Principle of General Covariance : Applications to Inelastic Continuum and Relative Gravitation

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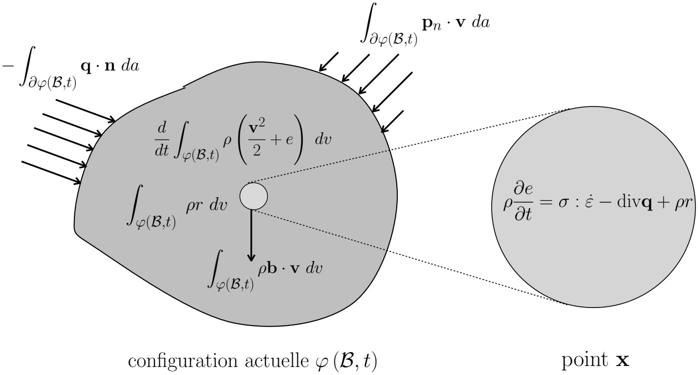
Planning

1. Classical Conservations Laws
2. Relative Gravitation and Principle of General Covariance
3. Description of some Inelastic Deformation
4. Principle of General Covariance and Applications
5. Concluding remarks

I. Conservation laws : Classical Thermomechanics



Conservation of energy in global formulation and localization.

Conservation laws : unified framework

1. $\mathscr{B}$**-derivative** (particular Lie derivative) (R 2003, 2022)

$$\left(\frac{d^{\mathscr{B}}}{dt}\mathcal{T}\right)\left(\mathbf{u}_1,...\mathbf{u}_p,\omega^1,...\omega^q\right) := \frac{d}{dt}\left[\mathcal{T}\left(\mathbf{u}_1,...\mathbf{u}_p,\omega^1,...\omega^q\right)\right]$$

2. **Conservation laws** (Poincaré's integral invariance) not covariant

$$\frac{d^{\mathscr{B}}}{dt}\left(\rho\omega_n\right) = 0$$

$$\frac{d^{\mathscr{B}}}{dt}\left(\rho v^i\omega_n\right) = \rho b^i\omega_n + \mathrm{div}\mathbf{p}_n^i\ \omega_n, \quad i = 1,2,3$$

$$\frac{d^{\mathscr{B}}}{dt}\left[\rho\left(u + \frac{\mathbf{v}^2}{2}\right)\omega_n\right] = \rho\mathbf{b}\cdot\mathbf{v}\ \omega_n + \mathrm{div}\left(\sigma^{\mathrm{T}}(\mathbf{v})\right)\omega_n + \rho r\ \omega_n - \mathrm{div}\mathbf{J}_H\ \omega_n$$

3. Entropy inequality

$$\frac{d^{\mathscr{B}}}{dt}\left(\rho s\omega_n\right) \geq \rho\frac{r}{\theta}\ \omega_n - \mathrm{div}\left(\frac{\mathbf{J}_H}{\theta}\right)\omega_n$$

Conservation laws : embedded base $(\mathbf{u}_\alpha, \alpha = 1, 3)$

1. Mass

$$\rho_0 = \rho\ J, \quad \text{with} \quad J := \frac{\omega_n\left(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\right)}{\omega_n\left(\mathbf{u}_{10}, \mathbf{u}_{20}, \mathbf{u}_{30}\right)}$$

2. Linear momentum:

$$\rho_0 \frac{d}{dt}\mathbf{v} = \rho_0 \mathbf{b} + \mathrm{Div}\mathbf{P}$$

3. Energy

$$\rho_0 \frac{d}{dt} u = \mathbf{P} : \nabla\mathbf{v} + \rho_0 r - \mathrm{Div}\mathbf{J}_{H0}$$

4. Entropy

$$\rho_0 \frac{d}{dt} s \geq \rho_0 \frac{r}{\theta} - \mathrm{Div}\left(\frac{\mathbf{J}_{H0}}{\theta}\right)$$

Conservation laws : "spatial" base $(\mathbf{e}_i, i = 1, 3)$

1. Mass

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \mathbf{v}) = 0$$

2. Linear momentum:

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \mathrm{div}\,(\rho \mathbf{v} \otimes \mathbf{v}) = \rho\ \mathbf{b} + \mathrm{div}\sigma$$

3. Energy

$$\rho \frac{d}{dt} u = \sigma : \mathbf{D} + \rho r - \mathrm{div}\mathbf{J}_H$$

4. Entropy

$$\rho \frac{d}{dt} s \geq \rho \frac{r}{\theta} - \mathrm{div}\left(\frac{\mathbf{J}_H}{\theta}\right)$$

Remarque

Base $(\mathbf{e}_i, i = 1, 3)$ is temporarily embedded base within the continuum at t (Goldstein "Lecture Notes on Fluid Mechanics").

Symmetries and conservation laws

1. Despite their **unique framework** for expressing conservation laws, Newton's laws of motion are not covariant (Galilean transformations) conversely to Lagrange's equations.

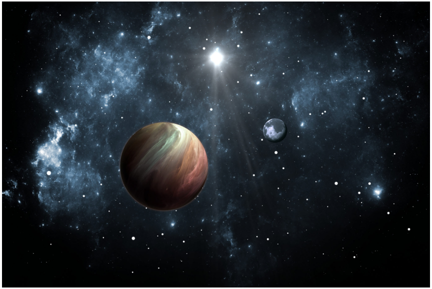
2. Einstein's great advance was to put **spacetime symmetry** first : SR makes particle mechanics and electromagnetism compatible (Lorentz transformations)

3. For RG, Einstein introduced the principle of equivalence (local symmetry) - **invariance of law expressions** under local changes of the spacetime coordinates. Symmetry Invariance for RG requires additional precisions: **it was a source of disputes** (Norton, 2005).

4. Indeed, Noether's theorems relate symmetry with conservation laws: **continuous symmetries generate conservation laws**.

5. **Constructing conservation laws and conserved quantities in an physics theory, including Einstein-Cartan Relative Gravitation and also other field theory, is still a main problem**.

II. Relative Gravitation and Principle of General Covariance



- Spacetime geometry
- Relative Gravitation as Gauge Theory

Given an affine **connection** ∇ on a manifold $\mathscr{B}$:

1. **Constants of structure** of Cartan (non coordinate base)[1]

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] := \aleph^\gamma_{0\alpha\beta}\ \mathbf{e}_\gamma$$

2. **Torsion tensor**

$$\begin{cases} \aleph(\mathbf{f}^\gamma, \mathbf{e}_\alpha, \mathbf{e}_\beta) = \mathbf{f}^\gamma \left(\nabla_{\mathbf{e}_\alpha}\mathbf{e}_\beta - \nabla_{\mathbf{e}_\beta}\mathbf{e}_\alpha - [\mathbf{e}_\alpha, \mathbf{e}_\beta]\right) \\ \aleph^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha} - \aleph^\gamma_{0\alpha\beta} \end{cases}$$

3. **Curvature tensor**

$$\begin{cases} \mathfrak{R}(\mathbf{f}^\gamma, \mathbf{e}_\alpha, \mathbf{e}_\beta, \mathbf{e}_\lambda) = \mathbf{f}^\gamma(\nabla_{\mathbf{e}_\alpha}\nabla_{\mathbf{e}_\beta}\mathbf{e}_\lambda - \nabla_{\mathbf{e}_\beta}\nabla_{\mathbf{e}_\alpha}\mathbf{e}_\lambda - \nabla_{[\mathbf{e}_\alpha,\mathbf{e}_\beta]}\mathbf{e}_\lambda) \\ \mathfrak{R}^\gamma_{\alpha\beta\lambda} = (\partial_\alpha\Gamma^\gamma_{\beta\lambda} + \Gamma^\mu_{\beta l}\Gamma^\gamma_{\alpha\mu}) - (\partial_\beta\Gamma^\gamma_{\alpha\lambda} + \Gamma^\mu_{\alpha\lambda}\Gamma^\gamma_{\beta\mu}) - \aleph^\mu_{0\alpha\beta}\Gamma^\gamma_{\mu\lambda} \end{cases}$$

[1] Orthonormal cylindrical : $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ and spherical $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$ are not coordinate bases.

Tool 2. : Covariant differential operators for $\mathscr{B}$

Two **covariant differential operators** necessary for any continuum theory. Let $\mathbf{T}$ be a tensor type (p, q):

1. Connection and **covariant derivative** ON $\mathscr{B}$

$$\boxed{\nabla_\gamma T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q}} = \partial_\gamma T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q} + \sum_{s=1}^{s=p} \Gamma^{\alpha_s}_{\gamma\mu} T^{\alpha_1\cdots\alpha_{s-1}\ \mu\ \alpha_{s+1}\cdots\alpha_p}_{\beta_1\cdots\beta_q} - \sum_{s=1}^{s=q} \Gamma^{\mu}_{\gamma\beta_s} T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_{s-1}\ \mu\ \beta_{s+1}\cdots\beta_q}$$

2. **Lie derivative** generates an **active diffeomorphism** OF $\mathscr{B}$

$$\boxed{\mathcal{L}_\xi T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q}} := \xi^\gamma \partial_\gamma T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q} - T^{\gamma\alpha_2\cdots\alpha_p}_{\beta_1\cdots\beta_q} \partial_\gamma \xi^{\alpha_1} - \cdots - T^{\alpha_1\cdots\alpha_{p-1}\gamma}_{\beta_1,\cdots\beta_q} \partial_\gamma \xi^{\alpha_p} + T^{\alpha_1\cdots\alpha_p}_{\gamma\beta_2\cdots\beta_q} \partial_{\beta_1} \xi^\gamma + \cdots + T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_{q-1}\gamma} \partial_{\beta_q} \xi^\gamma$$

Tool 3. : Derivative of an integral on $\mathscr{B}$

1. Definition of a **volume-form** (measure) and action on $\mathscr{B}$

$$\omega_n, \qquad \mathscr{S} := \int_{\mathscr{B}} \mathscr{L}\ \omega_n$$

where $\mathscr{L}$ is a **Lagrangian density** of a action.

2. **Poincaré** : Variation of integral under **active diffeomorphism**.
Let ω be a p-form ($p \leq n$) on $\mathscr{B}$. Then under the flow (with tangent ξ):

$$\int_{\mathscr{B}} \omega \quad \text{integral invariant} \quad \Longleftrightarrow \quad \boxed{\mathcal{L}_\xi\ \omega = 0}$$

3. Application on an action (remark presence of the **second term** ! e.g. Obukhov & Putzfield, 2014):

$$\mathscr{S} = \int_{\mathscr{B}} \mathscr{L} \omega_n \quad \Longrightarrow \quad \delta \mathscr{S} = \int_{\mathscr{B}} \mathcal{L}_\xi \left(\mathscr{L}\right) \omega_n + \boxed{\int_{\mathscr{B}} \mathscr{L}\ \mathcal{L}_\xi \omega_n}$$

1. **Continuum** is a compact, connected, oriented manifold :
   - a metric tensor $\mathbf{g}$ with components $g_{\alpha\beta}(\mathbf{x})$
   - an affine connection ∇ with coefficients $\Gamma^{\gamma}_{\alpha\beta}(\mathbf{x})$
   - a **volume-form** ω_n with components $\omega_{n\;1\cdots n}(\mathbf{x})$
2. ∇ and $\mathbf{g}$ are **compatible** if

$$\nabla_{\gamma} g_{\alpha\beta} \equiv 0$$

3. ∇ and ω_n are **compatible** if (e.g. Saa 1995)

$$\boxed{\mathcal{L}_{\xi}\omega_n = \left(\nabla_{\alpha}\xi^{\alpha}\right)\omega_n}$$

ξ tangent to flow (active diffeomorphism). ω_n exists !

1. **Continuum** is a compact, connected, oriented manifold :
   - a metric tensor $\mathbf{g}$ with components $g_{\alpha\beta}(\mathbf{x})$
   - a Levi-Civita connection $\overline{\nabla}$ with coefficients $\overline{\Gamma}^{\gamma}_{\alpha\beta}(\mathbf{x})$
   - a **volume-form** $\overline{\omega}_n := \sqrt{\mathrm{Det}\mathbf{g}}\, dx^1 \wedge \cdots \wedge dx^n$

2. **Symmetric** connection $\overline{\nabla}$ and metric $\mathbf{g}$ are **compatible**

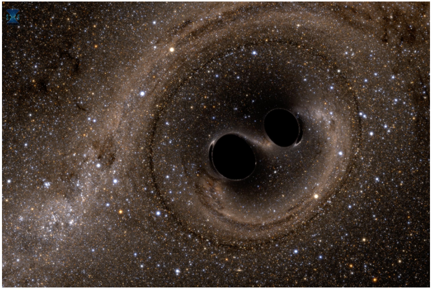
$$\overline{\nabla}_{\gamma} g_{\alpha\beta} \equiv 0, \qquad \overline{\Gamma}^{\gamma}_{\alpha\beta} = (1/2)\, g^{\gamma\lambda} \left(\partial_{\beta} g_{\alpha\lambda} + \partial_{\alpha} g_{\lambda\beta} - \partial_{\lambda} g_{\alpha\beta}\right)$$

3. Symmetric connection $\overline{\nabla}$ and $\overline{\omega}_n$ are **compatible**

$$\mathcal{L}_{\xi}\overline{\omega}_n = \left(\overline{\nabla}_{\alpha}\xi^{\alpha}\right)\overline{\omega}_n$$

ξ tangent to flow (active diffeomorphism). (**Divergence theorem**)

Relative Gravitation : Einstein spacetime (Riemann continuum)



1. The **Hilbert-Einstein** action governs the (simplest) theory for **vacuum spacetime** in Relative Gravitation :

$$\mathscr{S} := \frac{1}{2\chi}\int_{\mathscr{M}} \overline{\mathfrak{R}}\,\overline{\omega}_n, \quad \overline{\mathfrak{R}} := g_{\alpha\beta}\,\overline{\mathfrak{R}}^{\alpha\beta}, \quad \overline{\mathfrak{R}}_{\alpha\beta} := \mathfrak{R}^{\gamma}_{\gamma\alpha\beta}$$

1. The **Euler variation** $\delta g_{\alpha\beta}$ induces the Einstein field equation:

$$\Delta \mathscr{S} = (1/2\chi) \int_{\mathscr{M}} \left(\overline{\Re}^{\alpha\beta} - (\overline{\Re}/2) g^{\alpha\beta} \right) \delta g_{\alpha\beta} \, \varpi_n + \mathrm{BT} = 0$$

with $\delta \varpi_n = -(1/2) g^{\alpha\beta} \; \delta g_{\alpha\beta} \; \varpi_n$

2. Einstein **field equation** (not a conservation laws) holds:

$$\boxed{\overline{G}^{\alpha\beta} := \overline{\Re}^{\alpha\beta} - (\overline{\Re}/2) g^{\alpha\beta} = 0}$$

3. **Principle of General Covariance.** Considering active diffeomorphism engendered by **Lie derivative** along ξ,

$$\delta g_{\alpha\beta} \longrightarrow \mathcal{L}_\xi g_{\alpha\beta} = \overline{\nabla}_\alpha \xi_\beta + \overline{\nabla}_\beta \xi_\alpha$$

we deduce the vacuum spacetime **conservation laws**:

$$\boxed{\overline{\nabla}_\beta \overline{G}^{\alpha\beta} = 0}$$

2. Gravitation : Palatini approach e.g. Ferraris et al. 1982

1. **Hilbert-Einstein** action (metric, connection and derivatives):

$$\mathscr{S} = (1/2\chi) \int_{\mathscr{B}} \mathscr{L} \left(g_{\alpha\beta}, \overline{\Gamma}^{\gamma}_{\alpha\beta}, \partial_\lambda \overline{\Gamma}^{\gamma}_{\alpha\beta} \right) \varpi_n$$

2. Variations of metric and connection

$$\begin{aligned} \delta\overline{\Gamma}^{\gamma}_{\alpha\beta} &= (1/2)\, g^{\gamma\lambda} \left(\overline{\nabla}_\beta \delta g_{\alpha\lambda} + \overline{\nabla}_\alpha \delta g_{\lambda\beta} - \overline{\nabla}_\lambda \delta g_{\alpha\beta} \right) \\ \delta\overline{\Re}_{\alpha\beta} &= \overline{\nabla}_\lambda (\delta\Gamma^{\lambda}_{\beta\alpha}) - \overline{\nabla}_\beta (\delta\Gamma^{\lambda}_{\lambda\alpha}) \end{aligned}$$

3. Euler-Lagrange equations as **field equations**:

$$\begin{cases} G_{\alpha\beta} := \Re_{(\alpha\beta)} \left[\Gamma^{\sigma}_{\mu\nu}, \partial_\epsilon \Gamma^{\sigma}_{\mu\nu} \right] - \frac{1}{2} g_{\alpha\beta} \left(g^{\eta\theta} \Re_{(\eta\theta)} \left[\Gamma^{\sigma}_{\mu\nu}, \partial_\epsilon \Gamma^{\sigma}_{\mu\nu} \right] \right) = 0 \\ \overline{\nabla}^{\gamma} g_{\alpha\beta} = 0 \end{cases}$$

Remarque

Conservation laws *follow from* $\delta g_{\alpha\beta} \longrightarrow \mathcal{L}_\xi g_{\alpha\beta} = \overline{\nabla}_\alpha \xi_\beta + \overline{\nabla}_\beta \xi_\alpha$

3. Gravitation: Integral Invariance & Bianchi's identities

1. **Principle of General Covariance** (active diffeomorphism)

$$\delta \mathscr{S} = \int_{\mathscr{B}} \mathcal{L}_\xi(\mathscr{L})\, \overline{\omega}_n + \mathscr{L}(\overline{\nabla}_\alpha \xi^\alpha)\, \overline{\omega}_n = \frac{1}{2\chi} \int_{\mathscr{B}} g^{\beta\lambda} \overline{\nabla}_\nu \overline{\mathcal{R}}_{\beta\lambda}\, \overline{\omega}_n = 0$$

The second term vanishes for **divergence-free gauge** ξ.

2. **Bianchi**'s first identities

$$\overline{\nabla}_\nu \overline{\Re}^\gamma_{\alpha\beta\lambda} + \overline{\nabla}_\beta \overline{\Re}^\gamma_{\nu\alpha\lambda} + \overline{\nabla}_\alpha \overline{\Re}^\gamma_{\beta\nu\lambda} = 0$$

3. Technique : Putting $\alpha = \gamma$ and multiplying by $g^{\beta\lambda}$:

$$\overline{\nabla}_\nu \overline{\mathcal{R}} - \overline{\nabla}_\beta \overline{\mathcal{R}}^\beta_\nu - \overline{\nabla}_\alpha \overline{\mathcal{R}}^\alpha_\nu = 0$$

4. Deduction of the vacuum spacetime **conservation laws**:

$$\boxed{\overline{\nabla}_\beta \overline{G}^\alpha_\beta = 0, \qquad \overline{G}^\alpha_\beta := \overline{\mathcal{R}}^\alpha_\beta - \frac{\overline{\mathcal{R}}}{2}\, \delta^\alpha_\beta}$$

1. Variational formulation of Hilbert-Einstein action with **source**:

$$\delta \mathscr{S} := \frac{1}{2\chi} \int_{\mathscr{B}} \left(\overline{\mathfrak{R}}^{\alpha\beta} - \frac{\overline{\mathcal{R}}}{2} \, g^{\alpha\beta} \right) \mathcal{L}_\xi g_{\alpha\beta} \, \overline{\omega}_n + \int_{\mathscr{B}} \sigma^{\alpha\beta} \mathcal{L}_\xi g_{\alpha\beta} \, \overline{\omega}_n = 0$$

2. Field equation and **energy-momentum** tensor $\sigma^{\alpha\beta}$:

$$(1/2\chi) \left(\overline{\mathfrak{R}}^{\alpha\beta} - \overline{\mathcal{R}}/2 \, g^{\alpha\beta} \right) + \sigma^{\alpha\beta} = 0,$$

3. **Conservation laws** following $\delta g_{\alpha\beta} \longrightarrow \mathcal{L}_\xi g_{\alpha\beta} = \overline{\nabla}_\alpha \xi_\beta + \overline{\nabla}_\beta \xi_\alpha$:

$$\overline{\nabla}_\beta \sigma^{\alpha\beta} + (1/2\chi) \overline{\nabla}_\beta \left(\overline{\mathfrak{R}}^{\alpha\beta} - \overline{\mathcal{R}}/2 \, g^{\alpha\beta} \right) = 0 \quad \Longrightarrow \quad \boxed{\overline{\nabla}_\beta \sigma^{\alpha\beta} = 0}$$

The conservation laws of σ "do not contain" the gravitational term.

Some remarks

(1) Two-steps approach similar to use **substantial variation** which includes transformations of space(time) coordinates and of physical fields (e.g. Obukhov et al. 2014-5, Lompay & Petrov, 2013, ...)

(2) Spacetime in Einstein RG is a **second gradient continuum** :

$$\begin{aligned}
\overline{\Re}^{\lambda}_{\alpha\beta\mu} &= (1/2)g^{\lambda\sigma}\left(\partial_\mu\partial_\alpha g_{\sigma\beta} - \partial_\mu\partial_\beta g_{\sigma\alpha} + \partial_\sigma\partial_\beta g_{\mu\alpha} - \partial_\sigma\partial_\alpha g_{\mu\beta}\right) \\
&+ (1/4)g^{\lambda\sigma}\left(\partial_\alpha g_{\sigma\gamma} + \partial_\gamma g_{\alpha\sigma} - \partial_\sigma g_{\alpha\gamma}\right) g^{\gamma\kappa}\left(\partial_\beta g_{\kappa\mu} + \partial_\mu g_{\beta\kappa} - \partial_\kappa g_{\beta\mu}\right) \\
&- (1/4)g^{\lambda\sigma}\left(\partial_\beta g_{\sigma\gamma} + \partial_\gamma g_{\beta\sigma} - \partial_\sigma g_{\beta\gamma}\right) g^{\gamma\kappa}\left(\partial_\alpha g_{\kappa\mu} + \partial_\mu g_{\alpha\kappa} - \partial_\kappa g_{\alpha\mu}\right)
\end{aligned}$$

Remarque

The goal is now to extend the conservation laws which takes the generic expression $\nabla_\beta \sigma^{\alpha\beta} = 0$ *to more general geometries of* **Relative Gravitation** *and to some* **Inelastic continuum**.

III. Description of some Inelastic Deformation

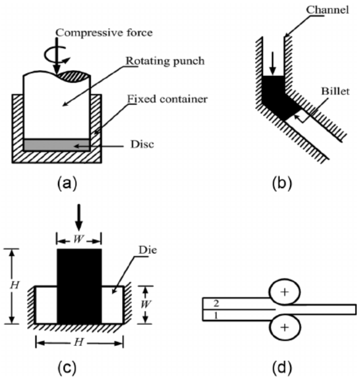


Figure: Material Processing with **Grain Refinement** by Severe Plastic Deformation : Strengthening of material.

Severe Plastic deformation : Example of HPT

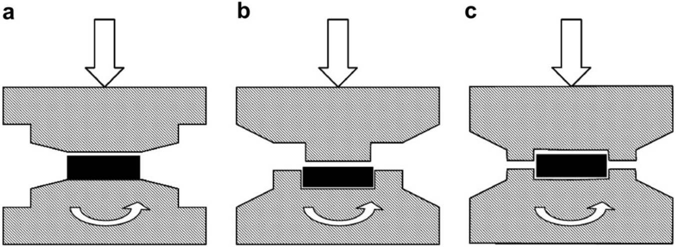


Figure: **Hight Pressure Torsion** Loadings : (a) Unconstrained; (b) and (c) Constrained.

Exemples of HPT plastification

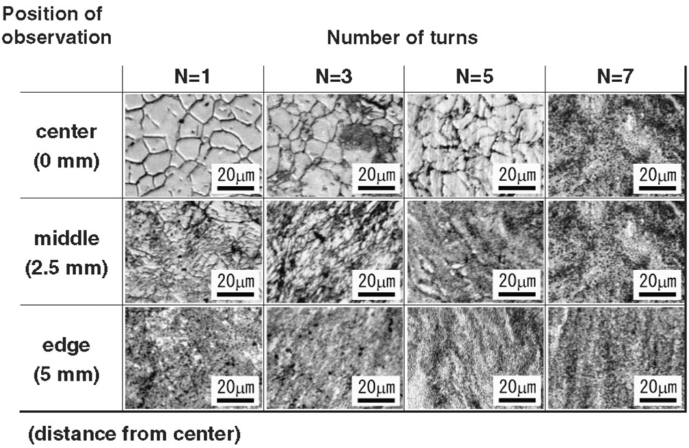


Figure: Optical Micrographs : Microstructures after HPT at the **center**, **half-way** position and **edge** of the disk in a magnesium AZ61 alloy after processing N turns at 423 K (Zhilayev & Langdon 2008).

Severe Plastic Deformation

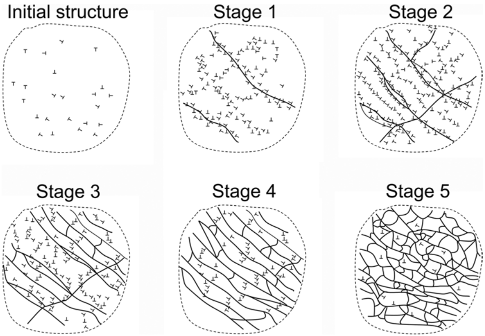


Figure: Grain size refinement under SPD (Cao et al. 2018):

(1) Initial : formation of large size dislocations cell blocks containing dislocations and dislocation cells structures;

(2) Formation of micro-bands and transformation of some nearly **dislocations cells** into cells blocks;

(3) Formation of **lamellar sub-grains** containing numerous dislocations;

(4) Formation of well-developed lamellar sub-grains and some **equiaxed sub-grains** ;

(5) Homogeneous distribution of **equiaxed ultrafine grains** or nano-grains..

Plastic deformation : Some conclusions

1. **Twinning** and **relative motions** of grain with refinement (e.g. Zhylaiev & Langdon 2008) are mostly the mechanisms underlying plastic deformation.
2. In addition, **density of dislocation** augments when the plastic deformation increases (e.g. Cao et al. 2018).
3. **Asympotic response**. For complicated deformation operating in SPD, microstructure mostly reach a steady state at which further deformation does not change the overall microstructure.

Dislocations density and grains relative motions

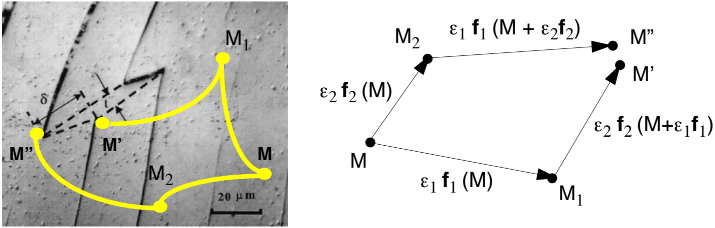


Figure: (L) Shear band formation. (R) Cartan parallelogram.

1. (Left) Illustration of a shear band formation resulting from a **defect discontinuity** (δ, ℓ) at points M' and M".
2. (Right) Cartan parallelogram with area $\epsilon_1 \times \epsilon_2$. At the continuum level, nucleation and migration of a **great number of defects** (discontinuities) occur, and give rise to **macroscopic plastic strain** (Lüders-Hartmann bands)

Dislocations density and grains relative motions

Schouten's relations, 1954:

Theorem

Say an affinely connected manifold $\mathscr{B}$. Let consider two arbitrary vectors $\mathbf{f}_1$ *and* $\mathbf{f}_2$ *at a point* M*, they define two paths of length* ϵ_1 *and* ϵ_2*. Then:*

$$\begin{cases} \lim_{(\epsilon_1,\epsilon_2)\to 0} \dfrac{(\theta' - \theta'')}{\epsilon_1\epsilon_2} & = \aleph(\mathbf{f}_1, \mathbf{f}_2)[\theta] \\ \lim_{(\epsilon_1,\epsilon_2)\to 0} \dfrac{(\mathbf{w}'' - \mathbf{w}')}{\epsilon_1\epsilon_2} & = \Re(\mathbf{f}_1, \mathbf{f}_2, \mathbf{w}) - \nabla_{\aleph(\mathbf{f}_1,\mathbf{f}_2)}\mathbf{w} \end{cases}$$

Proof (Partly R 1997, complete R 2021) With

$$\begin{aligned} \aleph(\mathbf{f}_1, \mathbf{f}_2)[\theta] &:= (\nabla_{\mathbf{f}_1}\mathbf{f}_2 - \nabla_{\mathbf{f}_2}\mathbf{f}_1)[\theta] - [\mathbf{f}_1, \mathbf{f}_2][\theta] \\ \Re(\mathbf{f}_1, \mathbf{f}_2, \mathbf{w}) &:= \nabla_{\mathbf{f}_1}\nabla_{\mathbf{f}_2}\mathbf{w} - \nabla_{\mathbf{f}_2}\nabla_{\mathbf{f}_1}\mathbf{w} - \nabla_{[\mathbf{f}_1,\mathbf{f}_2]}\mathbf{w} \end{aligned}$$

Sketch of the proof (1)

1) Consider a **scalar field** $\theta(M)$, we have respectively the relations:

$$\left\{\begin{array}{lcl} \theta(M_1) & = & \theta(M+\varepsilon_1\ \mathbf{f}_1) = \theta(M) + \nabla_{\varepsilon_1 \mathbf{f}_1(M)}\theta(M) \\ \theta(M') & = & \theta(M_1+\varepsilon_2 \mathbf{f}_2(M+\varepsilon_1 \mathbf{f}_1)) = \theta(M_1) + \nabla_{\varepsilon_2 \mathbf{f}_2(M+\varepsilon_1 \mathbf{f}_1)}\ \theta(M_1) \end{array}\right.$$

We obtain the value of scalar field at M' in terms of **its value at M**, by noticing $\theta(M') = \theta'$ and $\theta(M) = \theta$, :

$$\begin{array}{rcl} \theta' & = & \theta + \varepsilon_1 \nabla_{\mathbf{f}_1}\theta + \varepsilon_2 \nabla_{\mathbf{f}_2}\theta + \varepsilon_2\varepsilon_1 \nabla_{\nabla_{\mathbf{f}_1}\mathbf{f}_2}\theta + \varepsilon_2\varepsilon_1 \nabla_{\mathbf{f}_2}\nabla_{\mathbf{f}_1}\theta \\ & + & \varepsilon_2\varepsilon_1^2 \nabla_{\nabla_{\mathbf{f}_1}\mathbf{f}_2}\nabla_{\mathbf{f}_1}\theta \end{array} \quad (1)$$

2) Similarly, we also obtain:

$$\begin{array}{rcl} \theta'' & = & \theta + \varepsilon_2 \nabla_{\mathbf{f}_2}\theta + \varepsilon_1 \nabla_{\mathbf{f}_1}\theta + \varepsilon_1\varepsilon_2 \nabla_{\nabla_{\mathbf{f}_2}\mathbf{f}_1}\theta + \varepsilon_1\varepsilon_2 \nabla_{\mathbf{f}_1}\nabla_{\mathbf{f}_2}\theta \\ & + & \varepsilon_1\varepsilon_2^2 \nabla_{\nabla_{\mathbf{f}_2}\mathbf{f}_1}\nabla_{\mathbf{f}_2}\theta \end{array} \quad (2)$$

Sketch of the proof (2)

3) Following the same method, say a **vector field**, with $\mathbf{w} := \mathbf{w}(M)$,:

$$\begin{aligned}\mathbf{w}' &= \mathbf{w} + \varepsilon_1 \nabla_{\mathbf{f}_1}\mathbf{w} + \varepsilon_2 \nabla_{\mathbf{f}_2}\mathbf{w} + \varepsilon_2\varepsilon_1 \nabla_{\nabla_{\mathbf{f}_1}\mathbf{f}_2}\mathbf{w} + \varepsilon_2\varepsilon_1 \nabla_{\mathbf{f}_2}\nabla_{\mathbf{f}_1}\mathbf{w} \\ &+ \varepsilon_2\varepsilon_1^2 \nabla_{\nabla_{\mathbf{f}_1}\mathbf{f}_2}\nabla_{\mathbf{f}_1}\mathbf{w} \qquad (3)\end{aligned}$$

and

$$\begin{aligned}\mathbf{w}'' &= \mathbf{w} + \varepsilon_2 \nabla_{\mathbf{f}_2}\mathbf{w} + \varepsilon_1 \nabla_{\mathbf{f}_1}\mathbf{w} + \varepsilon_1\varepsilon_2 \nabla_{\nabla_{\mathbf{f}_2}\mathbf{f}_1}\mathbf{w} + \varepsilon_1\varepsilon_2 \nabla_{\mathbf{f}_1}\nabla_{\mathbf{f}_2}\mathbf{w} \\ &+ \varepsilon_1\varepsilon_2^2 \nabla_{\nabla_{\mathbf{f}_2}\mathbf{f}_1}\nabla_{\mathbf{f}_2}\mathbf{w} \qquad (4)\end{aligned}$$

4) Final step:

(1) - (2) induces the first relation on torsion.

(3) - (4) induces the second result on curvature (and torsion). $\square$

Remarque

Calculus is done **exculsively** *at point M (then $\mathscr{T}_M\mathscr{B}$). Without curvature but with torsion, we may have vector field discontinuities.*

Covariance of Lagrangian $\mathscr{L}$ (Antonio & R 2011)

Theorem

The model of continuum defined by the Lagrangian (Reichenbach, 1929)

$$\mathscr{L} = \mathscr{L}(g_{\alpha\beta},\ \Gamma^{\gamma}_{\alpha\beta},\ \partial_{\lambda}\Gamma^{\gamma}_{\alpha\beta})$$

is **covariant** *if and only if*

$$\mathscr{L} = \mathscr{L}(g_{\alpha\beta},\ \aleph^{\gamma}_{\alpha\beta},\ \mathfrak{R}^{\gamma}_{\alpha\beta\lambda})$$

Remarks :

- **Primal / internal variables** are metric $g_{\alpha\beta}$, torsion $\aleph^{\gamma}_{\alpha\beta}$, and curvature $\mathfrak{R}^{\gamma}_{\alpha\beta\lambda}$ (**Continuum physics** : elasticity, fluid mechanics, gravitation, electromagnetism, plastic deformation ...)
- This theorem extends **Cartan** (1922) and **Lovelock-Rund** (1971, 1975) theorems from Riemann to **Riemann-Cartan continuum**.

Sketch of proof

Proof: Different steps is given below (Antonio & R, 2011):

1. Consider an arbitrary **change of coordinates** (C^∞ diffeomorphism)

$$x^\alpha = x^\alpha\left(y^i\right), \quad \text{and} \quad J_i^\alpha := \partial_i x^\alpha, \cdots$$

   Write as follows : $g_{ij} = J_i^\alpha \; J_j^\beta \; g_{\alpha\beta}$, ...

2. Assume **diffeomorphism-invariance** (covariance) and the transformation rules for the metric $g_{\alpha\beta}$ and the connection $\Gamma_{\alpha\beta}^\gamma$

$$\mathscr{L}(g_{ij}, \Gamma_{ij}^k, \partial_l \Gamma_{ij}^k + \Gamma_{ij}^m \Gamma_{lm}^k) = \mathscr{L}(g_{\alpha\beta}, \Gamma_{\alpha\beta}^\gamma, \partial_\lambda \Gamma_{\alpha\beta}^\gamma + \Gamma_{\alpha\beta}^\mu \Gamma_{\lambda\mu}^\gamma)$$

3. Decompose $\Gamma_{\alpha\beta}^\gamma$ and $\partial_\alpha \Gamma_{\beta\lambda}^\kappa + \Gamma_{\beta\lambda}^\xi \Gamma_{\alpha\xi}^\kappa$ into **symmetric** and **skew symmetric** parts.

4. **Differentiate** $\mathscr{L}$ with respect to J_i^α, ...

5. Apply the **Quotient theorem** (**h** tensor):
   $A^{\alpha\beta} h_{\alpha\beta} + B^{\alpha\beta\gamma} \nabla_\gamma h_{\alpha\beta}$ scalar $\Rightarrow$ $A^{\alpha\beta}$, $B^{\alpha\beta\gamma}$ tensors $\square$.

Remarque

$\mathscr{L}$ satisfies the second axiom of Hilbert on the invariance of the action.

- **Elastic Continuum** (deduced from Cauchy-Weyl theorem, 1850, 1939)[2]

$$\mathscr{L} = \mathscr{L}(g_{\alpha\beta}), \qquad g_{\alpha\beta} := \mathbf{g}\left(\mathbf{f}_\alpha, \mathbf{f}_\beta\right)$$

where $\{\mathbf{f}_\alpha, \alpha = 1, n\}$ is an embedded base of $\mathscr{B}$ (Pfaffian **F**!).

- **Einstein** Relative Gravitation

$$\mathscr{L} = \mathscr{L}(g_{\alpha\beta},\ \overline{\mathfrak{R}}^{\gamma}_{\alpha\beta\lambda}) \qquad \text{as} \qquad \mathscr{L} = (1/2\chi)\ g^{\alpha\beta}\ \overline{\mathfrak{R}}^{\lambda}_{\lambda\alpha\beta}$$

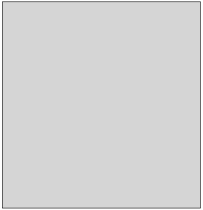
extended to **Einstein-Cartan** $\mathscr{L} = \mathscr{L}\left(g_{\alpha\beta}, \aleph^{\gamma}_{\alpha\beta}, \mathfrak{R}^{\gamma}_{\alpha\beta\lambda}\right)$.

- Class of **Inelastic Continua** $\mathscr{L} = \mathscr{L}\left(g_{\alpha\beta}, \aleph^{\gamma}_{\alpha\beta}, \mathfrak{R}^{\gamma}_{\alpha\beta\lambda}\right)$

[2] Complete proof in R2003 for rotational transformations

Some Inelastic continuum models $\mathscr{L} = \mathscr{L}(g_{\alpha\beta}, \aleph^{\gamma}_{\alpha\beta}, \Re^{\lambda}_{\alpha\beta\gamma})$

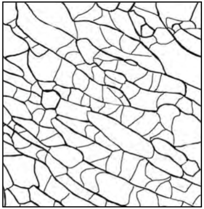
$g_{\alpha\beta}(\mathbf{X}, 0)$



$$\mathbf{x} = \varphi(\mathbf{X}, t)$$

$$d\mathbf{x} = \mathbf{F}(d\mathbf{X})$$

$g_{\alpha\beta}(\mathbf{X}, t) = F^{\mu}_{\alpha}(\mathbf{X}, t)\ g_{\mu\nu}(\mathbf{X}, 0)\ F^{\nu}_{\beta}(\mathbf{X}, t)$

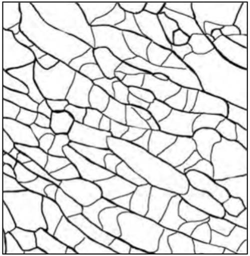


$$\aleph^{\gamma}_{\alpha\beta}(\mathbf{X}, 0) \equiv 0 \quad \rightarrow \quad \aleph^{\gamma}_{\alpha\beta}(\mathbf{X}, t) \neq 0$$

$$\Re^{\gamma}_{\alpha\beta\lambda}(\mathbf{X}, 0) \equiv 0 \quad \rightarrow \quad \Re^{\gamma}_{\alpha\beta\lambda}(\mathbf{X}, t) \neq 0$$

Figure: Inelastic deformation includes : (1) variation of the metric components; (2) variations of the **torsion** and **curvature**.

IV. Principle of General Covariance and Applications



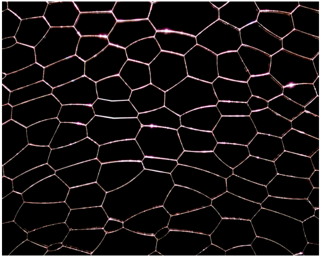


Figure: **Left**: Plastic deformation of (Metallic alloy, set of **microcosms**); **Middle**: Spacetime in Loop Quantum Gravitation (set of "**quanta**").

For the general case, we have.

1. **Principle of General Covariance.** [3] (e.g. Souriau 1975, Duval & Künzle 1978, R 2018) The principle dwells upon Poincaré's invariant integral involving **Lie derivatives**,:

$$\delta \mathscr{S} = \int_{\mathscr{B}} \left(\left(\sigma^{\alpha\beta} \mathcal{L}_\xi g_{\alpha\beta} + \Sigma_\gamma^{\alpha\beta} \mathcal{L}_\xi \aleph^\gamma_{\alpha\beta} + \Xi_\lambda^{\alpha\beta\mu} \mathcal{L}_\xi \Re^\lambda_{\alpha\beta\mu} \right) \omega_n = 0 \right.$$

∇ **compatible** with ω_n and ξ **divergence-free**.

2. **Conservation laws** are obtained when the trajectory is shifted while the action is left unchanged. Active diffeomorphisms are defined by **Lie derivative** variations

$$\{ \mathcal{L}_\xi g_{\alpha\beta}, \quad \mathcal{L}_\xi \aleph^\gamma_{\alpha\beta}, \quad \mathcal{L}_\xi \Re^\lambda_{\alpha\beta\mu} \}$$

[3] It can be extended to Noether-Klein method to provide Klein identities by considering ξ, $\nabla\xi$, $\nabla\nabla\xi$, ...)e.g. Lompay & Petrov 2013)

Application (1) : Lagrangian of the form $\mathscr{L}(g_{\alpha\beta})$

Constitutive laws. The "stress" σ is defined as :

$$\sigma^{\alpha\beta} := \frac{\partial \mathscr{L}}{\partial g_{\alpha\beta}}(g_{\mu\nu}, \underbrace{\aleph^{\gamma}_{\mu\nu}, \Re^{\gamma}_{\mu\nu\epsilon}}_{\text{frozen}})$$

Theorem

Consider a Riemann-Cartan continuum $(\mathscr{B}, \mathbf{g}, \nabla)$ with a Lagrangian function $\mathscr{L}(g_{\alpha\beta})$. Then :

$$\nabla_\alpha \sigma^\alpha_\gamma = 0$$

Proof: 1. Local invariance. The PGC of the Lagrangian holds:

$$\begin{aligned} \delta\mathscr{L} &:= \int \sigma^{\alpha\beta}\ \mathcal{L}_\xi g_{\alpha\beta}\ \omega_n \\ \mathcal{L}_\xi\ g_{\alpha\beta} &= \xi^\gamma\ \nabla_\gamma g_{\alpha\beta} + g_{\gamma\beta}\ \nabla_\alpha \xi^\gamma + g_{\alpha\gamma}\ \nabla_\beta \xi^\gamma \\ &+ \xi^\gamma\ \left(g_{\alpha\nu}\ \aleph^\nu_{\gamma\beta} + g_{\nu\beta}\ \aleph^\nu_{\gamma\alpha}\right) \end{aligned}$$

Integrate by parts and shift the boundary terms to obtain:

$$\nabla_\alpha \sigma^\alpha_\gamma + \aleph^\lambda_{\alpha\gamma}\ \sigma^\alpha_\lambda = 0$$

Proof (2)

2. Global Invariance. By considering a uniform vector field ξ such that $\nabla\xi \equiv 0$, we obtain a **Klein Identity**:

$$\sigma^{\alpha\beta}\left(g_{\alpha\nu}\ \aleph^{\nu}_{\gamma\beta} + g_{\nu\beta}\ \aleph^{\nu}_{\gamma\alpha}\right) = 2\ \sigma^{\alpha}_{\mu}\ \aleph^{\mu}_{\gamma\alpha} \equiv 0$$

We deduce the classical continuum conservation laws:

$$\nabla_{\alpha}\sigma^{\alpha}_{\gamma} = 0 \quad \square$$

- The implicit presence of the **torsion** within the covariant derivative should be accounted for (e.g. Futhazar et al. 2014). (Spatial fading of waves)
- Other identities might exist according to the choice of $\xi(\mathbf{x})$. (e.g. Lompay & Petrov 2013)

Example of $3D$-elasticity with non evolving defects

1. Lagrangian $\mathscr{L}$

$$u(\varepsilon) = \frac{\lambda}{2}\,\mathrm{Tr}^2\varepsilon + \mu\mathrm{Tr}(\varepsilon^2), \qquad \mathscr{L} := \frac{\rho}{2}\|\frac{d\mathbf{v}}{dt}\|^2 - u(\varepsilon)$$

2. Small strain assumption:

$$g_{\alpha\beta} \simeq \delta_{\alpha\beta} + \nabla_\alpha u_\beta + \nabla_\beta u_\alpha, \qquad \varepsilon := \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}\right)$$

3. Extended Navier equation:

$$\partial_t^2 u^\alpha = \frac{\lambda+\mu}{\rho}\nabla^\alpha\nabla_\beta u^\beta + \frac{\mu}{\rho}\nabla^\beta\nabla_\beta u^\alpha + \frac{\mu}{\rho}\hat{g}^{\alpha\mu}\left(\Re_{\mu\nu}\ u^\gamma - \aleph^\gamma_{\beta\mu}\nabla_\gamma u^\beta\right)$$

Remarque

Wave behaviour similar to the bending of light (electromagnetic wave) waving through a gravitational field !

Application (2) : Inelastic continuum

1. Separate the torsion into skew-symmetric **Cartan's forms**:

$$\aleph^{\gamma}_{\alpha\beta} \quad \rightarrow \quad \left\{\omega^{(1)}_{\alpha\beta}, \omega^{(2)}_{\alpha\beta}, \omega^{(3)}_{\alpha\beta}\right\}$$

(dislocations density)

2. Reduction of curvature to **Ricci tensor** in $3D$:

$$\Re^{\gamma}_{\lambda\alpha\beta} \quad \rightarrow \quad \mathcal{R}_{\alpha\beta} := \Re^{\gamma}_{\gamma\alpha\beta}$$

(disclinations density)

3. (Covariant) **Lagrangian** of some class of inelastic continua

$$\boxed{\mathscr{L} = \mathscr{L}(g_{\alpha\beta}, \omega^{(n)}_{\alpha\beta}, \mathcal{R}_{\alpha\beta})}$$

Principle of General Covariance

1. Gauge Invariance (divergence-free gauge ξ)

$$\delta\mathscr{S} = \int_{\mathscr{B}} \left(\sigma^{\alpha\beta} \mathcal{L}_\xi g_{\alpha\beta} + \Sigma^{\alpha\beta}_{(n)} \mathcal{L}_\xi \omega^{(n)}_{\alpha\beta} + \Xi^{\alpha\beta} \mathcal{L}_\xi \mathcal{R}_{\alpha\beta} \right) \omega_n = 0$$

2. Constitutive laws

$$\sigma^{\alpha\beta} := \frac{\partial \mathscr{L}}{\partial g_{\alpha\beta}}, \qquad \Sigma^{\alpha\beta}_{(n)} := \frac{\partial \mathscr{L}}{\partial \omega^{(n)}_{\alpha\beta}}, \qquad \Xi^{\alpha\beta} := \frac{\partial \mathscr{L}}{\partial \mathcal{R}_{\alpha\beta}}$$

Remarque

Stress $\sigma^{\alpha\beta}$ is symmetric, $\Sigma^{\alpha\beta}_{(n)}$ are skew-symmetric. The dual of Ricci curvature $\Xi^{\alpha\beta}$ is not necessarily symmetric.

Conservation Laws from Gauge Theory

Theorem

Let $\mathscr{B}$ a continuum with Lagrangian $\mathscr{L}(g_{\alpha\beta}, \omega^{(n)}_{\alpha\beta}, \mathcal{R}_{\alpha\beta}), n = 1, 2, 3$. Then the conservation laws hold:

$$\boxed{\nabla_\alpha \tilde{\sigma}^\alpha_\gamma = 0}$$

where we define the **generalized stress** $\tilde{\sigma}^\alpha_\gamma$*:*

$$\boxed{2\tilde{\sigma}^\alpha_\gamma := 2\ \sigma^\alpha_\gamma + 2\ \Sigma^{\alpha\beta}_{(n)} \omega^{(n)}_{\gamma\beta} + 2\ \Xi^{\alpha\beta} \mathcal{R}_{\gamma\beta}}$$

Remarque

Presence of dislocations density in $\tilde{\sigma}^\alpha_\gamma$ seems giving new interesting insights for modeling strain hardening phenomenon during plastic deformation : "the first problem to be attempted by dislocation theory and may be the last to be solved" (e.g. Kocks & Mecking, 2003).

Sketch of the proof

1. **Lie derivatives of primal variables**:

$$
\begin{aligned}
\mathcal{L}_\xi g_{\alpha\beta} &= \xi^\gamma \left(\nabla_\gamma g_{\alpha\beta} + g_{\alpha\nu}\ \aleph^\nu_{\gamma\beta} + g_{\nu\beta}\ \aleph^\nu_{\gamma\alpha} \right) + \left(g_{\gamma\beta}\ \nabla_\alpha \xi^\gamma + g_{\alpha\gamma}\ \nabla_\beta \xi^\gamma \right) \\
\mathcal{L}_\xi \omega^{(n)}_{\alpha\beta} &= \xi^\gamma \left(\nabla_\gamma \omega^{(n)}_{\alpha\beta} + \omega^{(n)}_{\nu\beta}\ \aleph^\nu_{\gamma\alpha} + \omega^{(n)}_{\alpha\nu}\ \aleph^\nu_{\gamma\beta} \right) + \left(\omega^{(n)}_{\gamma\beta} \nabla_\alpha \xi^\gamma + \omega^{(n)}_{\alpha\gamma} \nabla_\beta \xi^\gamma \right) \\
\mathcal{L}_\xi \mathcal{R}_{\alpha\beta} &= \xi^\gamma \left(\nabla_\gamma \mathcal{R}_{\alpha\beta} + \mathcal{R}_{\nu\beta}\ \aleph^\nu_{\gamma\alpha} + \mathcal{R}_{\alpha\nu}\ \aleph^\nu_{\gamma\beta} \right) + \left(\mathcal{R}_{\gamma\beta} \nabla_\alpha \xi^\gamma + \mathcal{R}_{\alpha\gamma} \nabla_\beta \xi^\gamma \right)
\end{aligned}
$$

2. **The three Klein-Noether identities take the following form for** $\gamma = 1, 2, 3$:

$$
\begin{aligned}
0 &= \sigma^{\alpha\beta} \left(\nabla_\gamma g_{\alpha\beta} + g_{\alpha\nu}\ \aleph^\nu_{\gamma\beta} + g_{\nu\beta}\ \aleph^\nu_{\gamma\alpha} \right) \\
&+ \sum_{n=1}^{n=3} \left[\Sigma^{\alpha\beta}_{(n)} \left(\nabla_\gamma \omega^{(n)}_{\alpha\beta} + \omega^{(n)}_{\nu\beta}\ \aleph^\nu_{\gamma\alpha} + \omega^{(n)}_{\alpha\nu}\ \aleph^\nu_{\gamma\beta} \right) \right] \\
&+ \Xi^{\alpha\beta} \left(\nabla_\gamma \mathcal{R}_{\alpha\beta} + \mathcal{R}_{\nu\beta}\ \aleph^\nu_{\gamma\alpha} + \mathcal{R}_{\alpha\nu}\ \aleph^\nu_{\gamma\beta} \right)
\end{aligned}
$$

3. **By accounting fot these identities, it is easy to deduce the following equation by integrating by parts**:

$$
\int_{\mathscr{B}} \xi^\gamma \nabla_\alpha \left[2\ \sigma^\alpha_\gamma + 2\ \Sigma^{\alpha\beta}_{(n)} \omega^{(n)}_{\gamma\beta} + \left(\Xi^{\alpha\beta} \mathcal{R}_{\gamma\beta} + \Xi^{\beta\alpha} \mathcal{R}_{\beta\gamma} \right) \right] \omega_n = 0 \qquad \square
$$

Comment on Generalized Stress

Remarque

Different names of $\Sigma_{(n)}^{\alpha\beta}$ *and* $\Xi^{\alpha\beta}$ *: Hypermomentum in* **Relative Gravitation with spin** *(e.g. Hehl et al., 1975-76,77,78), or Total energy stress (e.g. Duval and Künzle, 1978); Micro-stress and Polar micro-stress in* **Strain Gradient Plasticity** *(e.g. Gurtin and Anand, 2004-2005); Currents in* **Generalized Gravity** *(e.g. Obukhov et al. 2013, 2015, Lompay et al. 2013) ....*

Remarque

Generalized stress $2\tilde{\sigma}_{\gamma}^{\alpha}$ *includes*

1. *a "reversible" part* σ_{γ}^{α}
2. *contribution of* **Cartan** *forms* $\omega_{\alpha\beta}^{(n)}$ *(corresponding to a spin, due to dislocations) and its dual* $\Sigma_{(n)}^{\alpha\beta}$*,*
3. *contribution of* **Ricci** *tensor* $\mathcal{R}_{\alpha\beta}$ *and its dual* $\Xi^{\alpha\beta}$ *(due to disclinations).*

V. Concluding remarks

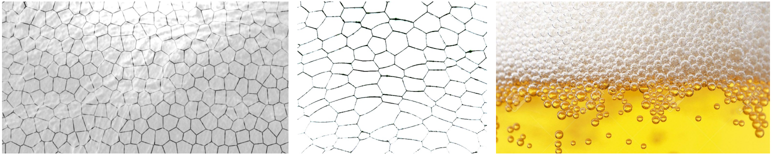


Figure: Plasticity, Gravitation and way of Life

The **Principle of General Covariance** (Gauge Theory) can be applied for Relative Gravitation spacetime and for Inelastic Continuum modeled as Riemann-Cartan manifold. Application in sequence the two invariance criteria is worth:

1. **Covariance** (passive diffeomorphism, necessary criterium for any physics theory, e.g. Norton 2005)
2. **Principle of General Covariance** (active gauge invariance, e.g. Souriau, 1975)

Some other applications (R 2023)

The same **sequential method** was applied to other various model :

1. **Riemann-Cartan** spacetime / continuum $\mathscr{L}(g_{\alpha\beta}, \aleph^{\gamma}_{\alpha\beta}, \Re^{\gamma}_{\alpha\beta\lambda})$ (MAG without nonmetricity) (e.g. Obukhov et al, 2015);
2. **Weitzeböck** Continuum with Lagrangian $\mathscr{L}(g_{\alpha\beta}, \mathcal{T}^{\gamma}_{\alpha\beta})$ where $\mathcal{T}^{\gamma}_{\alpha\beta} := \Gamma^{\gamma}_{\alpha\beta} - \overline{\Gamma}^{\gamma}_{\alpha\beta}$ is the contortion tensor (similar to plasticity);
3. "**Gradient Weitzeböck**" Continuum with Lagrangian $\mathscr{L}(g_{\alpha\beta}, \mathcal{T}^{\gamma}_{\alpha\beta}, \overline{\nabla}\mathcal{T}^{\gamma}_{\alpha\beta})$ which is very similar to the **Strain Gradient Plasticity** (e.g. review Voyiadjis & Song, 2019, Gurtin-Anand, 2005)

We get an **unified framework** for conservation laws :

$$\boxed{\nabla_{\alpha}\tilde{\sigma}^{\alpha}_{\gamma} = 0} \quad \boxed{\tilde{\sigma} = \sigma + \mathscr{C}(\aleph, \Re, \Sigma, \Xi)} \quad \text{or} \quad \boxed{\tilde{\sigma} = \sigma + \mathscr{C}(\mathcal{T}, \overline{\nabla}\mathcal{T}, \Sigma, \Xi)}$$

Very similar generalized stress was obtained in e.g. **Medina et al, 2019**, for studying Einstein-Cartan cosmologies, avoiding singularity.

Merci pour votre attention !

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