds Ω_t of timelike normal **N** play the role of the configuration manifolds $\Omega = \Omega_{p(t)}$ of Continuum Mechanics. The restriction

$$\Psi_t : \Omega_t \to \mathscr{B} \qquad \mathbf{x} \mapsto \mathbf{X} = \Psi_t(\mathbf{x}) = \Psi(x^0 = ct, \mathbf{x})$$

is not necessarily a diffeomorphism,
but its differential $T\Psi_t$ is always a linear isomorphism.

Generalized Lorentz factor and spatial velocity

- Introducing

$$\gamma := -\langle \mathbf{U}, \mathbf{N} \rangle_g$$

and setting $\mathbf{U}^{\top} = \gamma \boldsymbol{u}/c$, the orthogonal decomposition of $\mathbf{U}$ is written as

$$\mathbf{U} = \mathbf{U}^N + \mathbf{U}^{\top} = \gamma \left(\mathbf{N} + \frac{\boldsymbol{u}}{c} \right),$$

- $\gamma := -\langle \mathbf{U}, \mathbf{N} \rangle_g$ is called the generalized Lorentz factor, since

$$\gamma = 1 \Big/ \sqrt{1 - \frac{\|\boldsymbol{u}\|_g^2}{c^2}}, \quad \text{because} \quad \|\mathbf{U}\|_g^2 = -1.$$

- The spatial velocity $\boldsymbol{u}$ can be expressed on Ω_t from $T\Psi.\mathbf{U} = 0$, as

$$\boldsymbol{u} = -c\, \mathbf{F}\, T\Psi.\mathbf{N} \quad \text{on} \quad \Omega_t,$$

where $\mathbf{F}$ is the inverse of $\mathbf{F}^{-1} = T\Psi_t$, the restriction of $T\Psi$ on Ω_t.

ORTHOGONAL DECOMPOSITION OF $\mathbf{T}$

Stress-energy tensor $\mathbf{T}$

$$\mathbf{T} = E_{\text{tot}}\, \mathbf{N} \otimes \mathbf{N} + \frac{1}{c}\left(\mathbf{N} \otimes \boldsymbol{p} + \boldsymbol{p} \otimes \mathbf{N}\right) + \mathbf{s}.$$

- E_{tot} is the total energy density,
- $\boldsymbol{p}$ is the momentum density vector field,
- and $\mathbf{s}$ is the spatial part of $\mathbf{T}$ (related to the stress field).

Following Eckart (1940) and Souriau (1958), there exists 2 ways to define a spatial stress $\boldsymbol{\sigma}$, which generalizes the 3D stress, using either $\mathbf{S}$ or $\boldsymbol{\Sigma}$. They provide two alternative Relativistic Hyperelasticity laws.

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where ρ_r is the rest mass density, and E is the internal energy density. Then,

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where

$$\mathbf{S} := E\, g^{-1} - 2g^{-1}(T\Psi)^\star \frac{\partial E}{\partial \mathbf{H}}(T\Psi)g^{-1}, \qquad \mathbf{S}.\mathbf{U}^\flat = E\,\mathbf{U},$$

$$\mathbf{\Sigma} := -2\rho_r\, g^{-1}(T\Psi)^\star \frac{\partial e}{\partial \mathbf{H}}(T\Psi)g^{-1}, \qquad \mathbf{\Sigma}.\mathbf{U}^\flat = 0,$$

correspond to two choices of a (4D) relativistic stress tensor.

First modeling choice

The 3D stress tensor $\boldsymbol{\sigma}$ is defined as the spatial part of

$$\mathbf{S} = -2\rho_r \, g^{-1} (T\Psi)^\star \frac{\partial e}{\partial \mathbf{H}} (T\Psi) g^{-1} - E\,\mathbf{U} \otimes \mathbf{U} \qquad \big(\text{such as } \mathbf{S}.\mathbf{U}^\flat = E\,\mathbf{U}\big),$$

and is given by

$$\boldsymbol{\sigma} := -\frac{2}{\gamma}\rho\,(g^{3D})^\sharp \mathbf{F}^{-\star} \frac{\partial e}{\partial \mathbf{H}} \mathbf{F}^{-1} (g^{3D})^\sharp - \frac{\gamma^2 E}{c^2}\,\boldsymbol{u} \otimes \boldsymbol{u},$$

Associated (3+1)-decomposition of $\mathbf{T} = \rho_r c^2 \mathbf{U} \otimes \mathbf{U} - \mathbf{S}$

$$\begin{cases} E_{\text{tot}} = \gamma\rho c^2 + E\left(1 + \frac{1}{c^2}\,\|\boldsymbol{u}\|^2\right) - \frac{1}{c^2}\boldsymbol{u}^\flat \cdot \boldsymbol{\sigma} \cdot \boldsymbol{u}^\flat, \\ \boldsymbol{p} = (\gamma\rho c^2 + E)\boldsymbol{u} - \boldsymbol{\sigma} \cdot \boldsymbol{u}^\flat, \\ \mathbf{s} = \gamma\rho\,\boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{\sigma}, \end{cases}$$

where $\rho = \gamma\rho_r$ is the relativistic mass density.

SECOND MODELING CHOICE

The 3D stress tensor $\boldsymbol{\sigma}$ is defined as the spatial part of

$$\boldsymbol{\Sigma} = -2\rho_r\, g^{-1}(T\Psi)^\star \frac{\partial e}{\partial \mathbf{H}}(T\Psi)g^{-1} \qquad \big(\text{such as } \boldsymbol{\Sigma}.\mathbf{U}^\flat = 0\big),$$

and is given by

$$\boldsymbol{\sigma} := -\frac{2}{\gamma}\rho\,(g^{3D})^\sharp \mathbf{F}^{-\star}\frac{\partial e}{\partial \mathbf{H}}\mathbf{F}^{-1}(g^{3D})^\sharp.$$

Associated (3+1)-decomposition of $\mathbf{T} = L\mathbf{U}\otimes\mathbf{U} - \boldsymbol{\Sigma}$

$$\begin{cases} E_{\text{tot}} = \gamma^2 L - \frac{1}{c^2}\boldsymbol{u}^\flat \cdot \boldsymbol{\sigma} \cdot \boldsymbol{u}^\flat, \\ \boldsymbol{p} = \gamma^2 L\,\boldsymbol{u} - \boldsymbol{\sigma}\cdot\boldsymbol{u}^\flat, \\ \mathbf{s} = \frac{\gamma^2}{c^2}L\,\boldsymbol{u}\otimes\boldsymbol{u} - \boldsymbol{\sigma}, \end{cases}$$

where $\gamma^2 L = \gamma^2\left(\rho_r c^2 + E\right) = \gamma\rho c^2 + E\big(1 + \frac{\gamma^2}{c^2}\left\|\boldsymbol{u}\right\|^2\big)$.

Outline

1. General framework of Relativistic Hyperelasticity
2. Conformation and associated stress tensors
3. Conformation and strains
4. Further assumption: foliation of the World tube
5. Further assumption: restriction Ψ_t of the matter field is a diffeomorphism

3D Riemannian metrics on the body $\mathscr{B}$

If we make the stronger assumption that $\Psi_t \colon \Omega_t \to \mathscr{B}$ is a diffeomorphism, then, the conformation induces a one-parameter family $\gamma(t)$ of three-dimensional Riemannian metrics on the three-dimensional body $\mathscr{B}$

$$\gamma(t)^{-1} := \mathbf{H} \circ j_t \circ \Psi_t^{-1}, \qquad \gamma(t) := \mathbf{H}^{-1} \circ j_t \circ \Psi_t^{-1}.$$

$j_t \colon \Omega_t \to \mathscr{M}$: canonical embedding of these submanifolds into Universe $\mathscr{M}$.

The metric $\gamma(t)$ is the true analogue of the right Cauchy–Green tensor $\mathbf{C} := \mathbf{F}^{\star} q\, \mathbf{F}$ when the body $\mathscr{B}$ is identified with a reference configuration Ω_0.

The following result relates $\gamma(t)$ with the degenerate quadratic form h.

Lemma

On Ω_t, we have

$$\Psi_t^{*}\gamma(t) = j_t^{*}h,$$

where $h = g + \mathbf{U}^{\flat} \otimes \mathbf{U}^{\flat}$ and $j_t^{*}h = (Tj_t)^{\star}(h \circ j_t)Tj_t$.

Any of the three following 3D symmetric covariant tensor fields

$$\begin{aligned} & \mathbf{H}^{-1} \circ j_t && (\mathbb{S}^2 V\text{-vector valued, on } \Omega_t), \\ & \boldsymbol{\gamma}(t) && (\text{on } \mathscr{B}), \\ \text{and} \quad & j_t^* h && (\text{on } \Omega_t), \end{aligned}$$

leads then to equivalent formulations of Relativistic Hyperelasticity models.

Indeed, these tensor fields are related to each other by

$$\begin{aligned} \text{on } \mathscr{B}: \quad & \boldsymbol{\gamma}(t) = \mathbf{H}^{-1} \circ j_t \circ \Psi_t^{-1} = \Psi_{t*} j_t^* h, \\ \mathbb{S}^2 V\text{-vector valued, on } \Omega_t: \quad & \mathbf{H}^{-1} \circ j_t = (\Psi_{t*} j_t^* h) \circ \Psi_t = \boldsymbol{\gamma}(t) \circ \Psi_t, \\ \text{on } \Omega_t: \quad & j_t^* h = (\Psi_t)^* \boldsymbol{\gamma}(t) = (\Psi_t)^* (\mathbf{H}^{-1} \circ j_t \circ \Psi_t^{-1}), \end{aligned}$$

THREE-DIMENSIONAL STRAINS

The associated (equivalent) definitions of strain tensors are the following

- on $\mathscr{B}$:

$$\frac{1}{2}\left(\gamma(t)-\gamma_0\right)=\mathfrak{E}\circ\Psi_t^{-1},\qquad \frac{1}{2}\log\left(\gamma_0^{-1}\gamma(t)\right)=\widehat{\mathfrak{E}}\circ\Psi_t^{-1},$$

- $\mathbb{S}^2V$-vector valued, on Ω_t:

$$\frac{1}{2}\left(\mathbf{H}^{-1}-\mathbf{H}_0^{-1}\right)\circ j_t=j_t^*\mathfrak{E},\qquad -\frac{1}{2}\log\left(\mathbf{H}\,\mathbf{H}_0^{-1}\right)\circ j_t=j_t^*\widehat{\mathfrak{E}},$$

- on Ω_t:

$$\frac{1}{2}\left(j_t^*h-j_t^*h_0\right)=j_t^*\mathbf{e},\qquad \frac{1}{2}\log\left((j_t^*h_0)^{-1}j_t^*h\right),$$

$h_0=\Psi^*\gamma_0$: the so-called frozen metric on the World tube $\mathscr{W}$,
$j_t^*h_0=\Psi_t^*\gamma_0$: its restriction to Ω_t,
$\mathbf{H}_0=\gamma_0^{-1}\circ\Psi$.

Conclusion

- Souriau's Lagrangian General relativity formulation of Hyperelasticity
- Key role of the body $\mathscr{B}$ (of the mass measure μ and the fixed metric γ_0)
- Key role of the conformation (as strain)
- Proper definitions of strains and stresses tensors (according to the assumptions made)
- Even if the full theory is 4D, Hyperelasticity constitutive laws are 3D

Mass conservation in Hyperelasticity

Lemma (Souriau, 1958)

Let γ_0 be a fixed Riemannian metric on the body $\mathscr{B}$. Then, the rest mass density ρ_r can be written as

$$\rho_r = \rho_{\gamma_0}(\Psi)\sqrt{\det\left[\mathbf{H}(\gamma_0 \circ \Psi)\right]},$$

where Ψ is the matter field, $\mathbf{H}$ is the conformation, and $\rho_{\gamma_0} = \dfrac{\mu}{\mathrm{vol}_{\gamma_0}}$.

The fixed mass density ρ_{γ_0} (w.r.t. γ_0) is defined on the body $\mathscr{B}$.

Mass conservation in Classical Continuum Mechanics (on Ω_0)

$$\rho_0 = (\rho \circ \phi) J, \qquad J := \sqrt{\det(q^{-1}\mathbf{C})},$$

$\phi\colon \Omega_0 \to \Omega$ is the deformation
$\mathbf{C} := \phi^* q$ is the right Cauchy–Green tensor.

Relativistic perfect fluid

The stress-energy tensor of a Relativistic perfect fluid,

$$\mathbf{T} = (L + P)\,\mathbf{U} \otimes \mathbf{U} + P\,g^{-1}, \qquad L = \rho_r c^2 + E,$$

corresponds to an internal energy density of the form

$$E = \rho_r e(\rho_r),$$

where

$$P = \rho_r^2 e'(\rho_r).$$

$$\mathbf{\Sigma} = -Pg^{-1} - P\,\mathbf{U} \otimes \mathbf{U} = -P\,h^\sharp,$$

$$\mathbf{S} = \mathbf{\Sigma} - E\,\mathbf{U} \otimes \mathbf{U} = -P\,g^{-1} - (E + P)\mathbf{U} \otimes \mathbf{U}.$$

Reference metric on the body $\mathscr{B}$

There are several choices for a reference metric on the body $\mathscr{B}$.

- One possibility is to endow the body with an arbitrary fixed metric γ_0 (for example $\gamma_0 = q$, the Euclidean metric, in (Souriau, 1958, 1964)).
- But when a spacetime and the associated spacelike hypersurfaces Ω_t are introduced, with in particular the choice of a reference configuration Ω_{t_0}, and when the restriction $\Psi_{t_0} = j_{t_0}^* \Psi$ of the matter field to Ω_{t_0} is a diffeomorphism, then two other —mechanistic— possibilities are offered:
    (a) either to consider as reference metric on the body $\mathscr{B}$, $\gamma(t = t_0)$,

$$\gamma_0^a := \gamma(t_0) = (\Psi_{t_0})_* j_{t_0}^* h.$$

    (b) or to endow the body $\mathscr{B}$ with the Riemannian metric

$$\gamma_0^b := (\Psi_{t_0})_* j_{t_0}^* g.$$

These two reference metrics do not coincide in general.