AU DELA DES 4 DIMENSIONS: MODELE 5D DE KALUZA ET KLEIN

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- The general lagrangian is computed and the resulting non-linear equations which generalize Maxwell's system in a quite unique way are investigated.
- A possibility of the existence of static solutions is presented, and the qualitative behaviour of such solutions as models for the electron is discussed.

Maxwell's equations

Let us start by recalling the standard Maxwell's electromagnetism and fixing the notations.

The simplest and the most elegant form of Maxwell's system is written in modern system of units as follows:

$$\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla} \times \mathbf{E}, \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{1}$$

$$\frac{\partial \mathbf{D}}{\partial t} + \mathbf{j} = \boldsymbol{\nabla} \times \mathbf{H}, \quad \boldsymbol{\nabla} \cdot \mathbf{D} = \rho, \tag{2}$$

Geometrical nature of fields

It should be underlined that the pairs of fields **(E, H)** and **(D, B)** represent different geometrical objects. This can be better understood if we look at the integral form of Maxwell's equations:

$$-\frac{\partial}{\partial t}\oint_S \mathbf{B}\cdot \mathrm{d}\boldsymbol{\sigma} = \oint_{\partial S} \mathbf{E}\cdot \mathrm{d}\mathbf{l}, \qquad \oint_{\partial V} \mathbf{B}\cdot\boldsymbol{\sigma} = 0. \tag{3}$$

$$\oint_S \left[\frac{\partial}{\partial t}\mathbf{D} + \mathbf{j}\right]\cdot \mathrm{d}\boldsymbol{\sigma} = \oint_{\partial S} \mathbf{H}\cdot \mathrm{d}\mathbf{l}, \qquad \oint_{\partial V} \mathbf{D}\cdot\boldsymbol{\sigma} = Q. \tag{4}$$

Here S is a surface and ∂S its boundary, which is a closed line; V is a volume and ∂V is its boundary, a closed surface.

Vector fields and streams (2-forms)

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Vector fields and streams (2-forms)

- In the integral form of Maxwell's equations, the entities **E** and **H** are genuine vector fields which can be integrated along curves, whereas **B** and **D** are in fact 2-**forms**, defining **streams**.
- Rate of change of fluxes of D and B through a surface is determined by the circulation of their conjugate fields H and E along the boundary, and vice versa.
- A problem arises with number of equations versus number of functions: 8 equations for $4 \times 3 = 12$ components. The *constitutive relations* $\mathbf{E} = \mathbf{E}(\mathbf{D}, \ \mathbf{B}) \quad \text{and} \quad \mathbf{H} = \mathbf{H}(\mathbf{D}, \ \mathbf{B})$ reduce the number of variables to 6, thus making the system seemingly overdetermined.

- **Things become straightened up in a four-dimensional notation, with 4-vector potential defined as a vector in 4-dimensional space-time endowed wth Minkowskian metric** $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-), \quad \mu,\nu,.. = 0,1,2,3$

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- We assume that the 6 variables corresponding to the fields **E** and **B** are the 6 independent components of an antisymmetric 2-**covariant** tensor (a 2-form) $F_{\mu\nu} = -F_{\nu\mu}$, with $F_{0k} = E_k, \;\; F_{ik} = \epsilon_{ikm} B_m, \;\; i, k, m = 1, 2, 3.$

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- The Poincaré Lemma states that if a 2-**form** - **e.g.** $F = \frac{1}{2} F_{\mu\nu} dx^{\mu} \wedge dx^{\nu}$ - is defined on an open subset of Minkowskian space-time M_4, then it is an exterior differential of some 1-**form**, in this case $A = A_{\mu} dx^{\mu}$:

$$F = dA \;\; \rightarrow \;\; F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}. \tag{5}$$

There are two independent relativistic invariant functions of the Faraday's 2-form $F_{\mu\nu}$:

$$S = -\frac{1}{4}\eta^{\mu\lambda}\eta^{\nu\rho}F_{\mu\nu}F_{\lambda\rho} = \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right), \quad (6)$$

$$P = -\frac{1}{8}\epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho} = F_{\mu\nu}\hat{F}^{\mu\nu} = \mathbf{E}\cdot\mathbf{B}, \quad (7)$$

with

$$\hat{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}$$

The choice of symbols is not accidental: S **stands for "scalar", and** P **stands for "pseudo-scalar").**

Lagrangian and variational principle

To ensure relativistic invariance, the variational principle should be derived from a Lagrangian depending on two invariants, $\mathcal{L}(S, P)$.
The equations of motion of the electromagnetic field form two groups: the homogeneous ones,

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0, \tag{8}$$

which are the consequence of the fact that
$F = dA \;\rightarrow\; dF = d^2 A = 0$, and the equations resulting from variational principle applied to $\mathcal{L}$,

$$\partial_\mu G^{\mu\nu} = 0, \quad \text{with} \;\; G^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} \tag{9}$$

$$G^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial S} F^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial P} \hat{F}^{\mu\nu} \tag{10}$$

The dual Faraday tensor

By definition,

$$G^{0i} = -G^{i0} = D^i, \quad G^{ik} = -G^{ki} = \epsilon^{ik}{}_l H^l \tag{11}$$

so the equations of motion become:

$$\frac{\partial \mathbf{D}}{\partial t} = \mathbf{\nabla} \times \mathbf{H}, \quad \mathbf{\nabla} \cdot \mathbf{D} = 0, \tag{12}$$

which coincide with Maxwell's second set of equations when the sources (the current density $\mathbf{j}$ and the charge density (ρ) are put to zero.

Dynamical properties

The dynamical properties of the electromagnetic field are described by the energy-momentum tensor $T^{\mu\nu}$:

$$T^{\mu\nu} = F^{\mu}{}_{\lambda} G^{\lambda\nu} - \eta^{\mu\nu} \mathcal{L}, \tag{13}$$

$$T^{00} = \mathbf{E} \cdot \mathbf{D} - \mathcal{L}, \tag{14}$$

$$T^{0i} = (\mathbf{E} \times \mathbf{H})^i, \quad T^{i0} = (\mathbf{D} \times \mathbf{B})^i, \tag{15}$$

$$T^{ik} = -E^i D^k - H^i B^k + \delta^{ik}(\mathcal{L} + \mathbf{H} \cdot \mathbf{B}). \tag{16}$$

Conservation laws

The energy-momentum tensor is symmetric and conserved:

$$T^{\mu\nu} = T^{\nu\mu}, \qquad \partial_\mu T^{\mu\nu} = 0, \tag{17}$$

The proof uses the following identity:

$$F_{\mu\lambda}\hat{F}^{\lambda\nu} = \delta_\mu^\nu P, \tag{18}$$

The (17) result in the following conserved quantities

$$P^\mu = \int T^{\mu 0}\mathrm{d}\mathbf{r}^3, \quad M^{\mu\nu} = \int (x^\mu T^{\nu 0} - x^\nu T^{\mu 0})\mathrm{d}\mathbf{r}^3 \tag{19}$$

Canonical variables and Legendre transformations

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- Out of four fields $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{H}$ we can pick four different pairs, each consisting of one electric and one magnetic field, and treating them as independent variables, making the remaining two functions of them.
- To start with, let us choose the fields $\mathbf{E}$ and $\mathbf{B}$ as independent variables, akin to generalized velocities $\dot{\mathbf{q}}$ and coordinates $\mathbf{q}$ in classical Lagrangean mechanics. Then $\mathbf{D}$ and $\mathbf{H}$ defined as

$$\mathbf{D} = \frac{\partial \mathcal{L}}{\partial \mathbf{E}} \quad \text{and} \quad \mathbf{H} = \frac{\partial \mathcal{L}}{\partial \mathbf{B}}, \tag{20}$$

play the role of canonical momentum and generalized force.

The Legendre transformations

- **Let us define the Hamiltonian function:**

$$\mathcal{H}(\mathbf{D},\mathbf{B}) = \mathbf{D}\cdot\mathbf{E}(\mathbf{D},\mathbf{B}) - \mathcal{L}(\mathbf{E}(\mathbf{D},\mathbf{B}),\mathbf{B}) \tag{21}$$

which is to be treated as function of canonical variables $(\mathbf{D},\ \mathbf{B})$.

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- With the help of eq. **(20)** we get:

$$\mathbf{E} = \frac{\partial\mathcal{H}}{\partial\mathbf{D}} \quad \text{and} \quad \mathbf{H} = \frac{\partial\mathcal{H}}{\partial\mathbf{B}} \qquad (22)$$

The Legendre transformation

The three remaining choices of independent pairs of fields are displayed in the following table:

$(\mathbf{E},\ \mathbf{B})$	$\mathcal{L}(\mathbf{E},\mathbf{B})$	$\mathbf{D}=\dfrac{\partial\mathcal{L}}{\partial\mathbf{E}},\ \ \mathbf{H}=\dfrac{\partial\mathcal{L}}{\partial\mathbf{B}}$
$(\mathbf{D},\ \mathbf{B})$	$\mathcal{H}(\mathbf{D},\mathbf{B})=\mathbf{E}\cdot\mathbf{B}-\mathcal{L}$	$\mathbf{E}=\dfrac{\partial\mathcal{H}}{\partial\mathbf{D}},\ \ \mathbf{H}=\dfrac{\partial\mathcal{H}}{\partial\mathbf{B}}$
$(\mathbf{D},\ \mathbf{H})$	$\tilde{\mathcal{L}}(\mathbf{D},\mathbf{H})=\mathbf{E}\cdot\mathbf{B}-\mathbf{B}\cdot\mathbf{H}-\mathcal{L}$	$\mathbf{E}=\dfrac{\partial\tilde{\mathcal{L}}}{\partial\mathbf{D}},\ \ \mathbf{B}=-\dfrac{\partial\tilde{\mathcal{L}}}{\partial\mathbf{H}}$
$(\mathbf{E},\ \mathbf{H})$	$\tilde{\mathcal{H}}(\mathbf{E},\mathbf{H})=\mathbf{B}\cdot\mathbf{H}+\mathcal{L}$	$\mathbf{D}=\dfrac{\partial\tilde{\mathcal{H}}}{\partial\mathbf{E}},\ \ \mathbf{B}=\dfrac{\partial\tilde{\mathcal{H}}}{\partial\mathbf{H}}$

The Hamiltonian function $\mathcal{H}$ coincides with the energy density. With its use, the equations of motion of the electromagnetic field can be presented in a quasi-canonical manner:

$$\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla} \times \frac{\partial \mathcal{H}}{\partial \mathbf{D}}, \qquad \frac{\partial \mathbf{D}}{\partial t} = \boldsymbol{\nabla} \times \frac{\partial \mathcal{H}}{\partial \mathbf{B}}, \tag{23}$$

This circumstance opens the path to canonical quantization of the electromagnetic field, which is beyond the scope of the present talk.

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- One of the ideas was to use non-linear generalizations of Maxwell's theory, deviating from it only at very short distances and for very strong fields in order to ensure a cut-off and to avoid singularity at $r \to 0$.

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- Their results were not convincing, mainly because the symmetry between the electric and magnetic fields was broken from the beginning. Roughly speaking, it seems improbable to obtain a configuration with only electric field present, putting $\mathbf{B} = 0$ everywhere.
- Their lagrangians depended quite arbitrarily on two invariants of the electromagnetic tensor, $S = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)$ and $P = \frac{1}{4} F_{\mu\nu} {}^* F^{\mu\nu} = \mathbf{E} \cdot \mathbf{B}$. The second invariant does not appear in Maxwell's linear theory and was also discarded in G. Mie's non-linear version.

Mie's model

- G. Mie's model was based on the assumption that the electric field $\mathbf{E}$ can not exceed the limiting value $\mathbf{E_0}$, and that the repulsive force should be proportional to the expression

$$\mathbf{F} \sim \frac{\mathbf{E}}{\sqrt{1 - \frac{\mathbf{E}^2}{\mathbf{E_0}^2}}} \tag{24}$$

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- It was possible to find in this model a nonsingular solution with finite energy and charge, and with the field $\mathbf{E}$ falling off as r^{-2} at great distances, but this solution was not covariant with respect to the Lorentz transformations.

The Born-Infeld lagrangian

In **1932** and in **1934** M. Born and L. Infeld have published by now celebrated version of non-linear electrodynamics, in whihc they proposed the following Lorentz-invariant lagrangian:

$$\mathcal{L} = \beta^2 \left[\sqrt{\det \left(\delta^\mu_\lambda + \beta^{-1} \, F^\mu_\lambda \right)} \right] \qquad (25)$$

The constant β appears for dimensional reasons, and plays the same role here as the limiting value of the electric field in G. Mie's non-linear electrodynamics.

When expressed in terms of two invariants of Maxwell-Faraday's tensor,

$$P = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \text{and} \qquad S = \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} \,, \quad \text{with} \quad \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}$$

this lagrangian can be written explicitly as

$$\mathcal{L}_{BI} = \beta^2 \left[1 - \sqrt{1 + 2P - S^2} \right], \text{ or as :} \tag{26}$$

$$\mathcal{L}_{BI} = \beta^2 \left[\quad 1 - \sqrt{1 + \frac{1}{2\beta^2} (\mathbf{B}^2 - \mathbf{E}^2) - \frac{1}{16\beta^4} (\mathbf{E} \cdot \mathbf{B})^2} \quad \right] \tag{27}$$

Quite obviously, when developed in negative powers of the dimensional parameter β, we get as first approximation the usual lagrangean of the linear electrodynamics:

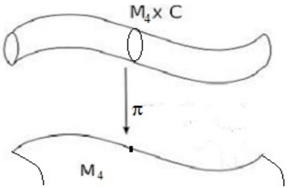
$$\mathcal{L}_{BI} \simeq \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2)\right) + \mathcal{O}(\beta^{-2}) \tag{28}$$

It has been proven by G. Boillat in the late 70-ties that the Born-Infeld lagrangian is the unique one that admits propagation of electromagnetic ways without bi-refringence (i.e. the appearance of two distinct light-cones).

The Kaluza-Klein theory

Theodor Kaluza (1885 – 1954) **and Oskar Klein** (1894 – 1977)

The Kaluza-Klein theory



The five-dimensional Kaluza-Klein space. The local coordinates are $x^A = (x^\mu, x^5)$; $A = 1, 2, ..5$, $\mu, \nu, .. = (0, i) = 0, 1, 2, 3$, **which under the projection** π **reduce to points in the Minkowski space-time:** $\pi(x^A) = \pi(x^\mu, x^5) = (x^\mu) \in M_4$.

In its first version proposed by Th. Kaluza, the fifth dimension was just an extra space coordinate, the entire space being isomorphic with $M_4 \times R^1 \sim [ct, x, y, z, x^5] \sim M_5$, a five-dimensional Minkowski space. Oskar Klein proposed to consider a compact fifth dimension, a circle with a very small radius. The dependence on the fifth dimension of functions defined on the "compactified" space must be then periodic, with a Fourier-like decomposition:

$$f(x^\mu, x^5) = \sum_{k=0}^{\infty} a_k(x^\mu) e^{ikmx^5}. \tag{29}$$

with dim (m) $= cm^{-1}$.

Then the eigenvalues of the fifth component of quantum momentum operator, $p_5 = -i\hbar\partial_5$ are integer multiples of mass m.

Kaluza-Klein metric

Let us recall the form of the Kaluza-Klein metric tensor:

$$\tilde{g}_{AB} = \begin{pmatrix} g_{\mu\nu} + \phi^2 A_\mu A_\nu & \phi^2 A_\mu \\ \phi^2 A_\nu & \phi^2 \end{pmatrix} \qquad (30)$$

or more explicitly,

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \phi^2 A_\mu A_\nu, \;\; \tilde{g}_{5\mu} = \tilde{g}_{\mu 5} = \phi^2 A_\mu, \;\; \tilde{g}_{55} = \phi^2. \qquad (31)$$

with A_μ and ϕ functions of space-time variables, identified as the 4-vector potential and an extra scalar field.

The inverse metric tensor $\tilde{g}^{AB}$ has the following components in 5-dimensional space-time:

$$\tilde{g}^{AB} = \begin{pmatrix} g^{\mu\nu} & -A^{\mu} \\ -A^{\nu} & \phi^{-2} + g_{\lambda\rho}A^{\lambda}A^{\rho} \end{pmatrix} \tag{32}$$

or more explicitly,

$$\tilde{g}^{\mu\nu} = g^{\mu\nu}, \;\; \tilde{g}^{5\mu} = \tilde{g}^{\mu 5} = -A_{\mu}, \;\; \tilde{g}^{55} = \phi^{-2} + g_{\lambda\rho}A^{\lambda}A^{\rho}. \tag{33}$$

- **A symmetric 2-covariant tensor in 5 dimensiona has** $(5 \times 6)/2 = 15$ **independent components versus** 10 **independent components of the** 4**-dimensional metric**
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- In fact the metric g_{AB} given by **(31)** has exactly 15 independent components, the 10 ones contained in the space-time part $g_{\mu\nu}$, the four ones given by the 4-dimensional vector (or rather the 1-form) A_μ and **one** component given by the scalar field ϕ.

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- **In fact the metric** g_{AB} **given by (31) has exactly** 15 **independent components, the** 10 **ones contained in the space-time part** $g_{\mu\nu}$**, the four ones given by the** 4-**dimensional vector (or rather the** 1-**form)** A_μ **and one component given by the scalar field** ϕ.
- **One easily calculates the determinants of the Kaluza-Klein metric:**

$$\det[g_{AB}] = \phi^2, \quad \det[g^{AB}] = \phi^{-2}. \tag{34}$$

Kaluza-Klein miracle

- **The original aim being the unification of Einstein's theory of gravity with Maxwell's electromagnetism, the scalar field ϕ was set equal to 1, thus leaving only 14 unknown functions, $g_{\mu\nu}$ and A_μ. However, the generalized Einstein's theory in 5 dimensions requires 15 equations to be satisfied:**

$$R_{AB} - \frac{1}{2} g_{AB} R = 0, \qquad (35)$$

so that the system becomes overdetermined if we suppress one of the 15 variables by fixing $\phi = 1$.

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- Nevertheless it turned out that this particular ansatz is still a solution to the full set of 15 equations, because in this case the last equation $R_{55} - g_{55} R = 0$ reduces to tautology $0 = 0$.

- **From now on, the metric tensor in five-dimensional Kaluza-Klein theory reads as follows:**

$$g_{AB} = \begin{pmatrix} g_{\mu\nu} + A_\mu A_\nu & A_\mu \\ A_\nu & 1 \end{pmatrix}, \tag{36}$$

where $A, B = 0, 1, 2, 3, 5$ **and** $\mu, \nu = 0, 1, 2, 3$

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where $A, B = 0, 1, 2, 3, 5$ **and** $\mu, \nu = 0, 1, 2, 3$

- **We deliberately discard the scalar field** $g_{55}(x^\mu)$**, putting** $g_{55} = 1$**, although the full system will be overconstrained. We shall also put** $g_{\mu\nu} = \eta_{\mu\nu} = diag(+, -, -, -)$**, saving only the electromagnetic part of the lagrangian.**

It is easy to compute all the non-vanishing components of the Riemann tensor R_{AB} of the metric **(36)** in the non-holonomous frame

$$\xi_\mu = \partial_\mu - A_\mu \partial_5, \quad \xi_5 = \partial_5. \tag{37}$$

They are as follows:

$$R_{\mu\nu\lambda\rho} = \frac{1}{4}\, F_{\mu\lambda} F_{\rho\nu} - F_{\nu\lambda} F_{\rho\mu} + 2\, F_{\mu\nu} F_{\rho\lambda}, \tag{38}$$

$$R_{\mu 5 \lambda\rho} = \frac{1}{2}\, \partial_\mu F_{\rho\lambda} \tag{39}$$

$$R_{\mu 55\lambda} = \frac{1}{4}\, \eta^{\nu\rho}\, F_{\mu\lambda} F_{\rho\nu}. \tag{40}$$

The Ricci tensor is then:

$$R_{\mu\nu} = \frac{1}{2}\,\eta^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho} \tag{41}$$

$$R_{\mu 5} = \frac{1}{2}\,\eta^{\nu\rho} \partial_\nu F_{\rho\mu} \tag{42}$$

$$R_{55} = \frac{1}{4}\,\eta^{\mu\nu} \eta^{\lambda\rho}\, F_{\mu\lambda} F_{\rho\nu} \tag{43}$$

and the scalar curvature is

$$R = -\frac{1}{4} \eta^{\mu\lambda} \eta^{\nu\rho}\, F_{\mu\nu} F_{\lambda\rho} \tag{44}$$

The full set of equations

The explicit form of the remaining 14 equations in the 5-dimensional Einstein's theory is then:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2} \eta^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho} - \frac{1}{8} \eta_{\mu\nu} \eta^{\sigma\lambda} \eta^{\kappa\rho} F_{\sigma\kappa} F_{\lambda\rho} = T_{\mu\nu} \quad (45)$$

$$R_{\mu 5} = \partial^{\nu} F_{\mu\nu} = 0 \quad (46)$$

- **Countless theories based on lagrangians depending on $F_{\mu\nu}F^{\mu\nu}$ and $(F_{\mu\nu}{}^{*}F^{\mu\nu})^2$ (the square is needed to keep the invariance under space reflections) can be produced if we lack a guiding principle to fix the form of the lagrangian.**

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- In the Kaluza-Klein theory, as well as in its improvements by P. Jordan and Y. Thiry was based on the Einstein-Hilbert variational principle in five-dimensional space, with lagrangian equal to R, the scalar curvature of the metric.
- This lagrangian is unique in four dimensions, because already the second invariant of the Riemann tensor,

$$I_2 = R_{ABCD} R^{ABCD} - 4\, R_{AB} R^{AB} + R^2 \qquad (47)$$

turns out to be a pure divergence and does not modify the equations of motion.

- The invariant (47) is the only quadratic combination of the Riemann tensor leading under variation to the second-order equations. In five dimensions this invariant is no more a divergence, therefore there is no reason to exclude it in the full theory.

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- The invariant (47) is the only quadratic combination of the Riemann tensor leading under variation to the second-order equations. In five dimensions this invariant is no more a divergence, therefore there is no reason to exclude it in the full theory.
- This fixes the lagrangian in five dimensions, leaving the place for the arbitrariness only in the choice of one dimensional parameter.
- This is the starting point for non-linear modification of the electrodynamics. In our calculations we shall discard the gravitational and scalar fields, both too weak to influence the behaviour of the electromagnetic field at short distances.

The invariant I_2 for the metric **(36)** is easily calculated and is found to be (discarding the pure divergence term equal to $\partial_\mu(F_{\rho\lambda}\partial^\mu F^{\rho\lambda}) - 2\partial^\nu(F_{\rho\lambda}\partial_\nu F^{\mu\lambda})$:

$$I_2 = \frac{3}{16}\left[(F_{\mu\nu}F^{\mu\nu})^2 - 2F_{\mu\lambda}F_{\nu\rho}F^{\mu\nu}F^{\lambda\rho}\right]. \tag{48}$$

For fixed Minkowskian metric $\eta_{\mu\nu}$ we can put $\sqrt{\mid g \mid} = 1$ and write the full lagragian as

$$\mathcal{L} = -\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} + \frac{3\varepsilon}{16e^2}\left[F_{\mu\nu}F^{\mu\nu})^2 - 2F_{\mu\lambda}F_{\nu\rho}F^{\mu\nu}F^{\lambda\rho}\right], \tag{49}$$

with ε a numerical parameter to be determined.

The equations of motion in vacuo are then

$$\partial_\lambda \left[F^{\lambda\rho} - \frac{3\varepsilon}{16e^2}(F_{\mu\nu}F^{\mu\nu})F^{\lambda\rho} + \frac{3\varepsilon}{e^2}F_{\mu\nu}F^{\lambda\mu}F^{\rho\nu} \right]. \qquad (50)$$

The identities

$$\partial_\mu F_{\lambda\rho} + \partial_\lambda F_{\rho\mu} + \partial_\rho F_{\mu\lambda} = 0 \qquad (51)$$

hold by definition (5), too.

Both lagrangian and equations of motion are more transparent when expressed by means of the fields E and B, D and H:

$$\mathcal{L} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) + \frac{3\varepsilon}{2e^2}(\mathbf{E}\cdot\mathbf{B})^2. \qquad (52)$$

The new term contains only the square of the second invariant of the electromagnetic field. The full set of modified Maxwell's equations is:

$$\mathrm{div}\ \mathbf{B} = 0, \qquad \mathbf{rot}\ \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \mathrm{div}\ \mathbf{D} = -\frac{3\varepsilon}{e^2}\ \mathbf{B}\cdot\mathbf{grad}(\mathbf{E}\cdot\mathbf{B}),$$

$$\mathbf{rot}\ \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \frac{3\varepsilon}{e^2}\left[\mathbf{H}\frac{\partial(\mathbf{E}\cdot\mathbf{B})}{\partial t} - \mathbf{E}\times\mathbf{grad}(\mathbf{E}\cdot\mathbf{B})\right]. \qquad (53)$$

In what follows we shall use the units in which $c = 1$, and in which we can put in the vacuum $\mathbf{E} = \mathbf{D}$ and $\mathbf{H} = \mathbf{B}$. Therefore the equations in vacuum will be

$$\mathrm{div}\ \mathbf{B} = 0, \qquad \mathbf{rot}\ \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \mathrm{div}\ \mathbf{E} = -\frac{3\varepsilon}{e^2}\ \mathbf{B} \cdot \mathbf{grad}(\mathbf{E} \cdot \mathbf{B}),$$

$$\mathbf{rot}\ \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \frac{3\varepsilon}{e^2}\left[\mathbf{B}\frac{\partial(\mathbf{E} \cdot \mathbf{B})}{\partial t} - \mathbf{E} \times \mathbf{grad}(\mathbf{E} \cdot \mathbf{B})\right]. \qquad (54)$$

When ε is put equal to zero, the equations recover their usual Maxwellian form. Two other possibilities, up to a scale that can be incorporated in e^2, are $\varepsilon = +1$ or -1.

The non-homogeneous couple of equations,

$$\mathrm{div}\ \mathbf{E} = -\frac{3\varepsilon}{e^2}\ \mathbf{B} \cdot \mathbf{grad}(\mathbf{E} \cdot \mathbf{B}) \tag{55}$$

and

$$\mathbf{rot}\ \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \frac{3\varepsilon}{e^2}\left[\mathbf{B}\frac{\partial(\mathbf{E} \cdot \mathbf{B})}{\partial t} - \mathbf{E} \times \mathbf{grad}(\mathbf{E} \cdot \mathbf{B})\right] \tag{56}$$

can be implemented by adding the charge density ρ to the right-hand side of **(55)** and the current density **j** to the right-hand side of (56).

However, even in the absence of these "external sources", the right-hand sides of the eqs. (55) and (56) behave like conserved induced charge and current densities; their conservation is independent of eventual other non-induced similar objects.

As a matter of fact, let us compare:

$$\frac{\partial}{\partial t}(\mathrm{div}\ \mathbf{E}) = -\frac{3\varepsilon}{e^2}\frac{\partial \mathbf{B}}{\partial t}\cdot \mathbf{grad}(\mathbf{E}\cdot\mathbf{B}) - \frac{3\varepsilon}{e^2}\mathbf{B}\cdot\mathbf{grad}\frac{\partial(\mathbf{E}\cdot\mathbf{B})}{\partial t} =$$

$$= -\frac{3\varepsilon}{e^2}(\mathbf{rot}\mathbf{E})\cdot\mathbf{grad}(\mathbf{E}\cdot\mathbf{B}) - \frac{3\varepsilon}{e^2}\mathbf{B}\cdot\mathbf{grad}\frac{\partial(\mathbf{E}\cdot\mathbf{B})}{\partial t} \qquad (57)$$

and

$$\text{div}\ \frac{\partial \mathbf{E}}{\partial t} = \text{div}(\mathbf{rot}\mathbf{B}) - \frac{3\varepsilon}{e^2}\text{div}\left(\mathbf{B}\ \frac{\partial(\mathbf{E}\cdot\mathbf{B})}{\partial t}\right) - \frac{3\varepsilon}{e^2}\text{div}(\mathbf{E}\times\mathbf{grad}(\mathbf{E}\cdot\mathbf{B})) =$$

$$= \frac{3\varepsilon}{e^2}(\text{div}\mathbf{B})\frac{\partial(\mathbf{E}\cdot\mathbf{B})}{\partial t} - \frac{3\varepsilon}{e^2}\mathbf{B}\cdot\mathbf{grad}\frac{\partial(\mathbf{E}\cdot\mathbf{B})}{\partial t} + \frac{3\varepsilon}{e^2}(\mathbf{rot}\mathbf{E})\cdot\mathbf{grad}(\mathbf{E}2\cdot\mathbf{B}). \quad (58)$$

because

$$\text{div}\mathbf{B} = 0, \quad \mathbf{rot}(\mathbf{grad} f) = 0, \quad \text{div}(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{rot}\mathbf{a}) - \mathbf{a}\cdot(\mathbf{rot}\mathbf{b}),$$

therefore

$$\frac{\partial}{\partial t}\left[-\frac{3\varepsilon}{e^2}\mathbf{B}\cdot\mathbf{grad}(\mathbf{E}\cdot\mathbf{B})\right]+\mathrm{div}\left[\frac{3\varepsilon}{e^2}\mathbf{B}\frac{\partial(\mathbf{E}\cdot\mathbf{B})}{\partial t}-\mathbf{E}\times\mathbf{grad}(\mathbf{E}\cdot\mathbf{B})\right]=0. \tag{59}$$

We shall denote the induced charge density by ρ_{ind}:

$$\rho_{ind}=-\frac{3\varepsilon}{e^2}\mathbf{B}\cdot\mathbf{grad}(\mathbf{E}\cdot\mathbf{B}), \tag{60}$$

and the induced current density by $\mathbf{j}_{ind}$:

$$\mathbf{j}_{ind}=\frac{3\varepsilon}{e^2}\mathbf{B}\frac{\partial(\mathbf{E}\cdot\mathbf{B})}{\partial t}-\mathbf{E}\times\mathbf{grad}(\mathbf{E}\cdot\mathbf{B}) \tag{61}$$

- **with**

$$\frac{\partial \rho_{ind}}{\partial t} + \mathrm{div}(\mathbf{j}_{ind}) = 0. \tag{62}$$

The theory will not need any non-induced charges if we can prove the existence of charged stable static solutions, localized in space (solitons). If we form the sum:

- with

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-

$$\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \tag{63}$$

we shall easily find another conservation law:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + \frac{3\varepsilon}{e^2}(\mathbf{E}\cdot\mathbf{B})^2\right] = \mathrm{div}\,(\mathbf{E}\times\mathbf{B}) \tag{64}$$

The Poynting vector in this theory is the same as in the linear electrodynamics, whereas the energy density contains a new term, as compared with the classical theory:

$$\mathcal{H} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + \frac{3\varepsilon}{e^2}(\mathbf{E} \cdot \mathbf{B})^2 \tag{65}$$

Note that the parameter ε has to be positive, in order to ensure the positivity of the energy. From now on we shall set $\varepsilon = 1$, leaving only the coupling constant e^2 to be determined.

- **The hamiltonian $\mathcal{H}$ can be also found from the lagrangian $\mathcal{L}$ directly. The only time derivatives present in $\mathcal{L}$ are $\partial_0 A_k, \; k = 1, 2, 3$ which enter through the combination $E_k = \partial_0 A_k - \partial_k A_0$.**

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- **Therefore E_k, the components of the electric field, can be chosen as generalized velocities, so that**

$$\mathcal{H} = \mathbf{E} \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{E}} - \mathcal{L}. \tag{66}$$

which yields the same result.

- Whenever the fields **E** and **B** are orthogonal to each other, our system in vacuum **(55, 56)** coincides with Maxwell's equations. Such is the case of the electromagetic waves, which are also solutions to the equations (55, 56).

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- Moreover, these solutions are stable with respect to perturbations. As a matter of fact, any deviation from the usual solution in which **E** is everywhere orthogonal to **B**, leads automatically to the rise of the energy $\mathcal{H}$, ensuring stability.

Let us also note that the energy-momentum tensor could be obtained directly as

$$T_{\mu\nu} = \frac{\partial(\sqrt{\mid g \mid}\mathcal{L}}{\partial g^{\mu\nu}} \tag{67}$$

yielding the same expressions for the Hamiltonian T_{00} and the Poynting vector $P_k = T_{0k}$.

Let us rewrite the equations (55 and 56) in the stationary case, when all the time derivatives vanish:

$$\mathrm{div}\ \mathbf{B} = 0, \qquad \mathbf{rot}\ \mathbf{E} = 0,$$

$$\mathrm{div}\ \mathbf{E} = -\frac{3}{e^2}\mathbf{B}\cdot\mathbf{grad}(\mathbf{E}\cdot\mathbf{B}), \quad \mathbf{rot}\ \mathbf{B} = -\frac{3}{e^2}\mathbf{E}\times\mathbf{grad}(\mathbf{E}\cdot\mathbf{B}). \quad (68)$$

It would be very interesting to obtain a static and non-singular solution of this system, having finite energy and behaving like a soliton.

- This is excluded in the linear case, therefore, if such solution exists, both fields **E** and **B** must be different from zero and non-orthogonal at least in some finite domain of space. We should also impose the rapid enough vanishing of both fields at infinity.

- This is excluded in the linear case, therefore, if such solution exists, both fields **E** and **B** must be different from zero and non-orthogonal at least in some finite domain of space. We should also impose the rapid enough vanishing of both fields at infinity.
- Spherical symmetry for **B** leads immediately to the singularity at the origin; so, if the condition div **B** is to be maintained everywhere, the lines of force of the field **B** have to be closed.

- **The lines of the local current**

$$\mathbf{j}_{ind} = \mathbf{rot}\ \mathbf{B} = -\frac{3\varepsilon}{e^2}\,\mathbf{E} \times \mathbf{grad}(\mathbf{E} \cdot \mathbf{B})$$

must be closed, too.

- The lines of the local current

$$\mathbf{j}_{ind} = \mathbf{rot}\ \mathbf{B} = -\frac{3\varepsilon}{e^2}\ \mathbf{E} \times \mathbf{grad}(\mathbf{E} \cdot \mathbf{B})$$

must be closed, too.

- This suggests the axial symmetry in which the current would have only the azimuthal component, and the field $\mathbf{B}$ would be everywhere perpendicular to the azimuthal unit vector $\mathbf{e}_\varphi$ (in cylindrical coordinates $(\rho = \sqrt{x^2 + y^2}),\ z,\ \varphi)$), i.e. $\mathbf{B}$ having its components along $\mathbf{e}_z$ and $\mathbf{e}_\rho$ only. Also the field $\mathbf{E}$ should have only the z and ρ components; then the Poynting vector $\mathbf{P} = \mathbf{E} \times \mathbf{B}$ will have the azimuthal component only.

- **Such a configuration has some remarkable symmetry properties:**
  The trilinear combinations on the right-hand sides of equations (68) produce induced charge and current densities.

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  The trilinear combinations on the right-hand sides of equations (68) produce induced charge and current densities.
- The current having only the azimuthal component will produce magnetic field which at great distances is similar to that of a circular distribution of currents, i.e. the one of a magnetic dipole.
- At the same time, one can expect a non-vanishing charge concentration falling off quite rapidly with distance from the origin, at large distances E should be then similar to the electric Coulomb field.

All these conditions put together lead to the following symmetry properties of the components **E** and **B**:

$$E_z(\rho, z) = -E_z(\rho, -z); \quad E_\rho(\rho, z) = E_\rho(\rho, -z), \tag{69}$$

and

$$B_z(\rho, z) = B_z(\rho, -z); \quad B_\rho(\rho, z) = -B_\rho(\rho, -z). \tag{70}$$

Let us evaluate the behaviour of charge and current distributions far away from the origin. We can take the field of a magnetic dipole and of concentrated charge as zeroth approximation satisfying Maxwell's equations, then insert them into the right-hand sides of eqs. **(68)** and compute the first corrections, supposing that the fields **E** and **B** develop as:

$$\mathbf{E} = \overset{(0)}{\mathbf{E}} + \frac{1}{e^2}\overset{(1)}{\mathbf{E}} + ..., \qquad \mathbf{B} = \overset{(0)}{\mathbf{B}} + \frac{1}{e^2}\overset{(1)}{\mathbf{B}} + ... \tag{71}$$

if we put

$$\overset{(0)}{\mathbf{E}} = \frac{Q\ \rho}{(\rho^2 + z^2)^{\frac{3}{2}}}\mathbf{e}_\rho + \frac{Q\ z}{(\rho^2 + z^2)^{\frac{3}{2}}}\mathbf{e}_z, \tag{72}$$

and

$$\overset{(0)}{\mathbf{B}} = \frac{3\mu\ \rho z}{4(\rho^2 + z^2)^{\frac{5}{2}}}\mathbf{e}_\rho + \frac{\mu\ (2z^2 - \rho^2)}{4(\rho^2 + z^2)^{\frac{5}{2}}}\mathbf{e}_z, \tag{73}$$

where Q is the total charge, μ the total magnetic moment.

As the first correction, we obtain

$$\mathrm{div}\ \overset{(1)}{\mathbf{E}} = \frac{3\mu^2 Q}{8(\rho^2+z^2)^{\frac{11}{2}}}(\rho^2+10z^2), \tag{74}$$

and

$$\mathbf{rot}\ \overset{(1)}{\mathbf{B}} = \frac{3\mu Q^2\ \rho}{2(\rho^2+z^2)^{\frac{9}{2}}}\mathbf{e}_\varphi \tag{75}$$

which shows that the charge density falls off as R^{-9} and the current density as R^{-8} ($R=\sqrt{\rho^2+z^2}$), i.e. really fast.

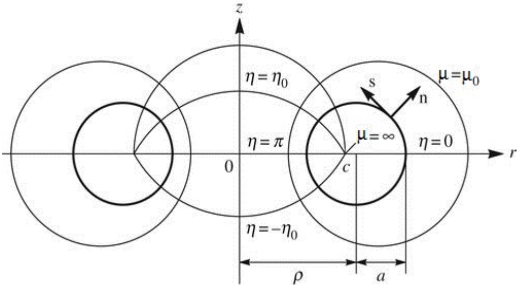
The lines of force of the field B form a family of closed curves which can be transformed into a family of circles by a suitable coordinate transformation; the toroidal coordinates are best adapted to describe the situation.

Let us introduce toroidal coordinates $(\mu, \ \eta, \ \phi)$:

$$\rho = \frac{a \ \sinh \mu}{\cosh \mu - \cos \eta}, \quad z = \frac{a \ \sin \eta}{\cosh \mu - \cos \eta}, \quad \phi = \varphi, \qquad (76)$$

with $0 \leq \phi \leq 2\pi, \quad 0 \leq \eta \leq 2\pi$ **and** $0 \leq \mu \leq \infty$; a **is the constant of dimension of length fixing the scale;** $\mu, \ \eta$ **and** ϕ **are dimensionless.**

Toroidal coordinates



Constant coordinate lines $\mu =$ Const. **and** $\eta =$ Const. **in the** (ρ, z)**-plane.**

A surface $\mu = \mu_0 =$ Const. is a torus with he external radius $a \coth \mu_0$ and internal radius $a/\sinh \mu_0$. When $\mu \to \infty$ it reduces to a circle of radius a in the (x, y)-plane. When $\mu \to 0$, the corresponding circle approaches the z-axis.

The lines of force of B coincide with circles $\mu =$ Const., i.e. in new coordinates (76)

$$\mathbf{B} = B_\eta(\mu, \eta)\, \mathbf{e}_\eta. \tag{77}$$

while $B_\mu(\mu, \eta) = 0$. **This determines the dependence of B on η:**

$$\text{as} \quad B_\mu(\mu, \eta) = (\mathbf{rot A}) \cdot \mathbf{e}_\mu \quad \text{with} \quad \mathbf{A} = A_\phi(\mu, \eta)\mathbf{e}_\phi, \tag{78}$$

we have

$$B_\mu(\mu,\eta) = \frac{(\cosh\mu - \cos\eta)^2}{a\ \sinh\mu}\ \frac{\partial}{\partial\eta}\left(\frac{\sinh\mu}{\cosh\mu - \cos\eta} A_\phi\right) = 0. \quad (79)$$

Therefore

$$A_\phi(\mu,\eta) = (\cosh\mu - \cos\eta)\ G(\mu), \quad (80)$$

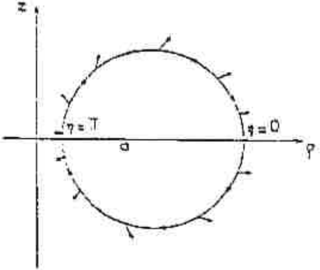
and

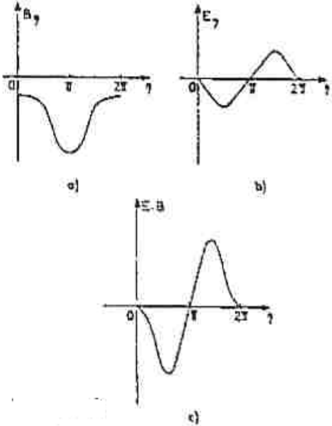
$$B_\eta(\mu,\eta) = -\frac{(\cosh\mu - \cos\eta)^2}{a\ \sinh\mu}\frac{\partial}{\partial\eta}(\sinh\mu\ G(\mu)) \quad (81)$$

with yet unknown function $G(\mu)$.

- **Putting aside the problem of eventual singularity, we can at this point see quite well what the induced charge and current distributions look like. Consider one of the lines of force of B, i.e. a circle $\mu =_m u_0, \; \phi = \phi_0$ in the (ρ, z) plane** (Figure 1, left).

- **Putting aside the problem of eventual singularity, we can at this point see quite well what the induced charge and current distributions look like. Consider one of the lines of force of B, i.e. a circle $\mu =_m u_0,\ \phi = \phi_0$ in the (ρ, z) plane** (Figure 1, left).
- **The symmetry properties of the fields E impose the vanishing of its η-component for $z = 0$, i.e. for $\eta = 0$ or π, because $E_\eta(\eta) = -E_\eta(2\pi - \eta)$. On the other hand, $B_\eta(\eta) = B_\eta(2\pi - \eta) >$, so that the scalar product $\mathbf{E} \cdot \mathbf{B} = E_\eta B_\eta$ on the circle $\mu = \mu_0$ is an odd function of η (Figure 1, right).**

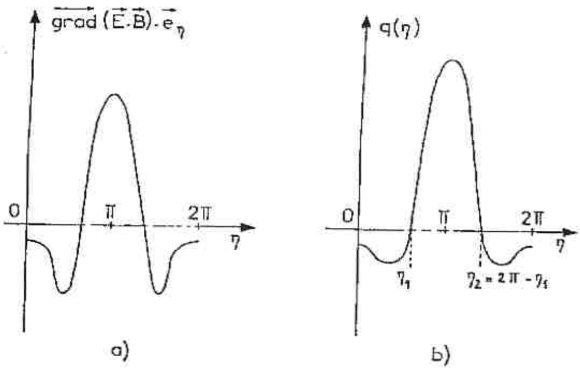



a)
b)
c)

In order to obtain the charge distribution along this circle, we have to compute $-\mathbf{B}\cdot\mathbf{grad}(\mathbf{E}\cdot\mathbf{B})$, which reduces to the expression

$$-B_\eta \frac{(\cosh\mu - \cos\eta)}{a}\frac{\partial}{\partial\eta}\left(\mathbf{E}\cdot\mathbf{B}\right). \quad (82)$$

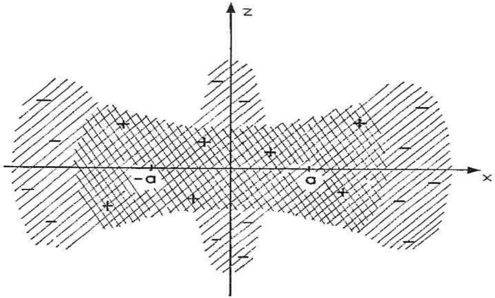
The corresponding functions are displayed in **Figure 3:**



a) The projection of $\mathbf{grad}(\mathbf{E} \cdot \mathbf{B})$ on the unit vector $\mathbf{e}_\eta$ as a function of η; b) The charge density distribution $q(\eta)$ as function of η.

- **The charge density changes its sign between η_1 and $\eta_2 = 2\pi - \eta_1$. This phenomenon describes vacuum polarization: if at the core of the static solution there is an accumulation of charge density of a given sign, it must be surrounded by a cloud of charge density of opposite sign.**

- The charge density changes its sign between η_1 and $\eta_2 = 2\pi - \eta_1$. This phenomenon describes vacuum polarization: if at the core of the static solution there is an accumulation of charge density of a given sign, it must be surrounded by a cloud of charge density of opposite sign.
- The value of η_1 at which the change of sign occurs depends on the line (i.e. the value of μ). Reproducing similar reasoning for all circles $\mu =$ Const. we obtain the picture of the overall charge density (Figure 4):



The cross-section $(x, z, \phi =$ **Const.**$)$ of the charge density distribution. The strongest vacuum polarization is on the z-**axis** and in the symmetry plane (x, y), around the axially symmetric charge distribution at the core.

- If at any point of this distribution we would like to interpret the azimuthal current density obtained from the last equation **(68)** as being produced by a rotational movement of the charge density around the z-**axis**, then it is easy to see, just comparing the units (remember that we have put $c = 1$), that the induced charge has to "move" with the speed of light.

- If at any point of this distribution we would like to interpret the azimuthal current density obtained from the last equation **(68)** as being produced by a rotational movement of the charge density around the z-**axis**, then it is easy to see, just comparing the units (remember that we have put $c = 1$), that the induced charge has to "move" with the speed of light.
- Of course, nothing is moving here: there is just a distribution of static fields $\mathbf{E}$ and $\mathbf{B}$ which produces this illusion, because the Poynting vector $\mathbf{E} \times \mathbf{B}$ has only the azimuthal component. Nevertheless, the illusion produced is the same as for the electron as a whole submitted to the *"zitterbewegung"* with the speed c as it comes out from the Dirac equation.

- There is also another striking similarity between the predictions of this model and those of the Dirac equation. The equations **(68)** are invariant with respect to the independent changes of sign, $\mathbf{E} \to -\mathbf{E}$ and $\mathbf{B} \to -\mathbf{B}$.

- There is also another striking similarity between the predictions of this model and those of the Dirac equation. The equations (68) are invariant with respect to the independent changes of sign, $\mathbf{E} \to -\mathbf{E}$ and $\mathbf{B} \to -\mathbf{B}$.
- This means that any static solution generates automatically three other ones, obtained by the inversions of $\mathbf{E}$ and $\mathbf{B}$. Now, the total charge is linear in $\mathbf{E}$, while the total magnetic moment is linear in $\mathbf{B}$; the Poynting vector is proportional to $\mathbf{E} \times \mathbf{B}$, and so will be the total kinetic angular momentum obtained by the integration of $\mathbf{r} \times (\mathbf{E} \times \mathbf{B})$ over entire space.

The four solutions so obtained can be put together in Table 1:

Solution	Energy	Charge	Magnetic μ	Spin
$\mathbf{E}, \quad \mathbf{B}$	m	q	$\boldsymbol{\mu}$	$\mathbf{S}$
$\mathbf{E}, \ -\mathbf{B}$	m	q	$-\boldsymbol{\mu}$	$-\mathbf{S}$
$-\mathbf{E}, \quad \mathbf{B}$	m	$-q$	$\boldsymbol{\mu}$	$-\mathbf{S}$
$-\mathbf{E}, \ -\mathbf{B}$	m	$-q$	$-\boldsymbol{\mu}$	$\mathbf{S}$

- **Any static solution is, as a matter of fact, a quadruplet of solutions with the same rest mass. The first two solutions describe a particle with electric charge q and magnetic moment μ** *parallel* **to spin S, in states with spin up or down** (with respect to the *z*-axis).

- **Any static solution is, as a matter of fact, a quadruplet of solutions with the same rest mass. The first two solutions describe a particle with electric charge** q **and magnetic moment** μ *parallel* **to spin S, in states with spin up or down** (with respect to the z-axis).
- **The second pair of solutions describes a particle with the opposite charge** $-q$ **and magnetic moment** *antiparallel* **to the spin S, also in two states with spin up or down. This result is identical with the predictions of Dirac's equation for the electron, which leads to the existence of the positron and a half-integer spin, which seems to be good news.**

The bad news is that unfortunately no C^{∞}-**class** solution of the system **(68** does exist. The proof is simple and goes as follows:

Knowing that $\operatorname{div} \mathbf{B} = 0$, we can write

$$\mathbf{B} \cdot \mathbf{grad}(\mathbf{E} \cdot \mathbf{B}) = \operatorname{div} \left(\mathbf{B} \left(\mathbf{E} \cdot \mathbf{B}\right)\right) \tag{83}$$

Similarly,

$$\mathbf{E} \times \mathbf{grad}(\mathbf{E} \cdot \mathbf{B}) = \mathbf{rot}(\mathbf{E} \left(\mathbf{E} \cdot \mathbf{B}\right)), \tag{84}$$

because $\mathbf{rot}\ \mathbf{E} = 0$. This leads to

$$\operatorname{div} \left(\mathbf{E} + \frac{3}{e^2} \mathbf{B}(\mathbf{E} \cdot \mathbf{B})\right) = 0, \quad \mathbf{rot} \left(\mathbf{B} - \frac{3}{e^2} \mathbf{E}(\mathbf{E} \cdot \mathbf{B})\right) = 0. \tag{85}$$

If the space we are working in has the topology of R^3, and all the functions are supposed to be C^∞-**smooth**, then the Poincaré lemma states that

$$\mathbf{E} + \frac{3}{e^2}\,\mathbf{B}(\mathbf{E}\cdot\mathbf{B}) = \mathbf{rot}\ \mathbf{C} \ \ ; \ \ \text{and} \quad \mathbf{B} - \frac{3}{e^2}\,\mathbf{E}(\mathbf{E}\cdot\mathbf{B}) = \mathbf{grad}\psi. \quad (86)$$

with $\mathbf{C}(\mathbf{r})$ and $\psi(\mathbf{r})$ supposed to be C^∞ smooth (vector and scalar, respectively) functions of $\mathbf{r}$.

Taking the scalar product of the first equation in **(86)** by **E** and of the second equation by **B** we get (supposing that $\mathbf{E} = -\mathbf{grad}\, V$):

$$\mathbf{E}^2 + \frac{3}{e^2}\,(\mathbf{E}\cdot\mathbf{B})^2 = \mathbf{E}\cdot\mathbf{rotC} = -(\mathbf{grad}\, V)\cdot\mathbf{rotC} = -\mathrm{div}\,(V\mathbf{rotC}), \tag{87}$$

and

$$\mathbf{B}^2 - \frac{3}{e^2}\,(\mathbf{E}\cdot\mathbf{B})^2 = \mathbf{B}\cdot\mathbf{grad}\psi = \mathrm{div}(\psi\,\mathbf{B}). \tag{88}$$

Combining equations (87) and (88) together, we have

$$\mathbf{E}^2 + \mathbf{B}^2 = \mathrm{div}(\psi \mathbf{B} - V \,\mathbf{rot}\, \mathbf{C}). \qquad (89)$$

If we want the total energy, as well as the total charge, to be finite, then both **E** and **B** must decrease at infinity at least as R^{-2}, so that the right-hand side of **(89)** must be of the order of R^{-4}, which means in turn that the vector field $\psi\,\mathbf{B} - V\,\mathbf{rot}\,\mathbf{C}$ is decreasing at infinity as R^{-3}. Applying the Gauss-Ostrogradsky theorem to a finite 3-**volume** Ω and its 2-dimensional boundary $\partial\Omega$:

$$\int_\Omega \mathrm{div}(\psi\,\mathbf{B} - V\,\mathbf{rot}\,\mathbf{C})\, d^3\mathbf{r} = \int_{\partial\Omega} (\psi\,\mathbf{B} - V\,\mathbf{rot}\,\mathbf{C}) \cdot \mathrm{d}\boldsymbol{\Sigma}, \qquad (90)$$

- we see that the integral of $\mathbf{E}^2 + \mathbf{B}^2$ over a spherical volume of radius R behaves as R^{-1}, i.e. it vanishes when taken over the whole space.

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- The impossibility of obtaining a C^∞ solution with finite energy can be also seen if we try to construct it by applying the method of successive approximations in toroidal coordinates.

Now the problem can be reduced down to two equations for two unknown functions, the azimuthal component of the vector potential A_ϕ and the scalar potential V. We can believe that in basic state the dependence on the azimuthal angle ϕ is trivial, therefore we may set

$$A_\phi = A_\phi(\mu, \eta) \quad \text{and} \quad V = V(\mu, \eta) \tag{91}$$

The dependence of both potentials on the toroidal angle η must be of the form $\sin(k\eta)$ or $\cos(k\eta)$, $k = 1, 2, ...$; using the substitution

$$A_\phi = u(\eta)\,\sqrt{\cosh\mu - \cos\eta} = (\cosh\mu - \cos\eta)\; G(\mu) \tag{92}$$

we make the μ-component of the magnetic field vanish, $B_\mu = 0$.

Along with another substitution

$$V = v(\eta)\,\sqrt{\cosh\mu - \cos\eta} \tag{93}$$

the laplacians appearing on the left-hand side of equations **(68)** will have their variables separated. For example, the equation

$$\mathrm{div}\,\mathbf{E} = -\frac{3}{e^2}\,\mathbf{B}\cdot\mathbf{grad}(\mathbf{E}\cdot\mathbf{B}) \tag{94}$$

will take on the form:

$$\frac{1}{\sinh\mu}\frac{\partial}{\partial\mu}\left(\sinh\mu\frac{\partial v}{\partial\mu}\right)+\frac{\partial^2 v}{\partial\eta^2}+\frac{1}{4}v=$$

$$\frac{3}{a^2e^2}\frac{(\cosh\mu-\cos\eta)}{\sinh^2\mu}\left[\frac{\partial}{\partial\mu}(\sinh\mu\ G(\mu))\right]^2[W(\mu,\eta)], \qquad (95)$$

with

$$W=[\cosh\mu-\cos\eta)\frac{\partial^2 v}{\partial\eta^2}+4\sin\eta\frac{\partial v}{\partial\eta}+\frac{(5\sin^2\eta+2\cosh\mu\cos\eta-\cos^2\eta)}{4(\cosh\mu-\cos\eta)}. \qquad (96)$$

Similarly, the laplacian of the function $u(\mu, \eta)$ is equal to some non-linear terms mutiplied by $3/(a^2 e^2$.

Developing functions u and v as e.g.

$$\sum_{n=1}^{\infty} [\overset{(1)}{v}_n(\mu) \sin(n\eta) + \overset{(2)}{v}_n(\mu) \cos(n\eta)]$$

the second derivatives in **(95)** will be replaced by $n^2 v$, and the solutions of the *homogeneous equations*, which correspond to the zeroth approximation ($\frac{2}{a^2 e^2} = 0$) are given as a series in spherical harmonics of half-integer order (cf. Morse and Feshbach).

$$P_{n+\frac{1}{2}}(\cosh \mu) \quad \text{and} \quad Q_{n-\frac{1}{2}}(\cosh \mu) \tag{97}$$

- **The functions $P_{n+\frac{1}{2}}$ display a logarithmic singularity for $\mu = \infty$, i.e. on the circle $\rho = a$, whereas the functions $Q_{n-\frac{1}{2}}$ have a logarithmic singularity for $\mu = 0$ (i.e. $\rho, z \to \infty$).**

- The functions $P_{n+\frac{1}{2}}$ display a logarithmic singularity for $\mu = \infty$, i.e. on the circle $\rho = a$, whereas the functions $Q_{n-\frac{1}{2}}$ have a logarithmic singularity for $\mu = 0$ (i.e. $\rho, z \to \infty$).
- In order to avoid singularity we may use the combination of both, but the price to pay is a discontinuity for some value of μ (on some toroidal surface). If we feed in such a solution to the right-hand side and use the Green functions in order to compute the first correction, we shall be faced with exactly the same problem, because any Green function has at least one singularity of the same kind.

- The failure of producing a non-singular soliton is probably due to the fact that we have projected everything onto three space dimensions, discarding the fifth circular one. It seems possible to obtain solitons using the fifth dimension in a non-trivial way, like in the case of Kaluza-Klein monopoles of Sorkin and Gross and Perry.

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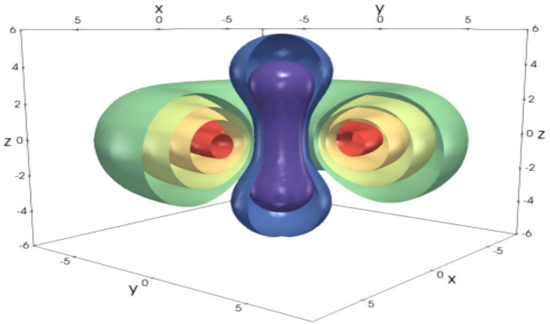
- Another development should include the non-abelian generalization of the Kaluza-Klein theory into more dimensions, in which also higher order invariants of the Riemann tensor might be included to the generalized lagrangian.

Recently toroidal solutions for the Higgs-'t Hooft $SU(2) \times U(1)$ monopole were produced numerically by M.S. Volkov *et al.*.



The constant energy density surfaces are represented in cartesian coordinates.

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